Introduction to the Forward Problem: Solving the Harmonic Oscillator Equation

The mathematical model for the Harmonic Oscillator or Spring-Mass-Dashpot system is given by

\[ m \frac{d^2 y(t)}{dt^2} + c \frac{dy(t)}{dt} + ky(t) = 0 \quad \text{or} \quad m\ddot{y}(t) + c\dot{y}(t) + ky(t) = 0 \]

with initial conditions

\[ y(t_0) = y_0, \quad \dot{y}(t_0) = v_0 \]

where \( y(t) \) is the vertical displacement of the mass about the equilibrium position. \( m \) is the mass, \( c \) is the damping constant, and \( k \) is the spring constant. With \( m \neq 0 \), we can set \( C = \frac{c}{m} \) and \( K = \frac{k}{m} \) and rewrite the model as

\[ \ddot{y}(t) + C\dot{y}(t) + Ky(t) = 0 \]

where the initial conditions remain the same.

1 Undamped Case: \( c = 0 \)

Because there is no damping term for the undamped case, the formula we use is

\[ m\ddot{y}(t) + ky(t) = 0. \]

Using the characteristic equation \( mr^2 + k = 0 \), we get

\[ y(t) = A \cos \omega t + B \sin \omega t, \quad \omega = \sqrt{k/m}. \]

This can also be written

\[ y(t) = A \sin(\omega t + \phi) \]

where \( A = \sqrt{A^2 + B^2} \) and \( \phi = \tan^{-1}(A/B) \). \( A \) and \( B \) can be found using the initial conditions, where \( A = y_0 \) and \( B = \frac{v_0}{\omega} \).

Figure 1: This is an undamped harmonic oscillator.
2 Damped Case: \( c \neq 0 \)

Using the characteristic equation \( mr^2 + cr + k = 0 \), we get

\[ y(t) = e^{-\frac{c}{2m}t} \left[ A \cos \omega t + B \sin \omega t \right], \quad \omega = \frac{\sqrt{4mk - c^2}}{2m} \]

where \( c^2 < 4mk \). This can also be written

\[ y(t) = e^{-\frac{c}{2m}t} A \sin(\omega t + \phi) \]

where \( A = \sqrt{A^2 + B^2} \) and \( \phi = \tan^{-1}(A/B) \). \( A \) and \( B \) can be found using the initial conditions, where \( A = y_0 \) and \( B = \frac{v_0 + \omega \left( y_0 \right)}{\omega} \).

![Figure 2: This is a damped harmonic oscillator.](image1)

![Figure 3: This is also a damped harmonic oscillator, but with a higher damping term.](image2)