An Introduction to Mathematical Finance

SAMSI/CRSC Undergraduate Workshop
May 30, 2006

Mathematical Finance

Mathematical Finance is the study of the mathematical models of financial markets.

- Types of Financial Markets:
  - Stock Markets
  - Bond Markets
  - Currency Markets
  - Commodity Markets
  - Futures and Options Markets

Fields

Mathematical Finance lies at the intersection of

- Applied Probability
- Partial Differential Equations
- Stochastic Differential Equations
- Economics
- Statistics
- Numerical Analysis

What's with the lingo?

- At the heart of mathematical finance is the analysis and pricing of derivatives using mathematical models

- Derivative: An instrument whose price depends on, or is derived from, the price of another asset.

Example

- An example of a derivative

- Let S_t denote the value of IBM stock at time t. Suppose today is time 0 and S_0= $60.

- Then, the payout of a European Call Option with strike price $62 and maturity T is given by

$$(S_T - 62)^+ = \begin{cases} S_T - 62 \text{ if } S_T \geq 62 \quad \text{at} \quad \text{time} \quad T \\ 0 \qquad \text{if } S_T < 0 \quad \text{at} \quad \text{time} \quad T \end{cases}$$

- Question: How much would you pay for this option today?

Example con't.

One powerful feature of derivatives is leverage

- Suppose you have \$3,000 to invest and you think IBM stock is going to increase from it's current level, S_0= \$60. The price of one European Call Option with strike price \$62 and a maturity of one year is \$2.

- You have two options

1. You buy \$3,000/\$60= 50 shares of IBM2.
2. You buy \$3,000/\$2= 1500 European Call Options

Payout

- After one year, suppose there are two cases
  - $S_1 = \$56$ or $S_1 = \$65$.

Let's look at our profit.

- If we bought the shares,

Stock Price	Action	Profit from Selling 50 shares
\$56	Do nothing	0
\$65	Sell the stock	$50 \cdot (\$65 - \$60) = \$250$

Payout

If we bought the options,

Stock Price	Action	Profit from Selling 1500 options
\$56	Options expires worthless	$-\$3000$
\$65	Exercise the options	$1500 \cdot (\$65 - \$62) = \$4500$

•Needless to say, the profit from buying options is much higher, but you have to have a cast-iron stomach for risk

Black-Scholes Formula

- Formula is a solution to a "Stochastic Differential Eqn." (SDE) that defines movement of value of option over time
- SDE's have a long history in math
- Specific form of equation solved…
  - embodies principle of replicating portfolio
  - makes specific assumptions about nature of movement value in competitive market

Random Walks

- A "Standarized Normal Random Variable", e(t)
  - It has a normal distribution (I.e. bell-shaped)
  - It has mean=0 and Standard Deviation=1

- A random walk is a process defined by
  - $Z(t+1) = Z(t) + e(t)(\Delta t)^{0.5}$

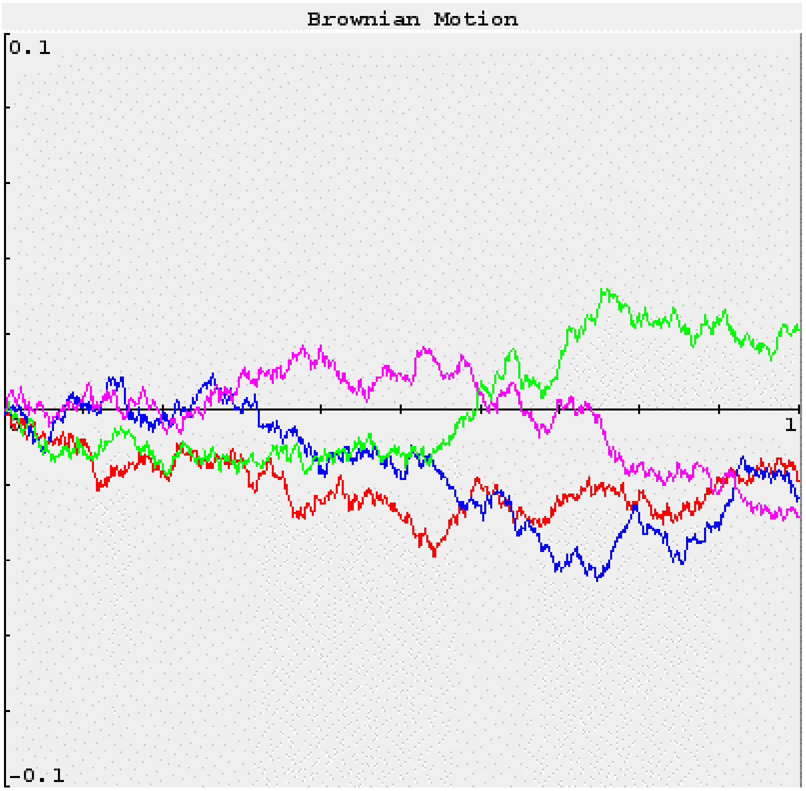
- Difference between 2 periods: Z(kt) – Z(jt)…
  - Expected Value =0 , Variance = kt – jt
  - Differences of non-overlapping periods are "uncorrelated"

Brownian Motion

- BM is what results when we let $(\Delta t) \to 0$.
- Formally, $Z(t+1) = Z(t) + e(t)(\Delta t)^{0.5}$ becomes
$$dZ = e(t)\ (\Delta t)^{0.5}$$
- Also known as Weiner Process
  - Traces back to Einstein
- Retains random walk properties
  - $Z(t)$-$Z(s)$ is a normal random variable
  - If $t_1 < t_2 < t_3 < t_4$ then $[Z(t_4) - Z(t_3)]$ and $[Z(t_2) - Z(t_1)]$ are independent.

Is this reasonable?

- These are simulated paths of BM
- Do they “look like” a graph of a stock price?
- Mathematicians make a career out of studying these paths!

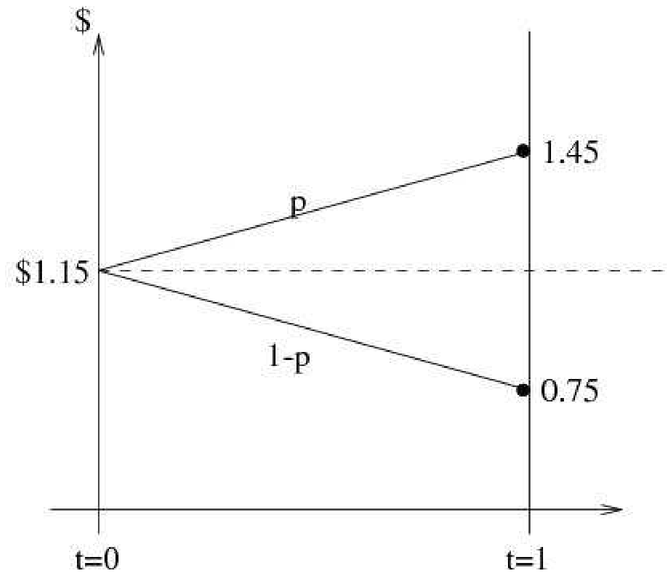


Arbitrage

- We take the interest rate r =0

- Consider a world with two time steps
    - At time t =0, 1 Euro = $1.15

At time t=1, there are two possibilities…
1. Euro goes to $1.45 with probability, p.
2. Euro goes to $.75 with probability 1-p



- What is the value of a European call with strike price K = $1.15?

Good Guess?

- If the price goes up (*with prob* **p**) we exercise the option
  - Ie. We buy the Euro for \$1.15 and sell it back to the market for \$1.45
- If the price of the Euro goes down to \$.75 we don't exercise our option, and it expires worthless..
- Mathematically, $H=(X_T - \$1.15)^+$
- A fair price would be

$$E\{H\} = (1.45 - 1.15)p + 0\ (1-p) = .3p$$

Action at time $t = 0$	Result
Sell the option at price $\pi(H)$	$+\pi(H)$
Borrow \$ $\dfrac{9}{28}$	$+\$0.32$
Buy $\dfrac{3}{7}$ Euros at \$1.15	-0.49

The balance at time $t = 0$ is $\pi(H) - 0.17$

At time T there are two possibilities:

(i) The Euro has risen:

Option is exercised	-0.30
Sell $\frac{3}{7}$ Euros at 1.45	$+0.62$
Pay back loan	$\underline{-0.32}$
	0

(ii) The Euro has fallen:

Option is worthless	0
Sell $\frac{3}{7}$ Euros at 0.75	$+0.32$
Pay back loan	$\underline{-0.32}$
	0

Black-Scholes PDE

- Assume stock price, S, follows GBM
- Also, f is the price of a derivative whose price depends on S. Then the B-S differential eqn. is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

•This has many solutions. The one of interest depends on the boundary condition.

•In particular, for a European call option, the key boundary condition is

$$f = \max(S-X, 0) \quad \text{when } t=T$$

Insight

- Risk-neutral valuation was introduced earlier in our example.
- Most important tool in analysis of derivatives!

- What appears in the differential equation?
  - Current price, time, stock price volatility, and interest rate
  - Expected return of stock, μ, drops out…
- So risk preference (correlated with μ) doesn't enter into the solution.
- "Everybody's price" can then be calculated in the "risk neutral" world.
- In fact, in this "world" the expected rate of return for all stocks is simply the interest rate, r.

European Call Option Price

- The solution to the B-S PDE is given by

$$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

•These terms have a nice economic interpretation.

•What is it?

Assumptions

- Black-Scholes theory is elegant, and the results were groundbreaking!!

Some of the assumptions made in their work include…

1. The stock price follows previously mentioned Geometric Brownian Motion with μ and σ constant.
2. There are no transaction costs or taxes.
3. There are no dividends during the life of the derivative.
4. There are no riskless arbitrage opportunities.
5. Security trading is continuous.
6. The risk-free rate of interest is, r, is constant and the same for all maturities.

The World could be Risk-Neutral, But Are You?

- I propose a game…

1. I have $5 in my pocket.
2. I need a buyer to give me one dollar.
3. What I'm selling is a piece of paper promising to pay you the $5 if I flip a coin twice, and you "call it" correctly both times.
4. If you guess wrong one time, I keep the dollar, and you get nothing.

Are You Risk-Neutral…

- There were many takers for that game???

Are You Risk-Neutral…

- There were many takers for that game???
- What is the B-S risk neutral price for this piece of paper I'm selling?
  - Your investment of \$1 can either go up to \$5 with probability .25 or down to \$0 with probability .75.
  - So the fair price for the game was \$1.25
  - The buyer gets the paper for cheaper than fair value

game…..

- Are there any takers for the same game if I now promise to pay $1024 to anybody (one person at a time to avoid insider trading) that can call "heads or tails" correctly 10 consecutive times.

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- Now….

What if I offer you $1,024,000 for getting 10 guesses correct, but it will cost you $1,000 to play, would you still do it.

- Some might, but most probably won't…

Moral

- Who would?
  - Rich people, gambling addicts, crazy people….
- Who wouldn't….
  - People that don't have access to $1,000, people with families to support, conservative spenders…
- This brings us to one of the drawbacks of B-S, and that is the issue, that "fair price" might not be enough.
- This area of research is called Risk Aversion, and was studied by researchers at SAMSI in the fall.

Why does Financial Mathematics Exist?

Answer:

- Because financial institutions are selling extremely complex financial derivatives to clients to hedge their risk exposure and to speculate on the direction of the markets.

- These financial institutions have to make sure they price these derivatives correctly and manage them effectively.

- This has created a booming area of research in applied probability and other fields to try to answer very complicated mathematical questions.

SAMSI

- To facilitate research into financial mathematics, SAMSI offered a semester long program in Financial Mathematics, Statistics, and Econometrics.
- Workshop Activities
  - Opening Workshop
  - Credit Risk Workshop
  - Transition Workshop
- Two classes offered
  - Advanced Topics in Financial Econometrics
  - Advanced Topics in Financial Mathematics

Main Activities

The formation of working groups

1. Credit Risk
2. Computational Issues
3. Levy Processes
4. Model Uncertainty
5. Portfolio Management

Options available to students

- Masters Level
  - Masters of Mathematical Finance, Masters of Financial Engineering, etc
    - Math Department
    - Operations Research Department
    - Statistics Department
- Ph.D. Level
  - Ph.D. in Applied Math
  - Ph.D. in Statistics
  - Ph.D. in Economics

Jobs?

There is tremendous demand for students with a mathematical finance degree

- Industry
  - Wall Street
  - Hedge Funds
  - Energy Companies
  - Financial Software Companies
- Academia
  - Postdoctoral Positions
  - Professor and Researcher

Thank You