Solving the Harmonic Oscillator Equation

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Consider a mass attached to a wall by means of a spring. Define $y=0$ to be the equilibrium position of the block. $y(t)$ will be a measure of the displacement from this equilibrium at a given time. Take $y(0) = y_0$ and $\frac{dy(0)}{dt} = v_0$. 
Basic Physical Laws

- Newton’s Second Law of motion states tells us that the acceleration of an object due to an applied force is in the direction of the force and inversely proportional to the mass being moved.

- This can be stated in the familiar form:

\[ F_{net} = ma \]

- In the one dimensional case this can be written as:

\[ F_{net} = m\ddot{y} \]
Relevant Forces

\[ F_H = -ky \]

- Hooke’s Law (k is called Hooke’s constant)

\[ F_F = -c\dot{y} \]

- Friction is a force that opposes motion. We assume a friction proportional to velocity.
Harmonic Oscillator

Assuming there are no other forces acting on the system we have what is known as a Harmonic Oscillator or also known as the Spring-Mass-Dashpot.

\[ F_{net} = F_H + F_F \]

or

\[ m\ddot{y}(t) = -ky(t) - c\dot{y}(t) \]
Solving the Simple Harmonic System

\[ m\ddot{y}(t) + c\dot{y}(t) + ky(t) = 0 \]

If there is no friction, \( c=0 \), then we have an “Undamped System”, or a Simple Harmonic Oscillator. We will solve this first.

\[ m\ddot{y}(t) + ky(t) = 0 \]
Simple Harmonic Oscillator

Notice that we can take $K = \frac{k}{m}$ and look at the system :

$$\ddot{y}(t) = -K y(t)$$

We know at least two functions that will solve this equation.

$$y(t) = \sin(\sqrt{K} t) \quad \text{and} \quad y(t) = \cos(\sqrt{K} t)$$
Simple Harmonic Oscillator

The general solution is a linear combination of sin and cos.

\[ y(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) \]

Where \( \omega_0 = \sqrt{\frac{k}{m}} \).
We can rewrite the solution as

\[ y(t) = A \sin(\omega_0 t + \phi) \]

where \( \omega_0 = \sqrt{\frac{k}{m}} \) and

\[ A = y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2 \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{y_0 \omega_0}{v_0}\right) \]
Visualizing the System

Undamped SHO

\[ m=2 \]
\[ c=0 \]
\[ k=3 \]
Alternate Method: 
The Characteristic Equation

Again we take $\omega_0 = \sqrt{\frac{k}{m}}$ and look at the system:

$$\ddot{y}(t) + \omega_0^2 y(t) = 0$$

Assume an exponential solution, $y(t) = e^{rt}$.

$$\Rightarrow \dot{y}(t) = re^{rt} \text{ and } \ddot{y}(t) = r^2 e^{rt}$$

Subbing this into the equation we have:

$$r^2 e^{rt} + \omega_0^2 e^{rt} = 0$$
Alternate Method:
The Characteristic Equation

To have a solution we require that

\[ r^2 = -\omega_0^2 \]

\[ \Rightarrow r = \pm i \omega_0 \]

Thus we have two solutions

\[ y(t) = e^{i\omega_0} \quad \text{and} \quad y(t) = e^{-i\omega_0} \]
Alternate Method:
The Characteristic Equation

As before, any linear combination of solutions is a solution, giving the general solution:

\[ y(t) = A^* e^{i\omega_0 t} + B^* e^{-i\omega_0 t} \]
Euler’s Identity

- Recall that we have a relationship between $e$ and sine and cosine, known as the Euler identity.

$$e^{i\omega_0 t} = \cos(\omega_0 t) + i \sin(\omega_0 t)$$

and

$$e^{-i\omega_0 t} = \cos(\omega_0 t) - i \sin(\omega_0 t)$$

- Thus our two solutions are equivalent.
Rewrite the solution

We now have two solutions to:

\[ m\ddot{y}(t) + ky(t) = 0 \]

One with complex exponentials

\[ y(t) = A^* e^{i\omega_0 t} + A^* e^{-i\omega_0 t} \]

and one with sine

\[ y(t) = A \sin(\omega_0 t + \phi) \]
Damped Systems

- If friction is not zero then we cannot use the same solution. Again, we find the characteristic equation.

Assume $y(t) = e^{rt}$

Then,

$\dot{y}(t) = re^{rt}$ and $\ddot{y}(t) = r^2 e^{rt}$
Damped Systems

Remember that we are now looking for a solution to:

\[ \ddot{y}(t) + C\dot{y}(t) + Ky(t) = 0 \]

Subbing in \( \dot{y}, \ddot{y} \) and \( y \) we have,

\[ r^2 e^{rt} + Cre^{rt} + Ke^{rt} = 0 \]

Which can only work if

\[ r^2 + Cr + K = 0 \]
Damped Systems

This gives us that:

\[ r = \frac{-C \pm \sqrt{C^2 - 4\omega_0^2}}{2} \text{ where } \omega_0 = \sqrt{\frac{k}{m}} \]

Let \( \alpha = C^2 - 4\omega_0^2 \)

We have three options

1. \( \alpha > 0 \)  overdamped
2. \( \alpha = 0 \)  critically damped
3. \( \alpha < 0 \)  underdamped
Underdamped Systems

- The case that we are interested in is the underdamped system.

\[ C^2 - 4\omega_0^2 < 0 \]

The solution is

\[ y(t) = Ae^{-C/2} \cos \alpha + Be^{-C/2} \sin \alpha \]

or

\[ y(t) = e^{-C/2} (A \cos(\omega t) + B \sin(\omega t)) \]

\[ \omega = \frac{1}{2} \sqrt{4K - C^2} = \frac{4mk - e^2}{2m} \]
Underdamped Systems

We can rewrite this in a more intuitive form,

\[ y(t) = Ae^{\frac{-ct}{2m}} \sin(\omega t + \phi) \]

Where \( \omega = \frac{\sqrt{4mk - c^2}}{2m} \),

\[ A = \sqrt{A^2 + B^2}, \quad \phi = \tan^{-1}(\frac{A}{B}), \]

\[ A = y_0 \text{ and } B = \frac{v_0 + \frac{c}{2m}y_0}{\omega} \]
Visualizing Underdamped Systems
Visualizing Underdamped Systems

Underdamped Harmonic Oscillator

\[ y(t) = Ae^{\left(\frac{-c}{2m}\right)t} \]

\[ y(t) = -Ae^{\left(\frac{-c}{2m}\right)t} \]

\[ m = 2 \]
\[ c = 0.5 \]
\[ k = 3 \]
Visualizing Underdamped Systems

Another Underdamped System

$m=2$
$c=2$
$k=3$
Visualizing Underdamped Systems

Underdamped Harmonic Oscillator

\[ y(t) = Ae^{-\frac{ct}{2m}t} \]

\[ y(t) = -Ae^{-\frac{ct}{2m}t} \]

- \[ m = 2 \]
- \[ c = 2 \]
- \[ k = 3 \]
Will this work for the beam?

- The beam seems to fit the harmonic conditions.
  - Force is zero when displacement is zero
  - Restoring force increases with displacement
  - Vibration appears periodic

- The key assumptions are
  - Restoring force is linear in displacement
  - Friction is linear in velocity
Writing as a First Order System

- Matlab does not work with second order equations
- However, we can always rewrite a second order ODE as a system of first order equations
- We can then have Matlab find a numerical solution to this system
Writing as a First Order System

Given this second order ODE
\[ \ddot{y}(t) + C\dot{y}(t) + Ky(t) = 0, \text{ with } y(0) = y_0 \text{ and } \dot{y}(0) = v_0 \]
We can let \( z_1(t) = y(t) \) and \( z_2(t) = \dot{y}(t) \).
Clearly,
\[ \dot{z}_1(t) = z_2(t) \]
and
\[ \dot{z}_2(t) = -Kz_1(t) - Cz_2(t) \]
Writing as a First Order System

Now we can rewrite the equation in matrix vector form

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix}(t) = \begin{bmatrix}
0 & 1 \\
-K & -C
\end{bmatrix}\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}(t)
\]

or

\[
z(t) = Az(t) \quad \text{where} \quad A = \begin{bmatrix}
0 & 1 \\
-K & -C
\end{bmatrix}
\]

with

\[
z(0) = \begin{bmatrix}
y_0 \\
v_0
\end{bmatrix}
\]

This is now in a form that will work in Matlab.
Constants Are Not Independent

- Notice that in all our solutions we never have c, m, or k alone. We always have c/m or k/m.
- The solution for y(t) given (m,c,k) is the same as y(t) given (αm, αc, αk).
- Very important for the inverse problem
Summary

- We can use Matlab to generate solutions to the harmonic oscillator.
- At first glance, it seems reasonable to model a vibrating beam.
- We don’t know the values of m, c, or k.
- Need the inverse problem.