OUR PLAN

Review Pertinent Linear Algebra Topics
Forward Problem for Linear Systems
Inverse Problem for Linear Systems
Discuss Well-posedness and Overdetermined Systems
Formulate a least squares solution for an overdetermined system
Introduction

Linear Algebra Review
Represent $m$ linear equations with $n$ variables:

\[
\begin{bmatrix}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m
\end{bmatrix}
\]

$A = m \times n$ matrix, $x = n \times 1$ vector $b = m \times 1$ vector

If $A = m \times n$ and $B = n \times p$ then $AB = m \times p$
(number of columns of $A = \text{Number of rows of } B$)
Matrix Properties

Matrix Properties:
A = Square matrix has \( n \) rows and \( n \) columns
if A is square then \( A^{-1} \) exists iff the determinant \( \neq 0 \)

What is a determinant??

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\]

Determinant = \( ad - bc \) How do we find the inverse once we determine that the determinant is \( \neq 0 \)?

\[
\frac{1}{ad - bc} \begin{bmatrix}
    d & -b \\
    -c & a
\end{bmatrix}
\]
Solve the following system of linear equations:

Refer to your Worksheet problem 1

\[
\begin{bmatrix}
2 & 1 \\
1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
\end{bmatrix}
\]

Solving this by hand is simple...
Let \( b_1 = 1 \) and \( b_2 = 3 \)

Then our system of linear equations is:
\[
2x_1 + x_2 = 1 \\
x_1 + 3x_2 = 3
\]
if $A^{-1}$ exists then $A^{-1}A = I$

$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
EXAMPLE: Square Matrix

\[
\begin{bmatrix}
  x_1 - x_2 + 3x_3 + x_4 = 2 \\
  3x_1 - 3x_2 + x_3 = -1 \\
  x_1 + x_2 - 2x_4 = 3 \\
  x_1 + x_2 + x_3 - x_4 = 1
\end{bmatrix}
\]

Solving this by hand is tedious and very time consuming,
especially as m and n grow larger

HOW DO WE SOLVE?
OUR EXAMPLE

\[
A = \begin{bmatrix}
1 & -1 & 3 & 1 \\
3 & -3 & 1 & 0 \\
1 & 1 & 0 & -2 \\
1 & 1 & 1 & -1 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
2 \\
-1 \\
3 \\
1 \\
\end{bmatrix}
\]

SOLVE USING MATLAB and the INVERSE of A
Forward Problem

The forward problem is fairly straightforward

$$Ax=b$$

If we have $A$ an $n \times n$ matrix and $x$ an $n \times 1$ vector then it is clear how we will solve for $b$

The forward problem consists of finding $b$ for a given $A$ and $x$
Example

What if

\[ A = \begin{bmatrix} 47 & 28 \\ 89 & 53 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

Solving for \( b \) is fairly straightforward.

\[ b_1 = 47 - 28 \]
\[ b_2 = 89 - 53 \]
Inverse Problem

For the Vibrating Beam, we are given data
(done in the lab tomorrow)
and we must determine $m$, $c$ and $k$.

In the case of the linear system $Ax = b$ we are provided
with $A$ and $b$ and must determine $x$
Example

\[ Ax = b \rightarrow A^{-1}Ax = A^{-1}b \rightarrow x = A^{-1}b \]

\[
\begin{bmatrix}
0 & 1 & -1 \\
-2 & 4 & -1 \\
-2 & 5 & -4
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
1 \\
-2
\end{bmatrix} \rightarrow
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & -1 \\
-2 & 4 & -1 \\
-2 & 5 & -4
\end{bmatrix}
^{-1}
\begin{bmatrix}
3 \\
1 \\
-2
\end{bmatrix}
\]

We will discuss in the following section, using the BACKSLASH operator
Alternative Method to solve

Matrix Calculations

Ax=b

MULTIPLY BOTH SIDES BY $A^{-1}$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

NOTE: $Ix = x$
These calculations can be performed using the BACKSLASH OPERATOR in MATLAB

MATLAB- \( x = A \backslash b \)

!!!NOTE: This is not the same as \( A \) divided by \( b \)!!!
PROBLEM

What if A is not a square matrix???

A is no longer directly invertible

(no longer a set of unique solutions to the problem)

BUT we can minimize the equation using method of least squares to be shown later this afternoon
PROBLEM:

A is no longer a square matrix
We wish to solve $Ax = b$

• Focus on an overdetermined System: (i.e. A is $m \times n$ where $m > n$)

• Usually no exact solution exists when A is overdetermined

• Definition. A linear system is called inconsistent or overdetermined if it does not have a solution. In other words, the set of solutions is empty. Otherwise the linear system is called consistent.

• In our experiment this week, the number of data points will exceed the number of parameters to solve $m > n$
Well-Posedness

Well Posed

What does it mean to be Well-Posed?

Ax = b is well posed when

**Existence** - for every b there exists a x such that Ax = b

**Uniqueness** - if Ax₁ = Ax₂ → x₁ = x₂

**Stability** - A⁻¹ is continuous

The solution technique x = A⁻¹b produces the correct answer when Ax = b is **well-posed**

Ax = b is ill-posed if it is not well-posed
Example

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\]

x is the solution to

See Matlab worksheet:
Ill-Posedness

$x$ is the solution to $Hx = \vec{1}$ where $H$ is the HILBERT MATRIX

TRY IT OUT IN MATLAB (WORKSHEET)
What is an Ill conditioned System

A system is ill-conditioned if some small perturbation in the system causes a relatively large change in the exact solution

EXAMPLE

Try entering this into your matrix and solving:

\[
\begin{bmatrix}
0.835 & 0.667 \\
0.333 & 0.266
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0.168 \\
0.067
\end{bmatrix}
\rightarrow
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.835 & 0.667 \\
0.333 & 0.266
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0.168 \\
0.066
\end{bmatrix}
\rightarrow
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

YOU WILL NOTICE MAJOR CHANGES HERE!
Overdetermined System

• Example $\|\bar{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

• MINIMIZE $\|Ax - b\|_2^2 = (Ax - b)^T(Ax - b)$
Obtaining the Normal Equations

• We want to minimize

\[ \phi(x) = (Ax - b)^T (Ax - b) : \]

\[ \nabla \phi(x) = A^T (Ax - b) + ((Ax - b)^T A)^T \]

\[ = A^T (Ax - b) + A^T (Ax - b) \]

\[ = A^T Ax - A^T b + A^T Ax - A^T b \]

\[ = 2(A^T Ax - A^T b) \]
• $\phi(x)$ is minimized when $x$ solves $A^T Ax = A^T b$

• $x = (A^T A)^{-1} A^T b$ provides the least squares solution

Which is a topic that will be presented later on today!