GOAL-Oriented Problem Solving

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Problem solving skills, although an important goal of many introductory science courses, are not easily mastered by many students.[2-5] The GOAL approach is easy to recall and encourages students to *Gather* information about the problem, *Organize* an approach to the solution, *Analyze* the problem, and then *Learn* from their efforts. This mnemonic is based on the findings of other researchers who have studied how students solve problems and was conceived during development of an introductory physics text[6] as a mental aid to avoid formula-centered problem solving and to teach skills employed by experienced problem solvers.

I. INTRODUCTION

The GOAL problem-solving strategy is based on the numerous research studies that have been published in recent years on how students learn to solve physics problems.[7-19] Maloney's chapter in the *Handbook of Research on Science Teaching and Learning* is a thorough summary.[14] Many of these studies, and most introductory physics textbooks, suggest a framework for solving physics problems that is similar to the problem solving methods described in Polya's book, *How to Solve It*,[7] originally published in 1945. Polya advocated a four-step strategy for problem solving: 1) understand the problem, 2) devise a plan, 3) execute the plan, and 4) look back to review the results. The GOAL strategy is essentially this same model in mnemonic form so that it is easy for students to remember.

One of the earliest studies of problem solving in physics used a strategy similar to Polya's to demonstrate that students who received specific problem-solving instruction showed improvement in their ability to solve problems compared to students who did not receive special instruction.[8] The researchers organized the details of the strategy into a checklist, but they found that a problem-solving checklist did not work well for many students who find such a tool to be cumbersome and dull, they may misplace it, or they simply do not use it. Better success was found with a "template" that has the key steps to solving a problem on a sheet of paper with blank spaces to be filled in by the student. A similar problem-solving template has been used successfully in our Physics Tutorial Center at North Carolina State University.[9] These templates provide a guide that students can use to plan and organize their solutions, which is beneficial since mastering how to plan a solution is one of the most difficult skills for novice problem solvers.[10] The GOAL strategy serves as a general mental guide that helps students plan their solution when they are faced with "problem-solver's block." The advantage of a mnemonic over a paper template is that students will always have their problem solving tools with them.

One significant difference between novices and experts in their approach to problem solving is the context in which they examine the problem. Experts naturally have a larger knowledge base and more experience solving problems, and this enables them to have a better qualitative understanding of a problem than a novice who often begins working with equations without a broader sense of purpose and comprehension.[14] Novices generally lack the ability to identify the major physics principles that apply to a problem, and instead see the problem in terms of its surface features.[11] For example, novices might classify all questions involving an inclined plane as "ramp problems" regardless of whether Newton's Laws or energy conservation might be the most direct approach to a solution. These inexperienced problem solvers also tend to search for equations that contain variables that appear to relate to the problem, often without a firm understanding of what the equations mean or whether they apply to the given problem.[12] Overall, beginning students view physics problem solving as mostly memorization and manipulation of equations to get a specific, numerical answer, while physicists perceive problem solving as applying a few central concepts to a wide range of scenarios.[13]

A primary purpose of the GOAL strategy is to assist novice problem solvers by providing a procedure that suits their current abilities while training them to develop the skills and techniques typically used by expert problem solvers. As most experienced teachers know, and as was verified in Larkin's research, novice problem solvers sometimes work backwards from the unknown problem solution to the given quantities, while an expert usually works forward from the given to the desired quantities.[8] The GOAL strategy accommodates both methods by encouraging the problem solver to first examine the form or estimated value of the desired quantity and then gather the necessary information to work forward toward the solution. In the same research it was also observed that the novice problem solver required goals and subgoals to direct the work toward a solution. The GOAL strategy provides such guidance and is easy for students to recall.

II. EXPLANATION OF THE *GOAL* STRATEGY

Gathering information about the problem: The first step in solving a problem is understanding the question. Quite often, a student may not know how to tackle a formal solution, but may at least be able to determine the objective of the problem. A careful reading of the problem for key words or phrases like “at rest,” “constant velocity,” or “freely falls” can provide insight into some of the variables that are will be needed in completing a solution. Reviewing any diagrams that accompany the question can help students begin to visualize the physical situation being examined. Recalling their own experience with similar physical phenomena is an extremely valuable early step toward solving a problem. Making an educated guess at the answer or its mathematical form encourages students to think through the problem qualitatively and estimate what kind of answer might be reasonable in a manner similar to an experienced physicist.[2] Another advantage of initially predicting a reasonable answer to a problem is that a student can more accurately check the final result without the potential danger of convincing himself that a wrong answer is correct (the psychological “I-knew-it-all-along” phenomenon[14]). Generally, students can make a guess from their physical intuition or common sense about what they expect to happen based on their own personal experiences with the world around them. If a student makes a guess based on a misconception, the incorrect answer will be confronted directly at the end of the process when the final answer is compared with the initial guess. This method of directly confronting student misconceptions has been shown to be effective in getting students to change their preconceptions.[15,16]

There are several aspects involved in making a good estimate. The simplest is to determine the proper units of the answer. If a numerical value is required for the solution, it can usually be confined to a range of possible values that might be reasonable for the given situation. Examining the limiting cases is often beneficial in determining the form or range of the solution and is a typical practice of experienced physicists.[10,17] Also, the units along with known variable dependencies (proportionalities) can often be used to estimate the algebraic form of the answer.

Insert Figure 1 near here

The primary purpose of this approximation part of the “Gathering” step is not to solve the problem completely, but to think through the problem and narrow the possibilities of the solution. Students should be cautioned against spending too much time estimating a solution. The details of the problem should be left for the “Analysis” step. Of course, for very simple problems, there may not be much additional work required beyond the initial estimate. And in some cases, an educated guess may be sufficient for the given situation. For instance, a narrowed range of possible values for the answer may be sufficient to eliminate all but one option on a multiple-choice test. (Experience has shown that the existence of this possibility acts as an excellent student motivator.) And for problems we encounter in our daily lives, we may only need an accurate estimate and not a precise answer.

The precision of the estimate will generally depend on the experience of the problem solver. A novice problem solver may only limit the answer to the proper order of magnitude while an expert might narrow the range of the answer to a more specific value (*i.e.* v = 10 to 20 m/s) based on familiarity with similar problems or real-life situations. With this preliminary thinking about what kind of answer might be reasonable for a given problem, it is less likely that a novice problem solver will be satisfied with an answer that is physically impossible.

One simple example of how the “Gathering information” step can be applied is the problem of determining the proper trigonometric function to use for finding the normal force for a block resting on a rough inclined plane (Fig. 2). Students often choose the wrong function simply because they do not think about the angle dependence. As part of the first step, students should consider the limiting cases of the inclined plane when it is horizontal and when it is vertical. For these two situations, it is clear that the normal force becomes the weight of the block: N = mg for _ = 0°, and N = 0 for _ = 90°. The trigonometric function that gives 1 for _ = 0° and 0 for _ = 90° is cosine. Therefore, N = mgcos _. This intuitive approach to problem solving gives many students the confidence they need to attempt more complicated problems, and discourages their reliance on using formulas without understanding their meaning.

Insert Figure 2 near here

Organize an approach to the problem: This second step is an extension of the first. Now that the problem objective is established and an attempt has been made to visualize the situation, the given information should be organized in a form that facilitates analysis. (Sometimes it is necessary to break a complicated problem into sub-problems that can be solved individually.) Making a sketch and labeling the critical pieces is almost always helpful.[2] If the problem involves vector quantities, the sketch should also indicate the orientation of the chosen coordinate system so there is no confusion as to the reference frame. Sometimes a free-body diagram, graph, or sketch of the sequence of events is also beneficial, depending on the problem. All relevant information stated in the problem should be listed and assigned variables consistent with standard physics conventions. Often this translation from words to symbols is a significant stumbling block for novice students because they do not know which equations apply simply because the symbols used in the equations are not familiar and they lack appropriate training on how to draw these kind of sketches and diagrams. Alan Van Heuvelen's motion sequence diagrams and multiple representation problem-solving techniques can be very useful.[15] At this point students should categorize the problem and recall the general approach that is taken for that type of problem. For example, if they recognize that they are dealing with a projectile motion question, then they should remember that the horizontal and vertical components of motion are independent except for time of flight. These two components often result in two equations that can be solved simultaneously.

Analyze the problem: Some students like to call this the “Algebra” step. At the end of the “Organize” step, the problem solver defined the relevant physics concepts for the situation being studied. For each concept, a fundamental equation that summarizes that concept should be written. These and possibly other derived equations should be arranged to express the desired quantity in terms of the known values. It is usually best to solve the problem algebraically in terms of the variables before inserting numbers into the equations to facilitate the “Learning” step.

During this analysis, students should examine the problem again qualitatively to ensure proper understanding of the diagrams. For example, check to make sure the force vectors are drawn in correct proportion to each other. Those students who do not review the physical meaning of a problem may not confront a misconception if they blindly solve equations without thinking about the meaning of the formulas.[2]

Learn from your efforts: There is a subtle bit of pedagogy hidden at the end of the Organize step and beginning of the Analysis step where the problem is classified and connections are made to similar problems that the student might have seen. Novice problem solvers do not often think of a long homework assignment as a series of different physical phenomena to understand; they simply see a list of questions with numerical answers that needs to be completed before the due date. The last stage of the GOAL procedure is designed to minimize this mindset.

This closing procedure involves the consideration of several questions: Does the answer from the analysis agree with the original guess, if one was made? Could there be any exceptions for special cases? Does the answer completely satisfy the question? What assumptions were needed to obtain the result? Are there times when these assumptions are not valid? During the analysis, were any equations derived that might prove useful for future problems? Especially important to address are metacognitive issues such as “Why was this problem assigned?” and “What knowledge have I gained by working on it?” Without prodding, students rarely consider these ideas, even though they may have been foremost in their professor’s mind when the problem was assigned.

III. AN EXAMPLE FROM ROTATIONAL DYNAMICS

In order to demonstrate the GOAL problem solving strategy, consider the following example: A bucket of water of mass m = 12 kg is tied to a rope of negligible mass. The rope is wrapped around a solid, uniform cylindrical flywheel of mass M = 25 kg and radius r = 0.50 m. The flywheel is free to rotate with negligible friction about its horizontal axis. The bucket is brought to a height h = 10 m and dropped from rest, making the flywheel rotate. Find the acceleration of the bucket as it falls.

Solution using the GOAL approach:

Gather information: A careful reading of the problem indicates that we can ignore friction and the mass of the rope. Since the mass of the flywheel is more than twice the mass of the bucket it probably cannot be excluded from the problem. We are given some parameters of the situation and told that the initial downward velocity of the bucket is zero. As the bucket falls it pulls on the rope which then rotates the flywheel. For this problem, it should be obvious that the acceleration of the bucket cannot exceed the acceleration due to gravity (g = 9.8 m/s^2). Also, the bucket's downward acceleration must be greater than zero if it falls at all since it is initially at rest. Therefore, the magnitude of the bucket's acceleration must lie within the range:

$$0 < a < g$$

At the very least, a student should know that the acceleration must have the units of m/s^2 in accordance with the SI units given in the problem data.

A more careful or experienced problem solver might examine the algebraic form of the acceleration to determine how the answer will depend on the relative masses of the bucket and flywheel. For this type of analysis, it is often helpful to examine the proportionalities of the variables and test for the extreme cases. Since the acceleration of the bucket should be nearly equal to *g* when the mass of the bucket *m* is much greater than the mass of the flywheel *M* (freefall case), and since the acceleration should be nearly zero when the flywheel is much more massive than the bucket (static case), then we know that:

$$a \propto \frac{m}{M}$$

Therefore, we might guess that:

$$a = \frac{mg}{M}$$

This equation gives the right units, but is not correct because it gives $a \to \infty$ for $m >> M$, which contradicts our common sense limit that a must be less than g. To avoid the singularity for $M = 0$, perhaps a different form would work:

$$a = \frac{mg}{m+M}$$

This equation gives the correct units, proportionalities, and limiting values for $M = 0$ and also $m << M$. While this formula makes sense, it should be verified by analyzing the problem from basic physics principles. In any case, we can see that this problem involves both linear acceleration and a rotating mass and so we will probably have to deal with the translational and rotational forms of Newton's Second Law.

Organize the solution: Drawing a sketch of the situation and then a free-body diagram that shows the forces acting on the bucket is helpful for this problem (Fig. 3). Note that we are defining the downward direction to be positive since this is the direction the bucket is falling. Since the only forces acting on the bucket are gravity pulling down and the tension in the rope pulling up, for the bucket this is simply a one-dimensional motion problem with constant acceleration. However, this linear motion is coupled to the rotational motion of the flywheel.

Insert Figure 3 near here

Analyze the problem: The problem should now be solved in terms of the variables, and then the given data can be inserted into our final equation to get a numerical answer.

Applying Newton's Second Law (and remembering that <u>down</u> is defined to be positive):

$$mg - T = ma \qquad (1)$$

But we do not know the tension T, so we must find this force from the dynamics of the flywheel. The flywheel has no translational motion, so we work with the rotational form of Newton's second law,

$$Torque = rT = I\_ \qquad (2)$$

Where __is the angular acceleration which can be found from:

$$a = r\_ \qquad (3)$$

and I is the moment of inertia of the flywheel, which is basically a solid, uniform cylinder, so:

$$I = \frac{1}{2} Mr^2 \qquad (4)$$

So, now we have all the pieces necessary to solve the problem.

$$mg - T = ma \qquad (1)$$

$$rT = I\_ = 1/2Mr^2(a/r) = Mra/2 \qquad (2, 3, \& 4)$$

therefore,

$$T = Ma/2 \qquad (5)$$

so,

$$ma = mg - Ma/2 \qquad (1 \& 5)$$

therefore,

$$a = \frac{mg}{m+M/2} \qquad (6)$$

Substituting the given values, we have:

$$a = 4.8\ m/s^2 \text{ (positive, which we defined as downward)}$$

Learn from your efforts: Our final algebraic expression for a is almost the same as our guess and at least has the same form so that the extreme cases of $M = 0$ and $m<<M$ give $a = g$ and $a = 0$ respectively. The primary difference between the actual answer and the initial estimate was that our guess did not include the proper effect from the rotational

inertia of the flywheel. Our original guess would be correct for a flywheel with a rotational inertia $I = Mr^2$ (like for a bicycle wheel), but for a solid flywheel, the effective rotational inertia is 1/2 of that. The numerical answer also appears reasonable since it lies within our predicted range $0 < a < g = 9.8\ m/s^2$, has the correct units, and appropriate direction, since acceleration is a vector.

There are several additional insights that can be learned from this problem. One interesting observation is that the acceleration of the bucket does not depend on the radius of the flywheel, so the problem can be solved without this information. The height of the bucket above the water was also not necessary to determine the acceleration, although this height could be used to determine the time it would take for the bucket to fall. If we were asked to find the tension in the rope, we would simply use equation (5) with our acceleration found from equation (6). For this problem, T = 60 N, or about half the weight of the bucket, which again makes sense since the acceleration was found to be about half that of gravitational acceleration g. Here is a good example of why it is most efficient to solve the problem algebraically; often intermediate results to one problem can be applied to another problem.

There were several approximations used for this problem that simplified the real world situation. A more careful analysis would require including the effects of friction of the flywheel and the weight of the rope. Fortunately, these effects tend to counteract each other. Air resistance was also ignored in our solution, but for the given situation, this was a reasonable omission. However, if the bucket and flywheel were much less massive, air resistance could be significant, especially if the bucket reached terminal velocity, then its acceleration would be zero!

From this problem, we learned how the acceleration of a falling object is retarded by the rotational inertia of the rotating mass to which it is connected. This knowledge could be applied to a problem where the mass of a pulley should not be ignored. It should also be noted that choosing the vertical axis to be positive downward reduced the use of negative signs. Many times, the analysis of a problem may be simplified by choosing a coordinate system that is convenient for the given situation.

By reviewing a problem and thinking about how it compares to other physics problems, students build a knowledge structure based on fundamental physics principles, similar to the knowledge structure of experts[2,8]

IV. EVIDENCE FROM USE OF *GOAL* IN THE CLASSROOM

The GOAL problem solving strategy is currently being tested in the experimental classrooms at NC State, in the Physics Tutorial Center[18] and in our general physics labs. It is too early to report much data, but evidence collected so far suggests a positive impact on student ability to solve problems. Students taught this protocol scored 83% correct on the same problems that their (demographically matched) peers only earned 68%. Similarly, a common final exam showed nearly half a letter grade improvement in performance. However, these results could be due to any one of a number of innovations being implemented in these courses. Additional work is needed to see how the GOAL approach by itself affects problem solving skills.

V. SUMMARY

The four steps of the GOAL protocol are not new. They correspond to proven methods of helping students solve problems. Unfortunately, students do not often utilize these known techniques. GOAL is our attempt to create a mnemonic that will be useful for students who “don’t know what to do next” and is applicable to a wide variety of physics problems. Practicing these steps often enough may help novices approach the skill level of experienced problem solvers and begin to think like physicists.

ACKNOWLEDGMENTS

The concepts presented in this paper are based on the work of many physicists who have concerned themselves with student problem solving ability. Some of the most important contributions to the literature are given in the following references. We would also like to thank David Maloney for a helpful discussion of some of these issues and Elizabeth Rieg for comments on a draft of this paper.

Steps in the GOAL Problem Solving Protocol

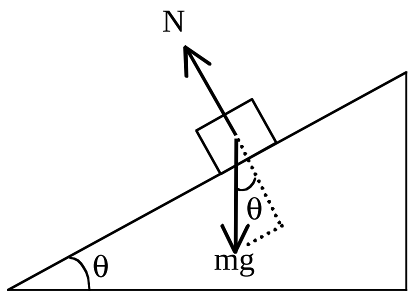
Gather information about the problem

Organize an approach to the solution

Analyze the problem

Learn from your efforts

Figure 1



$$N \overset{?}{=} mg\cos\theta$$

$$N \overset{?}{=} mg\sin\theta$$

Figure 2 Determining the proper trigonometric function for a standard problem

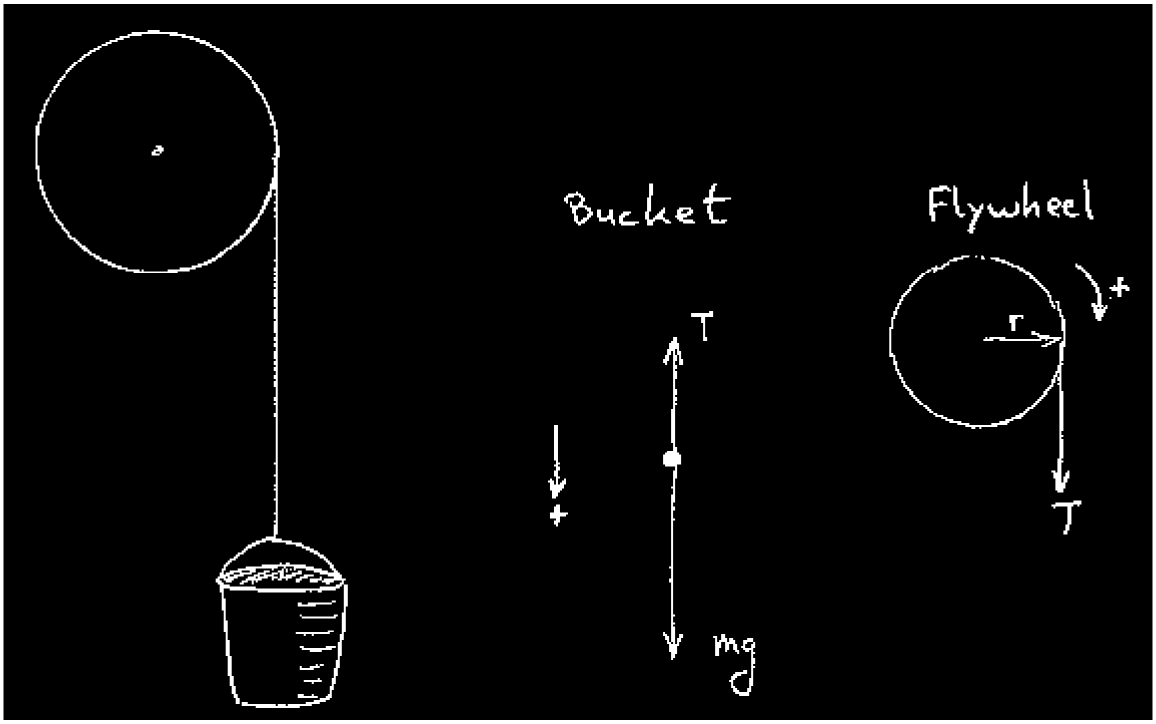


Figure 3 Sketch and free body diagram: a critical part of the organization step.

(Note: If the quality of this scanned image is not good enough for publication, a simple hand-drawn sketch would be sufficient. The idea is to show what a student might draw.)

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[2]A. Van Heuvelen, "Learning to think like a physicist: A review of research-based instructional strategies," *American Journal of Physics* **59**, 891-897 (1991).

[3]P. Heller, R. Keith, and R. Glaser, "Teaching problem solving through cooperative grouping. Part 1: Group versus individual problem solving," *American Journal of Physics* **60,** 627-636 (1992).

[4]F. Reif, "Millikan Lecture 1994: Understanding and teaching important scientific thought processes," *American Journal of Physics* **63,** 17-32 (1995).

[5]F. Reif, *Understanding Basic Mechanics* (Wiley, New York, 1995).

[6]R. Serway and R. Beichner, *Physics for Scientists and Engineers, 5th ed.* Saunders, Philadelphia (in progress).

[7]G. Polya, *How to Solve It, 2nd ed.* (Doubleday, Garden City, NY, 1957).

[8]F. Reif, J. Larkin, and G. Brackett, "Teaching general learning and problem-solving skills," *American Journal of Physics* **44**, 212-217 (1976).

[9]The Problem Solving Template used in the tutorial center at NCSU was developed by K. Johnston, L. Grable, J. Felder, and E. Rieg.

[10]J. Dalby, F. Tourniaire, M.C. Linn, "Making programming instruction cognitively demanding: An intervention study," *Journal of Research in Science Teaching* **23**, (1986) 427-436.

[11]P. Hardiman, R. Dufresne, J. Mestre, "The relation between problem categorization and problem solving among experts and novices," *Memory & Cognition* **17**, 627-638 (1989).

[12]M. Chi, P. Feltovich, and R. Glaser, "Categorization and representation of physics problems by experts and novices," *Cognitive Science* **5**, 121-152 (1981).

[13]W. Leonard, R. Dufresne, J. Mestre, "Using qualitative problem-solving strategies to highlight the role of conceptual knowledge in solving problems," *American Journal of Physics* **64**, 1495-1503 (1996).

[14]D. Myers, *Psychology, 1st ed.* (Worth Publishers, Hope College, 1986).

[15]L.C. McDermott, "Research on conceptual understanding in mechanics," *Physics Today* **37**, 24 (1984).

[16]L.C. McDermott, "Millikan Lecture 1990: What we teach and what is learned–closing the gap," *American Journal of Physics* **59**, 301-315 (1991).

[17] Similar to the famous Fermi problem solving methodology with its order of magnitude estimates and a series of guesses whose errors of estimating too high and too low tend to cancel out.

[18]The Physics Tutorial Center in the Department of Physics at NCSU is a drop-in study area where students can receive individual assistance with their general physics coursework from trained tutors.