Statistical Analysis: The Vibrating Beam Example

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Statistical Analysis

Our Goal

- Estimate $\sigma^2$ (the variance of the measurement error)
- Estimate the standard errors for the estimates of the parameters
  - **Two Parameter setup** $C$ and $K$
  - **Two Parameter setup** $C, K, y(0), v(0)$.
- Graphically examine whether the least squares assumptions hold.
Our Model

- The mass-spring-dashpot model

\[ \frac{d^2y(t)}{dt^2} + C \frac{dy(t)}{dt} + K y(t) = 0 \]

- Let \( a(t) = \frac{d^2y(t)}{dt^2} \) and \( v(t) = \frac{dy(t)}{dt} \), then

\[ a(t) = -Cv(t) - Ky(t) \] (1)
Recall that for the simple linear model
\[ Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \]
we estimated the covariance matrix of \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) using
\[
\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = (X'X)^{-1}\hat{\sigma}^2
\]
where
\[
X = \begin{bmatrix}
1 & X_1 \\
1 & X_2 \\
\vdots & \vdots \\
1 & X_n \\
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial Y_1}{\partial \beta_0} & \frac{\partial Y_1}{\partial \beta_1} \\
\frac{\partial Y_2}{\partial \beta_0} & \frac{\partial Y_2}{\partial \beta_1} \\
\vdots & \vdots \\
\frac{\partial Y_n}{\partial \beta_0} & \frac{\partial Y_n}{\partial \beta_1} \\
\end{bmatrix}.
\]
Estimation of the S. E.: Our Model

\[
\text{Cov}(\hat{C}, \hat{K}) = (X'X)^{-1}\hat{\sigma}^2
\]

where

\[
X = \begin{bmatrix}
\frac{\partial y(t_1)}{\partial C} & \frac{\partial y(t_1)}{\partial K} \\
\frac{\partial y(t_2)}{\partial C} & \frac{\partial y(t_2)}{\partial K} \\
\vdots & \vdots \\
\frac{\partial y(t_n)}{\partial C} & \frac{\partial y(t_n)}{\partial K}
\end{bmatrix}
\]

The standard errors of \( \hat{C} \) and \( \hat{K} \) are the square roots of the diagonal elements of \( \text{Cov}(\hat{C}, \hat{K}) \).
Estimation of the Standard Error (Change)

- To compute the standard errors of $\hat{C}$ and $\hat{K}$, first, we need to compute $\frac{\partial y(t)}{\partial C}$ and $\frac{\partial y(t)}{\partial K}$ (to get the columns of $X$ matrix).

- Using the chain rule for differentiation, we get the relation

$$\frac{\partial a(t)}{\partial C} = -v(t) - C \frac{\partial v(t)}{\partial C} - K \frac{\partial y(t)}{\partial C}$$

and

$$\frac{\partial a(t)}{\partial K} = -C \frac{\partial v(t)}{\partial K} - y(t) - K \frac{\partial y(t)}{\partial K}$$

- We need to compute $\frac{\partial y(t)}{\partial C}$, $\frac{\partial y(t)}{\partial K}$, $\frac{\partial v(t)}{\partial K}$ and $\frac{\partial v(t)}{\partial K}$. 
Sensitivity Equations (Change)

How do we compute $\frac{\partial y(t)}{\partial C}$, $\frac{\partial y(t)}{\partial K}$, $\frac{\partial v(t)}{\partial K}$, and $\frac{\partial v(t)}{\partial K}$, if we don’t have an analytical expression for $y(t)$ or $v(t)$?

- Solve a new system of differential equations, called the sensitivity equations.
Sensitivity Equations (Change)

To make the following derivation clearer, we will omit from our notation the dependence of \( y \) and \( v \) on \( t \).

\[
\frac{dy}{dt} = v \tag{2}
\]

and

\[
\frac{dv}{dt} = -Cv - Ky. \tag{3}
\]

Differentiating Equation (3) with respect to \( C \) and \( K \) and interchanging the order of derivatives on the left hand side gives

\[
\frac{d}{dt} \left[ \frac{\partial y}{\partial C} \right] = \frac{\partial v}{\partial C}. \tag{4}
\]

\[
\frac{d}{dt} \left[ \frac{\partial y}{\partial K} \right] = \frac{\partial v}{\partial K}. \tag{5}
\]
Differentiating Equation (3) with respect to $C$ and $K$ gives

$$\frac{d}{dt} \left[ \frac{\partial v}{\partial C} \right] = -v - C \left[ \frac{\partial v}{\partial C} \right] - K \left[ \frac{\partial y}{\partial C} \right]. \quad (6)$$

$$\frac{d}{dt} \left[ \frac{\partial v}{\partial K} \right] = -C \left[ \frac{\partial v}{\partial K} \right] - y - K \left[ \frac{\partial y}{\partial K} \right]. \quad (7)$$

Equations (5)-(8) are the four sensitivity equations.

The sensitivity equations, along with the original two equations for $y$ and $v$ can be solved by the Matlab function, `ode`. 
Checking the Model Assumptions

- If the model is appropriate for the data at hand, the observed residuals $e_i$ should reflect the properties assumed for the $\varepsilon_i$.

- Residuals can be used to detect departures from the model
  - A residual plot against the fitted values can be used to determine if the error terms have a constant variance.
  - A plot of the residuals with time can be used to check for non-independence over time. When the error terms are independent, we expect them to fluctuate in a random pattern around 0.
  - Plot of quantiles of residuals against the quantiles of a normal: **QQPlot** to check normality of errors.