8 Example 6: Cantilever Beam

We introduce another example which is ubiquitous in structural applications and is ideal for use in illustrating our weak formulations that follow.

8.1 The Beam Equation

We consider the beam equation given by

\[ \rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2}{\partial \xi^2} M = f(t, \xi) \]  

(1)

where the term \( \frac{\partial y}{\partial t} \) represents the external damping (the air or viscous damping), \( M \) is the internal moment, \( \rho \) is the mass density, and \( f \) is the external force applied. For a development of this equation from basic principles, see [BSW, BT]. Before discussing the equation further, we discuss boundary and initial conditions.

Boundary Conditions

We consider a beam with one end fixed and one end free.

**Fixed end:**

\[ y(t, 0) = 0 \]  zero displacement,
\[ \frac{\partial y}{\partial \xi}(t, 0) = 0 \]  zero slope.

**Free end:**

\[ M(t, l) = 0 \]  zero moment,
\[ \frac{\partial M}{\partial \xi}(t, l) = 0 \]  zero shear.

We observe that the shear force is represented by \( S(t, \xi) = -\frac{\partial M}{\partial \xi}(t, \xi) \).

Initial Conditions

We consider a beam with initial displacement and velocity given by \( \Phi \) and \( \Psi \), respectively.

\[ y(0, \xi) = \Phi(\xi) \]
\[ \dot{y}(0, \xi) = \Psi(\xi) \]

We observe that we have an equation (1) given with two unknowns; however we couple these with a constitutive relationship given by

\[ M = M(y). \]

Using Hooke’s law (\( \sigma = E\epsilon \) where \( \sigma \) is the stress and \( \epsilon \) is the strain, \( E \) is the Young’s modulus), we obtain (using basic principles [BT]) for linear elastic systems the relationship

\[ M = EI \frac{\partial^2 y}{\partial \xi^2} \]
where $I$ represents the moment of inertia of the cross-sectional area. We remark that either $E$ and/or $I$ may be a function of $\xi$, i.e., $E = E(\xi)$ and/or $I = I(\xi)$.

We remark that the above derivations use the assumptions of linear elasticity and small displacements. Without small displacements, $M$ could be a nonlinear function of $y$. For further discussions, see [BSW, BT].

We also have internal damping involved for which we must account. We will assume Kelvin-Voigt damping so that $M(t, \xi)$ is given by

$$M(t, \xi) = EI \frac{\partial^2 y}{\partial \xi^2} + cD I \frac{\partial^3 y}{\partial \xi^2 \partial t}.$$  \hspace{1cm} (2)

Combining equations (1) and (2), we have the beam equation given by

$$\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2}{\partial \xi^2} \left( EI \frac{\partial^2 y}{\partial \xi^2} + cD I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) = f(t, \xi),$$

with the modified boundary conditions

$$\left( EI \frac{\partial^2 y}{\partial \xi^2} + cD I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) |_{\xi=l} = 0,$$

$$\left[ \frac{\partial}{\partial \xi} \left( EI \frac{\partial^2 y}{\partial \xi^2} + cD I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) \right] |_{\xi=l} = 0.$$

### 8.2 The Beam Equation in the form $\dot{x} = Ax + F$

We wish to write the beam equation in the form $\dot{x} = Ax + F$. Let $x$ be represented by

$$x(t, \cdot) = \left( \begin{array}{c} y(t, \cdot) \\ \dot{y}(t, \cdot) \end{array} \right)$$

and take as state space $X = H^2(0, l) \times L^2(0, l)$ where $H^2(0, l) = \{ \varphi \in H^2(0, l) | \varphi(0) = 0, \varphi'(0) = 0 \}$. Thus for elements in $H^2(0, l)$, the function and the derivative both vanish at the left boundary. We rewrite the equation using the abbreviation $\partial = \frac{\partial}{\partial \xi}, \cdot = \frac{\partial}{\partial t}$, and have

$$\dot{y} = \frac{1}{\rho} \left[ -\partial^2 (EI \partial^2 y) - \partial^2 (cD I \partial^2 \dot{y}) \right] - \frac{\gamma}{\rho} \dot{y} + \frac{1}{\rho} f.$$
We rewrite this in first order vector form as

\[ \frac{d}{dt} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix} \begin{pmatrix} y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\rho} f \end{pmatrix}. \]

Let

\[ A = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix} \]

where we define \( A \) and \( B \) in the following way:

\[ A = \frac{1}{\rho} \partial^2 (EI \partial^2 \phi) \]

\[ B = \frac{1}{\rho} \partial^2 (cD I \partial^2 \psi) + \frac{\gamma}{\rho} \phi. \]

Then we have the form \( \dot{x} = Ax + F \) with

\[ \mathcal{D}(A) = \{(\varphi, \psi) \in X | \psi \in H^2_L(0, l), A\varphi + B\psi \in L^2(0, l), [EI \partial^2 \varphi + cD I \partial^2 \psi]_t = 0, [\partial(EI \partial^2 \varphi + cD I \partial^2 \psi)]_t = 0 \}. \]

### 8.2.1 \( A \) is an Infinitesimal Generator

We claim that \( A \) is an infinitesimal generator of a \( C_0 \) semigroup in \( X_\mathcal{E} \) where \( X_\mathcal{E} \) is an associated energy space. Specifically, we define \( X_\mathcal{E} \) as \( X \) with the energy product

\[ \langle \langle \varphi_1, \psi_1 \rangle, \langle \varphi_2, \psi_2 \rangle \rangle_{\mathcal{E}} = \int_0^l EI \partial^2 \varphi_1 \partial^2 \varphi_2 d\xi + \int_0^l \rho \psi_1 \psi_2 d\xi = \int_0^l EI \partial^2 \varphi_1 \partial^2 \varphi_2 d\xi + \langle \rho \psi_1, \psi_2 \rangle. \]

If \( 0 < \rho_1 \leq \rho(\xi) \leq \rho_2 < \infty \), then \( \langle \rho \psi_1, \psi_2 \rangle \) is equivalent to the norm in \( L^2(0, l) \) where the norm for \( H^2(0, l) \) is defined by

\[ |\varphi|_{H^2}^2 = |\varphi|_{L^2}^2 + |\varphi'|_{L^2}^2 + |\varphi''|_{L^2}^2. \]

An equivalent norm is

\[ |\varphi|_{\sim}^2 = |\varphi(a)| + |\varphi'(b)| + |\varphi''|_{L^2}^2, \]

where \( a, b \) are any values in \([0, l]\). In particular, on \( H^2_L \), we can take \( |\varphi|_{H^2}^2 \equiv |\varphi(0)| + |\varphi'(0)| + |\varphi''|_{L^2}^2 \). This defines a Hilbert space which is topologically equivalent to \( X \) so that we thus have that if \( A \) generates a \( C_0 \) semigroup in \( X_\mathcal{E} \), it is also a \( C_0 \) semigroup in \( X \).
8.2.2 Poincare (First) Inequality

To argue that $A$ is a generator, we need a standard and well-known $[A]$ inequality given by

Suppose there exists a set $\Omega$ which is a bounded subset of $\mathbb{R}^n$. Then there exists a $c = c(\Omega)$, such that for all $\varphi \in W_0^{1,2}(\Omega) = H_0^1(\Omega)$, one has

$$|\varphi|^2_{H^1(\Omega)} \leq c(\Omega) \sum_{|s|=l} \int_\Omega |\partial^s \varphi|^2$$

where $s = (s_1, \ldots, s_n)$.

8.2.3 Dissipativeness of $A$

We argue that $A$ is dissipative in $X$. Recalling that $M = EI\partial^2 \varphi + c_D I \partial^2 \psi$, we have

$$\langle A(\varphi, \psi), (\varphi, \psi) \rangle_X = \int_0^t EI\partial^2 \psi \partial^2 \varphi + \int_0^t [-\partial^2 (EI\partial^2 \varphi + c_D I \partial^2 \psi) - \gamma \psi] \psi$$

$$= \int_0^t EI\partial^2 \psi \partial^2 \varphi - \int_0^t \partial^2 (EI\partial^2 \varphi + c_D I \partial^2 \psi) \psi - \int_0^t \gamma \psi^2$$

$$= \int_0^t EI\partial^2 \psi \partial^2 \varphi - \int_0^t ((EI\partial^2 \varphi + c_D I \partial^2 \psi) \partial^2 \psi + \gamma \psi^2)$$

$$- \partial M \psi \big|_0^l + M \partial \psi \big|_0^l$$

$$= - \int_0^t c_D I |\partial^2 \psi|^2 - \int_0^t \gamma |\psi|^2$$

$$\leq -k |\psi|^2_{H^2(0,t)}$$

$$\leq 0$$

for $k = \min \{|c_D I|, |\gamma|\}$. Here we have integrated by parts twice and used the fact that $(\varphi, \psi) \in D(A)$ so that $\psi(0) = \partial \psi(0) = 0$ and $M(t, l) = \partial M(t, l) = 0$.

8.2.4 $\mathcal{R}(\lambda I - A) = X$ for some $\lambda$

To verify the range statement of Lumer Phillips, we need to show for some $\lambda$ that

$$\lambda(\varphi, \psi) - A(\varphi, \psi) = (h, g)$$

(3)
can be solved for \((\varphi, \psi)\) for a given \((h, g) \in X\). However, (3) reduces to finding, for any \((h, g) \in X = H^2_L(0, l) \times L^2(0, l)\), a solution \((\varphi, \psi)\) in \(\mathcal{D}(\mathcal{A})\) that satisfies

\[
-\psi + \lambda \varphi = h \quad (4)
\]

\[
A \varphi + B \psi + \lambda \psi = g \quad (5)
\]

where \(h \in H^2_L\) and \(g \in L^2\). We can rewrite (4) to obtain

\[
\psi = \lambda \varphi - h \quad (6)
\]

By substituting (6) into (5), we can reduce the above question to one of solving

\[
\lambda^2 \varphi + A \varphi + \lambda B \varphi = g + \lambda h + Bh \quad (7)
\]

for \(\varphi\), given any \((h, g) \in H^2_L(0, l) \times L^2(0, l)\). After solving for \(\varphi\), we can then obtain \(\psi\) using (6). If we obtain \((\varphi, \psi) \in \mathcal{D}(\mathcal{A})\), then we have proven the range statement of Lumer-Phillips. To complete the argument we will use the concepts and results for sesquilinear forms to be discussed below.
9 Gelfand Triple and Sesquilinear Forms

An easy way to solve the above problem is to use the Lax-Milgram theorem; however, we first introduce Gelfand triples. For relevant material, see also [Sh, T, W].

9.1 Concept of Gelfand Triple

The usual notation for a Gelfand triple is “$V \hookrightarrow H \hookrightarrow V^*$ with pivot space $H$”. This notation stands for $V, H$, complex Hilbert spaces, such that $V \subset H$ and $V$ is densely and continuously embedded in $H$. That is, $V$ is a dense subset of $H$ and

$$|v|_H \leq k|v|_V$$

for all $v \in V$ and some constant $k$. Therefore, you can identify elements in $V$ with elements in $H$ with an injection operator, $i$, where $i$ is continuous and $i(V)$ is a dense subset of $H$.

We denote by $V^*$ the conjugate dual of $V$. That is, $V^*$ consists of all conjugate linear continuous functionals on $V$. (Note that we use $V'$ to denote the algebraic dual of $V$ and $V^*$ to denote the topological dual. Frequently one encounters exactly the opposite notation in the literature.)

For $h \in H$, we define $\varphi(h) \in V^*$ by

$$\varphi(h)(v) = \langle h, v \rangle_H$$

for $v \in V$. We claim that $\varphi : H \to \varphi(H) \subset V^*$ is continuous, linear, one-to-one and onto. Moreover, $\varphi(H)$ is dense in $V^*$ in the $V^*$ topology.

By the Riesz theorem, every $h^* \in H^*$ can be represented by

$$h^*(h) = \langle \hat{h}^*, h \rangle_H \quad h \in H$$

for some $\hat{h}^* \in H$. Hence, it is readily argued that $H^*$ is isomorphic to $H$, $H^* \simeq H$. That is, we may identify $H^*$ with $H$ through $\varphi(H) = \tilde{H} \simeq H^*$ and write $h(v) = \langle h, v \rangle_H$ for $h \in H = H^*$.

This construction is commonly written as $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ for the pivot space $H$.

9.2 Duality Pairing

With a Gelfand triple, one frequently utilizes the duality pairing denoted by $\langle \cdot, \cdot \rangle_{V^*, V}$ given by the extension by continuity of the $H$ inner product from $H \times V$ to $V^* \times V$. That is, for $v^* \in V^*$,

$$v^*(v) = \langle v^*, v \rangle_{V^*, V} = \lim_n \langle h_n, v \rangle_H$$
where \( h_n \in H, h_n \to v^* \) in \( V^* \). Note that we have \( \langle h, v \rangle_{V^*,V} = \langle h, v \rangle_H \) if \( h \in V^* \) also satisfies \( h \in H \).

### 9.3 Sesquilinear Forms

**Definition 1** Let \( H_1 \) and \( H_2 \) be two complex Hilbert spaces, and let \( \sigma : H_1 \times H_2 \to \mathbb{C} \). Then we call \( \sigma \) a **sesquilinear form** if it satisfies

1. \( \sigma \) is linear/conjugate linear;
2. \( \sigma \) is continuous. More precisely, there exists \( \gamma > 0 \) such that
   
   \[ |\sigma(x, y)| \leq \gamma |x|_1 |y|_2 \]

   for \( x \in H_1 \) and \( y \in H_2 \).

### 9.4 Norm of a Sesquilinear Form

The norm of a sesquilinear form \( \sigma \) is defined by

\[
|\sigma| = \sup_{x, y \neq 0} \frac{|\sigma(x, y)|}{|x|_1 |y|_2}
\]

for \( x \in H_1 \) and \( y \in H_2 \).

### 9.5 Representation

We consider two different representations:

1. For \( y \in H_2, x \to \sigma(x, y) \) is continuous and linear on \( H_1 \). Therefore, by the Riesz Theorem, there exists a unique \( z \in H_1 \) such that
   
   \[ \sigma(x, y) = \langle x, z \rangle_{H_1}. \]

   Thus there exists a mapping \( B \in \mathcal{L}(H_2, H_1) \) defined by \( By = z \). Therefore,
   
   \[ \sigma(x, y) = \langle x, z \rangle_{H_1} = \langle x, By \rangle_{H_1}. \]

   We find that \( |\sigma| = |B|_{\mathcal{L}(H_2, H_1)}. \) To see this we argue as follows.
For $x$ in $H_1$ and $y$ in $H_2$, we have

$$ |\sigma| = \sup_{x,y \neq 0} \frac{|\sigma(x,y)|}{|x|_1|y|_2} = \sup_{x,y \neq 0} \frac{|(x,By)_{H_1}|}{|x|_1|y|_2} $$

$$ \leq \sup_{y \neq 0} \frac{|x|_1|By|_1}{|x|_1|y|_2} = \sup_{y \neq 0} \frac{|By|_1}{|y|_2} $$

$$ = |B|_{\mathcal{L}(H_1,H_2)}. $$

Therefore, we have $|\sigma| \leq |B|_{\mathcal{L}(H_2,H_1)}$. Thus we need only argue that $|\sigma| \geq |B|_{\mathcal{L}(H_2,H_1)}$. However, we have

$$ |\sigma| = \sup_{x,y \neq 0} \frac{|\sigma(x,y)|}{|x|_1|y|_2} = \sup_{x,y \neq 0} \frac{|(x,By)_{H_1}|}{|x|_1|y|_2} $$

$$ \geq \sup_{x=By} \frac{|(By,By)_{H_1}|}{|By|_1|y|_2} = \sup_y \frac{|By|_1}{|y|_2} $$

$$ = |B|_{\mathcal{L}(H_2,H_1)}. $$

Hence we have $|\sigma| = |B|_{\mathcal{L}(H_2,H_1)}$.}

2. For $x \in H_1$, $y \rightarrow \sigma(x,y)$ is continuous and antilinear on $H_2$, i.e., $\sigma(x,\cdot) \in H_2^*$. Therefore, by the Riesz Theorem, there exists a unique $z \in H_2$ such that

$$ \sigma(x,y) = \langle z,y \rangle_{H_2}. $$

Thus there exists a mapping $A \in \mathcal{L}(H_1,H_2)$ defined by $Ax = z$. Therefore,

$$ \sigma(x,y) = \langle z,y \rangle_{H_2} = \langle Ax,y \rangle_{H_2}, $$

with $|A| = |\sigma|$. Define $\tilde{\sigma}(y,x) = \overline{\sigma(x,y)}$, and interchange the roles of $H_1$ and $H_2$ in the above arguments to argue $|A| = |\sigma|$. 

We wish use our Gelfand triple and sesquilinear form formulations to consider elliptic equations of the form $Ax = f$. However, first, we begin by discussing operators of the form $A : H_1 \rightarrow H_2$ such that $A$ is bounded and continuous. This is very useful in integral equations. However, we will need to modify this formulation to account for the unbounded nature of PDE’s. We can think of our equation $Ax = f$ in the form $\sigma(x,y) = \langle f,y \rangle_H$ for all $y$ in $H$ where $\sigma$ is equivalent to $A$. In other words, $\langle Ax - f,y \rangle_H = 0$ for all $y$ in $H$. The useful results are given in the famous Lax-Milgram theorems, in bounded and unbounded formulations.
10 Lax-Milgram(bounded form)

Theorem 1 (Lax-Milgram Theorem (bounded form)) Suppose $\sigma : H \times H \to C$ is continuous and linear/conjugate linear, i.e., it is a sesquilinear form with

$$|\sigma(x, y)| \leq \gamma |x|_H |y|_H$$

for all $x$ and $y$ in $H$, and $\sigma$ is strictly positive, i.e.,

$$|\sigma(x, x)| \geq \delta |x|^2$$

for all $x$ in $H$ where $\gamma$ and $\delta$ are positive constants. Then there exists $A : H \to H, A \in \mathcal{L}(H, H)$ defined by

$$\sigma(x, y) = \langle Ax, y \rangle$$

with $|A| = |\sigma| = \gamma$ for all $x$ and $y$ in $H$. Moreover,

$$\sigma(A^{-1}x, y) = \langle x, y \rangle$$

with $|A^{-1}| \leq \frac{1}{\delta}$ for all $x$ and $y$ in $H$.

Proof of Theorem

By the Riesz Theorem, we know there exists $A \in \mathcal{L}(H)$ such that $|A| = |\sigma| = \gamma$ where $\sigma(x, y) = \langle Ax, y \rangle_H$ for all $x$ and $y$ in $H$.

We claim that $A$ is one-to-one. For this we need to show that $Ax = 0$ implies $x = 0$. Suppose $Ax = 0$. Then, by the definition of $A$ and the assumptions given, we have

$$0 = |\langle Ax, x \rangle| = |\sigma(x, x)| \geq \delta |x|^2.$$ 

Therefore, $\delta |x|^2 \leq 0$ implies $|x| = 0$, and hence $x = 0$. In other words, $A$ is one-to-one. Since $A$ is one-to-one, we know that $A^{-1}$ exists on $\mathcal{R}(A) \subset H$.

We next claim that $A^{-1}$ is bounded on $\mathcal{R}(A)$. By the proposition that $\sigma$ was strictly positive and the definition of $A$, we have

$$\delta |x|^2 \leq |\sigma(x, x)| = |\langle Ax, x \rangle| \leq |Ax||x|.$$ 

Therefore, $|Ax| \geq \delta |x|$ for all $x$ in $H$. In particular, for $x = A^{-1}y$, we have

$$|y| \geq \delta |A^{-1}y|.$$
Therefore,

\[ |A^{-1}y| \leq \frac{1}{\delta} |y| \]

for all \( y \) in \( \mathcal{R}(A) \). In other words, \( A^{-1} \) is bounded. Finally all we have to show is that \( \mathcal{R}(A) = H \). By assumption, we know that \( A \) is a continuous operator and we have argued that \( A^{-1} \) is bounded on \( \mathcal{R}(A) \). Therefore, \( \mathcal{R}(A) \) is closed. Now suppose \( \mathcal{R}(A) \neq H \). This implies there exists \( z \neq 0 \) such that \( z \perp \mathcal{R}(A) \). In other words,

\[ \langle Ax, z \rangle = 0 \]

for all \( x \) in \( H \). In particular, choosing \( x = z \), we have

\[ 0 = \langle Az, z \rangle = \sigma(z, z) \geq \delta |z|^2. \]

This implies \( z = 0 \); therefore, we have a contradiction. So, \( \mathcal{R}(A) = H \).

10.1 Discussion of \( Ax = f \) with \( A \) bounded

The bounded form of Lax-Milgram is very useful in linear integral equations. The Fredholm integral equations with kernel \( k \in L_2([a, b] \times [a, b]) \) are given by

\[
\int_a^b k(x, y)\varphi(y)dy = f(x) \quad x \in [a, b] \quad \text{(first kind)}
\]

\[
\varphi(x) - \int_a^b k(x, y)\varphi(y)dy = f(x) \quad x \in [a, b] \quad \text{(second kind)}.
\]

These can be written as operator equations in \( H = L^2(a, b) \),

\[
A\varphi = f
\]

\[
\varphi - A\varphi = f
\]

to be solved for \( \varphi \), given \( f \), where \( A \) is a bounded linear operator (discussed below) in \( H \). It also has applications in scattering theory (acoustic and electromagnetic radiation), far field radiation, and single and double layer potential theory.

For relevant material, see also [K], [CK1], and [CK2].

As we have noted, the bounded form of Lax-Milgram is adequate if one wants to solve \( Ax = f \) in \( H \) where \( A \) is bounded. Consider \( k \in L_2(\Omega) \) with \( \Omega = [0, 1] \times [0, 1] \) with \( H = L_2(0, 1) \). Then define \( A : H \to H \) by

\[
(A\varphi)(t) = \int_0^1 k(t, \xi)\varphi(\xi)d\xi.
\]
We can see that this operator is bounded, i.e.,

$$
\int_0^1 \left( \int_0^1 k(t, \xi) \varphi(\xi) d\xi \right)^2 dt < \gamma \int_0^1 |\varphi(\xi)|^2 d\xi.
$$

Therefore, there is a sesquilinear form $\sigma$ such that $A$ corresponds $\sigma$ as in the Lax-Milgram above with

$$
\sigma(\varphi, \psi) = \int_0^1 \left( \int_0^1 k(t, \xi) \varphi(\xi) d\xi \right) \psi(t) dt
$$

$$
= \langle A\varphi, \psi \rangle_{L^2(0,1)}.
$$

We can show that all the requirements of Lax-Milgram (bounded form) are met with this operator $A$; therefore, Lax-Milgram (bounded form) is applicable.

### 10.2 Example - The steady state heat equation

Let $\Omega = [0, 1] \times [0, 1]$. Then the heat equation is given by

$$
\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) + f
$$

where $\nabla = \frac{\partial}{\partial \xi_1} \hat{i} + \frac{\partial}{\partial \xi_2} \hat{j}$ with $(\xi_1, \xi_2) \in \Omega$. We assume boundary conditions that require $u \in H^1_0(\Omega)$ Then the steady state equation is given by

$$
-\nabla \cdot (D \nabla u) = f
$$

or

$$
\frac{\partial}{\partial \xi_1} (D \frac{\partial u}{\partial \xi_1}) + \frac{\partial}{\partial \xi_2} (D \frac{\partial u}{\partial \xi_2}) = f(\xi_1, \xi_2).
$$

In the weak or variational form, we have (by integration by parts and use of the boundary conditions)

$$
\langle D\nabla u, \nabla \varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2}
$$

for a test function $\varphi \in H^1_0(\Omega)$. We define $\sigma$ on $H^1_0 \times H^1_0$ by $\sigma(\psi, \varphi) = \langle D\nabla \psi, \nabla \varphi \rangle_{L^2}$ with $|D| \leq \gamma$, i.e., $D$ is bounded. Then $\sigma$ is not continuous on $H = L^2(\Omega)$. On the other hand, for $\varphi, \psi \in H^1_0$

$$
|\sigma(\psi, \varphi)| \leq \gamma |\nabla \psi|_{L^2} |\nabla \varphi|_{L^2}
$$

$$
\leq \gamma |\psi|_{H^1_0} |\varphi|_{H^1_0}
$$
implies $\sigma$ is continuous on $V = H^1_0(\Omega)$.

Moreover, we do have some type of positivity. If $|D(\xi_1, \xi_2)| \geq \delta$, then

$$|\sigma(\varphi, \varphi)| = |\langle D\nabla \varphi, \nabla \varphi \rangle|$$

$$\geq \delta |\nabla \varphi|^2_{L_2} \geq \delta |\varphi|_{H^1_0(\Omega)}.$$

In other words, in the $V = H^1_0(\Omega)$ norm, we would have both continuity and strict positivity of the sesquilinear form. But we don’t have continuity and strict positivity in the $H = L_2(\Omega)$ sense. Thus if we choose our $H$ space to be $H^1_0(\Omega)$, we would be guaranteed a unique solution of $\langle Au - f, \varphi \rangle_{H^1_0} = 0$. That is, $u = A^{-1}f$ in the $H^1_0$ sense (and of course $u \in H^1_0$ since $A^{-1}$ is bounded from $H^1_0$ to $H^1_0$. This also requires the input function $f$ to be in $H^1_0(\Omega)$ which is a strong requirement for $f$. Therefore, it would not be useful in some cases to choose our $H$ space to be $H^1_0(\Omega)$ in the Lax-Milgram theorem. Instead, we need an extension of Lax-Milgram to treat the heat equation in a more reasonable manner.
11 Lax-Milgram (unbounded form)

Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triple.

**Definition 2** A sesquilinear form $\sigma$ is said to be $V$-continuous if

$$|\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V$$

for all $\varphi$ and $\psi$ in $V$.

**Consequences of $V$-continuity**

For a fixed $\varphi$ in $V$, we consider the mapping $\psi \mapsto \sigma(\varphi, \psi)$ for a $V$-continuous $\sigma$. This mapping is continuous by the definition of continuity above, and its a conjugate linear mapping into $C$. Therefore, $\psi \mapsto \sigma(\varphi, \psi)$ is in $V^*$. This implies there exists an operator $A \in \mathcal{L}(V, V^*)$ such that $\sigma(\varphi, \psi) = \langle A\varphi, \psi \rangle_{V^*, V}$.

Conversely, if $A \in \mathcal{L}(V, V^*)$, we can define $\sigma : V \times V \to C$ by $\sigma(\varphi, \psi) = (A\varphi)\psi$ so that $\sigma$ is $V$-continuous and linear/conjugate linear.

In other words, if we have a $V$-continuous sesquilinear form $\sigma$, there is a one-to-one correspondence between $\sigma$ and $A \in \mathcal{L}(V, V^*)$. While the operator is not bounded in $V$, it will be useful in considering the equation $Au = f$ in $V^*$ will be useful. (Note the weak requirements on $f$ in this case.)

**Definition 3** A sesquilinear form $\sigma$ is $V$-coercive if there exists a constant $\delta > 0$ such that

$$|\sigma(\varphi, \varphi)| \geq \delta |\varphi|_V^2$$

for $\varphi \in V$.

**Theorem 2** (Lax-Milgram Theorem (unbounded form)) Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triple. Let $\sigma : V \times V \to C$ be a $V$-continuous, $V$-coercive sesquilinear form. Then $A : V \to V^*$ given by

$$\sigma(\varphi, \psi) = \langle A\varphi, \psi \rangle_{V^*, V}$$

is a linear (topological) isomorphism between $V$ and $V^*$. Moreover, $A^{-1}$ is continuous from $V^*$ to $V$ with

$$|A^{-1}|_{\mathcal{L}(V^*, V)} \leq \frac{1}{\delta}.$$
Proof of Theorem

Let $R : V^* \to V$ be a Riesz isomorphism, so that for $v^* \in V^*$ we have

$$v^*(v) = \langle v^*, v \rangle_{V^*, V} = \langle Rv^*, v \rangle_V.$$  \hfill (10)

Then $R^{-1} : V \to V^*$ is continuous.

Let $\sigma : V \times V \to C$ be a $V$-continuous, $V$-coercive sesquilinear form. Then $\psi \to \sigma(\varphi, \psi)$ in $V^*$ implies there exists a $z$ in $V$ such that $\sigma(\varphi, \psi) = \langle z, \psi \rangle_V$. From the bounded version of Lax-Milgram, this implies there exists $A \in \mathcal{L}(V)$ such that

$$\sigma(\varphi, \psi) = \langle A\varphi, \psi \rangle_V$$  \hfill (11)

for all $\varphi$ and $\psi$ in $V$, with $A$ one-to-one, $A$ onto, $|A|_{\mathcal{L}(V)} \leq \gamma$ and $|A^{-1}|_{\mathcal{L}(V)} \leq \frac{1}{\delta}$. In other words, $A$ is an isomorphism $V \to V$.

We have that $R^{-1} : V \to V^*$ is also an isomorphism; therefore, $R^{-1}A : V \to V^*$ is an isomorphism. The claim is that $A = R^{-1}A$. By (10) we have for all $\varphi, \psi \in V$

$$\langle R^{-1}A\varphi, \psi \rangle_{V^*, V} = \langle A\varphi, \psi \rangle_V.$$  \hfill (12)

However, by (11), this implies

$$\langle R^{-1}A\varphi, \psi \rangle_{V^*, V} = \sigma(\varphi, \psi).$$

Therefore, by definition of $A : V \to V^*$, $A$ must be given by $A = R^{-1}A$. So, we have

$$\langle A\varphi, \psi \rangle_{V^*, V} = \sigma(\varphi, \psi) \leq \gamma|\varphi|_V |\psi|_V$$  \hfill (13)

and

$$\langle A\varphi, \varphi \rangle_{V^*, V} \geq \delta|\varphi|^2_V.$$  \hfill (14)

Recall that $\sup_{|\psi| = 1} |\langle A\varphi, \psi \rangle_{V^*, V}| = |A\varphi|_{V^*}$. Letting $\varphi = A^{-1}\psi$ in (13) and combining this with (12) and (13), we have

$$\delta|\varphi|_V \leq |A\varphi|_{V^*} \leq \gamma|\varphi|_V.$$  \hfill (15)

Therefore, we have

$$|A|_{\mathcal{L}(V, V^*)} \leq \gamma$$

and

$$|A^{-1}|_{\mathcal{L}(V^*, V)} \leq \frac{1}{\delta}.$$
Implications of Lax-Milgram (unbounded form)

For $A$ as in the Lax-Milgram Theorem 2 and $f \in V^*$, we consider $Au = f$ in $V^*$. Then Lax-Milgram implies there exists a unique solution $u = A^{-1}f$ in $V$ that depends continuously on $f$. More precisely,

$$|u|_V = |A^{-1}f|_V \leq \delta|f|_{V^*}.$$ 

We revisit the steady-state heat equation (8) in the example above with $D \in L^\infty(\Omega)$ and $V = H^1_0(\Omega)$. This, of course, allows discontinuous coefficients. Then for any $f$ in $V^* = H^{-1}(\Omega)$ there exists a unique $u \in V$ satisfying $A u = f$. That is, $\langle Au - f, \varphi \rangle_{V^*, V} = \sigma(u, \varphi) - \langle f, \varphi \rangle_{V^*, V} = \langle D\nabla u, \nabla \varphi \rangle_{L^2} - \langle f, \varphi \rangle_{V^*, V} = 0$ for all $\varphi \in V$. Thus we say $u \in V$ satisfies $Au = f$ in the sense of $V^*$. Hence (8) holds in the $V^*$ sense which is precisely (9). This is also sometimes referred to as $u$ satisfying (8) in the “sense of distributions.”

11.1 The concept of $D_A$

We assume that we have a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ and a continuous sesquilinear form $\sigma : V \times V \rightarrow C$. As usual, $f(\psi) = \langle f, \psi \rangle_H$ defines, for $f \in H$, an element $f \in V^*$. Considering $(A\varphi)(\psi) = \sigma(\varphi, \psi)$ for $\varphi$ and $\psi$ in $V$, we define

$$D_A = \{ \varphi \in V | A\varphi \in H \}.$$ 

That is, $D_A$ is the set of $\varphi \in V$ such that $A\varphi \in V^*$ has the representation $(A\varphi)(\psi) = \langle \tilde{\varphi}, \psi \rangle_H$, for $\psi$ in $V$ and for some $\tilde{\varphi}$ in $H$.

We denote this element $\tilde{\varphi}$ by $-A\varphi = \tilde{\varphi}$, i.e., $A$ is linear from $D_A \subset V$ into $H$ and given by

$$\sigma(\varphi, \psi) = \langle -A\varphi, \psi \rangle_H$$

for $\psi$ in $V$ and $\varphi$ in $D_A$.

We note that the above can be interpreted as: $\varphi \in D_A \subset V$ if and only if $\varphi \in V$ and $A\varphi \in H^* \cong H$ so that $(A\varphi)(\psi) = \langle \tilde{\varphi}, \psi \rangle_H$, for all $\psi$ in $V$, and for some $\tilde{\varphi} \in H$. Alternatively, we may write $D_A = \{ \varphi \in V | |(A\varphi)(\psi)| = |\sigma(\varphi, \psi)| \leq k|\psi|_H, \psi \in V \}$ for some $k$. Moreover, we could write $D_A = \{ \varphi \in V | \psi \rightarrow \sigma(\varphi, \psi) \text{ is in } H^*, \text{ i.e, continuous on } H \}$ (continuous in the $H$ sense).

Note that we also have

$$\sigma(\varphi, \psi) = (A\varphi)(\psi) = \langle A\varphi, \psi \rangle_{V^*, V}$$

for all $\varphi$ and $\psi$ in $V$. If we restrict $\varphi \in D_A$ then this also equals $\langle -A\varphi, \psi \rangle_H$.

**Theorem 3** If $\sigma$ is a $V$-continuous $V$-coercive sesquilinear form on $V$, then $D_A$ is dense in $V$ and, hence, dense in $H$. 

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Proof of Theorem

Define \( \tilde{\sigma}(\varphi, \psi) = \overline{\sigma(\psi, \varphi)} \) (called the “adjoint” sesquilinear form).

If \( \sigma \) is \( V \)-coercive and \( V \) continuous, then \( \tilde{\sigma} \) is also \( V \)-coercive and \( V \) continuous. In other words, there exists an operator \( \tilde{A} : V \to V^* \) such that \( \tilde{A} \in \mathcal{L}(V, V^*) \) with

\[
\tilde{\sigma}(\varphi, \psi) = \langle \tilde{A}\varphi, \psi \rangle_{V^*,V} = (\tilde{A}\varphi)(\psi)
\]

for all \( \varphi \) and \( \psi \) in \( V \). Hence, \( \mathcal{R}(\tilde{A}) = V^* \) by the Lax-Milgram theorem.

Next we need to show that \( D_A \) is dense in \( V \). It suffices to show that if \( f \in V^* \) and \( f(v) = 0 \) for all \( v \in D_A \), then \( f \equiv 0 \).

Let \( f \in V^* \) such that \( f(v) = 0 \) for all \( v \in D_A \). Since \( f \in V^* \) and \( \mathcal{R}(\tilde{A}) = V^* \), then there exists \( \varphi \in V \) such that \( f = \tilde{A}\varphi \).

For \( v \in D_A \), we have

\[
(-Av)(\varphi) = \langle Av, \varphi \rangle = \sigma(v, \varphi)
\]

\[
= \overline{\tilde{\sigma}(\varphi, v)} = \overline{\langle \tilde{A}\varphi, v \rangle_{V^*,V}}
\]

\[
= \overline{\langle f, v \rangle_{V^*,V}} = \overline{f(v)}
\]

\[
= 0.
\]

Therefore, \( (Av)(\varphi) = 0 \) for all \( v \in D_A \).

But, we know \( \mathcal{R}(A|_{D_A}) = H \cong H^* \). Then, for every \( h \in H^* \), \( h(\varphi) = 0 \).

However, as \( H^* \) is dense in \( V^* \), then this implies \( \varphi = 0 \). Moreover, \( \tilde{A}\varphi = f \) implies \( f = 0 \).

Thus we find that any \( V \)-continuous \( V \)-coercive sesquilinear form on \( V \) gives rise to a densely defined operator \( A \) on \( \mathcal{D}(A) = D_A \) with

\[
\sigma(\varphi, \psi) = \langle A\varphi, \psi \rangle_{V^*,V} \quad \varphi, \psi \in V
\]

\[
= \langle -A\varphi, \psi \rangle_H \quad \varphi \in D_A, \psi \in V.
\]

11.2 V-elliptic

Definition 4 A sesquilinear form \( \sigma \) on \( V \) is \( V \)-elliptic if there exists a constant \( \delta > 0 \) such that

\[
\text{Re } \sigma(\varphi, \varphi) \geq \delta|\varphi|_V^2 \quad \varphi \in V.
\]
Discussion of V-elliptic

We note that $\sigma$ is V-elliptic implies $\sigma$ is V-coercive. Of course, if we are working in real spaces, $\text{Re} \; \sigma = \sigma$ and V-elliptic is equivalent to V-coercive. We also remark that the terminology among various authors is not standard. Some authors (e.g., Wloka in [W]) use our definition for V-coercive as the definition of V-elliptic and then use

$$|\sigma(\varphi, \varphi) + k|\varphi|^2_H| \geq \delta|\varphi|^2_V \; \varphi \in V, \delta > 0$$

as the definition of V-coercive.

Some of the terminology and usage of sesquilinear forms derives directly from that for PDE’s of the form

$$\frac{\partial y}{\partial t} = \sum_{i,j} \frac{\partial}{\partial \xi_i} (a_{ij} \frac{\partial y}{\partial \xi_j}) + \sum_j b_j \frac{\partial y}{\partial \xi_j}, \; t > 0, \xi \in G \subset \mathbb{R}^n.$$

**Definition 5** In classical PDE’s, an “operator” $\{a_{ij}\}$ is said to be strongly elliptic on $G$ if there exists $\delta > 0$ such that for $\xi \in G$

$$\text{Re} \; \sum_{i,j} a_{ij}(\xi) q_i \bar{q}_j \geq \delta \sum_i |q_i|^2$$

for all $q \in \mathbb{C}^n$.

An associated sesquilinear form on $V = H^1(G)$ can be defined by

$$\sigma(\varphi, \psi) = \int_G \left[ \sum_{i,j} a_{ij}(\xi) \frac{\partial \varphi}{\partial \xi_i} \frac{\partial \bar{\psi}}{\partial \xi_j} + \sum_k b_k \frac{\partial \varphi}{\partial \xi_k} \bar{\psi} \right] d\xi$$

**Discussion of Strongly elliptic**

It is a standard result that $\{a_{ij}\}$ strongly elliptic implies there exists $\delta > 0$ such that for $\lambda$ sufficiently large

$$\text{Re} \; \sigma(\varphi, \varphi) + \lambda |\varphi|^2_H \geq \delta|\varphi|^2_V, \; \varphi \in V.$$

Hence if $\{a_{ij}\}$ is strongly elliptic, then for some $\lambda_0$ sufficiently large,

$$\tilde{\sigma}(\varphi, \psi) = \sigma(\varphi, \psi) + \lambda_0 \langle \varphi, \psi \rangle_H$$

is V-elliptic and hence V-coercive.

A most useful result is that continuous V-elliptic forms give rise to operators that are infinitesimal generators of $C_0$ (actually analytic) semigroups.
12 Some Useful Theorems

Definition 6 A semigroup $T(t)$ on $H$ is called an analytic semigroup if $t \rightarrow T(t) \varphi$ is analytic for each $\varphi$ in $H$.

Theorem 4 Let $V$, $H$ be complex Hilbert spaces with $V \hookrightarrow H \hookrightarrow V^*$ and suppose that $\sigma : V \times V \rightarrow C$ is continuous and $V$-elliptic; i.e.

$$|\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V$$

and

$$\text{Re } \sigma(\varphi, \varphi) \geq \delta |\varphi|_V^2$$

$\delta > 0, \varphi \in V$.

Define $A : \mathcal{D}(A) \subset V \rightarrow H$ by

$$\mathcal{D}(A) = \{ \varphi \in V \mid \text{there exists } K_\varphi > 0 \text{ such that } \sigma(\varphi, \psi) \leq K_\varphi |\psi|_H, \psi \in V \}$$

and

$$\sigma(\varphi, \psi) = \langle -A \varphi, \psi \rangle_H, \ \varphi \in \mathcal{D}(A), \psi \in V.$$

Then $\mathcal{D}(A)$ is dense in $H$ and $A$ is the infinitesimal generator of a contraction semigroup in $H$ that actually is an analytic semigroup.

The proof of this theorem will come later.

Theorem 5 Suppose all the assumptions of Theorem 4 hold except that the $V$-ellipticity condition for $\sigma$ is replaced by

$$\text{Re } \sigma(\varphi, \varphi) + \lambda_0 |\varphi|_H^2 \geq \delta |\varphi|_V^2$$

for some $\lambda_0, \delta > 0, \varphi \in V$. Then defining $A$ as in Theorem 4, we have that $A$ is densely defined and is the infinitesimal generator of an analytic semigroup in $H$.

12.1 Return to Example 1

The system is given by

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial \xi} \left( D(\xi) \frac{\partial y}{\partial \xi} \right)$$

$$y(t, 0) = 0, \ \frac{\partial y}{\partial \xi}(t, l) = 0$$

$$y(0, \xi) = \Phi(\xi).$$
We choose the state space $X = L^2(0,l)$ as before. To obtain the weak variational form and the space $V$, we work backwards by multiplying the equation by a "test" function $\varphi$ and integrating.

$$\int_0^l \dot{y}\varphi = \int_0^l (Dy')'\varphi$$

$$= \int_0^l -Dy'\varphi' + Dy'|_0^l$$

Therefore we have

$$\langle \dot{y}(t), \varphi \rangle + \langle Dy'(t), \varphi' \rangle - Dy'|_0^l = 0$$  \hspace{1cm} (15)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L_2$. However, (15) is equivalent to

$$\langle \dot{y}(t), \varphi \rangle + \langle Dy'(t), \varphi' \rangle = 0$$

if $\varphi \in H^1_L(0,l) = \{ \varphi \in H^1(0,l) | \varphi(0) = 0 \}$ and $Dy'(t,l) = 0$.

Defining $V = H^1_L(0,l)$ and $\sigma$ on $V \times V$ by

$$\sigma(\varphi,\psi) = \langle D\varphi', \psi' \rangle,$$

we may write the equation in weak form as: find $y(t) \in V$ satisfying

$$\langle \dot{y}(t), \varphi \rangle + \sigma(y(t), \varphi) = 0$$

for all $\varphi \in V$. This equation is equivalent to the original system whenever $y(t) \in V \cap H^2(0,l)$ by using the reverse of the above arguments.

Next we consider the flux boundary condition of the original problem. Suppose $y(t)$ is a weak solution, i.e.,

$$\langle \dot{y}(t), \varphi \rangle + \sigma(y(t), \varphi) = 0 \quad \forall \varphi \in V$$

$$y(0) = \Phi(\xi)$$

and $y$ in $H^2(0,l)$. Then

$$\langle \dot{y}(t), \varphi \rangle + \int_0^l Dy' \varphi' = 0.$$

Integrating by parts, we find that the above equation is equivalent to

$$\int_0^l (\dot{y} - (Dy')') \varphi + Dy'(t,l) \varphi(l) = 0$$  \hspace{1cm} (16)$$

for all \( \varphi \in H^1_L \). However, \( H^1_0 \subset H^1_L \); therefore,

\[
\int_0^l (\dot{y} - (Dy')') \varphi = 0
\]

(17)

for all \( \varphi \in H^1_0 \). Since \( H^1_0 \) is dense in \( L^2(0, l) \), this implies \( \dot{y} - (Dy')' = 0 \). However, if we choose \( \varphi \in H^1_L \) such that \( \varphi(l) \neq 0 \), then (16) implies \( Dy'(t, l) = 0 \), i.e., the flux boundary condition is satisfied.

If we define the \( V \)-inner product as

\[
\langle \varphi, \psi \rangle_V = \int_0^l \varphi' \psi',
\]

and set \( H = X = L^2(0, l) \), then we readily see \( V \hookrightarrow H \hookrightarrow V^* \). Note that the \( V \) norm is equivalent to the usual \( H^1 \) norm on \( H^1_L(0, l) \). Furthermore, we have

\[
|\sigma(\varphi, \psi)| = |\langle D\varphi', \psi' \rangle| \\
\leq |D|_{\infty} |\varphi'|_{L^2} |\psi'|_{L^2} \\
= |D|_{\infty} |\varphi|_V |\psi|_V.
\]

Also,

\[
\text{Re } \sigma(\varphi, \varphi) = \text{Re } \langle D\varphi', \varphi' \rangle \geq \delta |\varphi'|_{L^2}^2 = \delta |\varphi|_V^2
\]

so that \( \sigma \) is bounded and \( V \)-elliptic.

We can define \( A : V \to V^* \) by

\[
\langle A\varphi, \psi \rangle_{V^*, V} = \sigma(\varphi, \psi) = \langle D\varphi', \psi' \rangle.
\]

Note that \( A\varphi \in H \hookrightarrow V^* \) if and only if \( \langle D\varphi', \psi' \rangle = \langle w, \psi \rangle \) for all \( \psi \in V \) for some \( w \in H \). However, integrating by parts we have

\[
\int_0^l D\varphi' \psi' = - \int_0^l (D\varphi')' \psi + D\varphi' \psi|_0^l \\
= \langle -(D\varphi')', \psi \rangle + D(l)\varphi'(l)\psi(l) \\
= \langle -(D\varphi')', \psi \rangle
\]

if \( \varphi'(l) = 0 \) and \( (D\varphi')' \in L^2(0, l) \). Thus we may define

\[
A\varphi = (D\varphi')'
\]
on
\[ \mathcal{D}(A) = \{ \varphi \in H^1_2(0, l) | (D \varphi')' \in L_2(0, l), \varphi'(l) = 0 \} \]
and obtain \( A \varphi = -A \varphi \in H \) exactly whenever \( \varphi \in \mathcal{D}(A) \).

The above results hence guarantee that \( A \) generates a \( C_0 \)-semigroup (actually an analytic semigroup) \( T(t) \) on \( H = X = L_2(0, l) \).

12.2 Return to Example 2

We consider again the transport equation given by
\[
\frac{\partial y}{\partial t} + \frac{\partial}{\partial \xi} (\nu y) = \frac{\partial}{\partial \xi} (D \frac{\partial y}{\partial \xi}) - \mu y \\
y(t, 0) = 0 \\
(D \frac{\partial y}{\partial \xi} - \nu y)_{\xi=l} = 0 \\
y(0, \xi) = \Phi(\xi).
\]
We can rewrite the transport equation by
\[
y_t = (Dy' - \nu y)' - \mu y.
\]
Multiplying by a test function and integrating from 0 to \( l \), we have
\[
\langle y_t, \varphi \rangle = \int_0^l ((Dy' - \nu y)' \varphi - \mu y \varphi) \, d\xi \\
= -\langle Dy' - \nu y, \varphi' \rangle + (Dy' - \nu y)\varphi|_0^l - \langle \mu y, \varphi \rangle.
\]
If we choose \( H = X = L_2(0, l) \) and \( V = H^1_2(0, l) \) as in Example 1 above, with the same \( V \) inner product, we have
\[
\langle y_t, \varphi \rangle = -\langle Dy' - \nu y, \varphi' \rangle - \langle \mu y, \varphi \rangle.
\]
As before, we have \( V \hookrightarrow H \hookrightarrow V^* \). Then we can define the sesquilinear form \( \sigma : V \times V \rightarrow C \) by
\[
\sigma(\varphi, \psi) = (D \varphi' - \nu \varphi, \psi') + \langle \mu \varphi, \psi \rangle.
\]
Therefore, we have the equation
\[
\langle \dot{y}, \varphi \rangle + \sigma(y, \varphi) = 0.
\]
Briefly, we discuss the various possibilities for boundary conditions and the effects on the choice of \( V \). If we had a no flux boundary condition at \( \xi = 0 \), we would choose \( V = H^2_0(0, l) \). On the other hand, if we had essential
boundary conditions at both boundaries, i.e., \( y = 0 \) at \( \xi = 0 \) and \( \xi = l \), we
would need to choose \( V = H^1_0(0,l) \). A third possibility is if we had the no
flux boundary conditions at both boundaries, \( \xi = 0 \) and \( \xi = l \). In that case,
as both boundary conditions were natural, we would choose \( V = H^1(0,l) \).

The \( V \)-continuity of \( \sigma \) is established by arguing

\[
|\sigma(\varphi, \psi)| \leq |D|_\infty |\varphi'|_H |\psi'|_H + |\nu|_\infty |\varphi|_H |\psi'|_H + |\mu|_\infty |\varphi|_H |\psi'|_H
\]

\[
\leq |D|_\infty |\varphi|_V |\psi|_V + |\nu|_\infty k |\varphi|_V |\psi|_V + |\mu|_\infty k^2 |\varphi|_V |\psi|_V
\]

\[
= (|D|_\infty + k |\nu|_\infty + k^2 |\mu|_\infty) |\varphi|_V |\psi|_V.
\]

As \( \sigma \) is \( V \)-continuous, we have

\[
\sigma(\varphi, \psi) = \langle A \varphi, \psi \rangle_{V^*, V} \quad \varphi \in V
\]

\[
= \langle -A \varphi, \psi \rangle_H \quad \varphi \in \mathcal{D}(A),
\]

where \( \mathcal{D}(A) \) is defined by

\[
\mathcal{D}(A) = \{ \varphi \in H^2(0,l) | \varphi(0) = 0, (D \varphi' - \nu \varphi) \in H^1(0,l), (D \varphi' - \nu \varphi)(l) = 0 \}.
\]

Note that \( V \) carries the essential boundary conditions, while the natural
boundary conditions are found in \( \mathcal{D}(A) \).

To show that \( \sigma \) is \( V \)-coercive, if we assume \( D \geq c_1 > 0 \) and \( \langle \mu \varphi, \varphi \rangle \geq -|\mu|_\infty |\varphi|_H^2 \), then we have

\[
\text{Re} \sigma(\varphi, \varphi) \geq c_1 |\varphi|_V^2 - \frac{|\nu|_\infty^2}{4\epsilon^2} |\varphi|_H^2 - \epsilon |\varphi|_V^2 - |\mu|_\infty |\varphi|_H^2
\]

\[
= (c_1 - \epsilon) |\varphi|_V^2 - (\frac{|\nu|_\infty^2}{4\epsilon^2} + |\mu|_\infty) |\varphi|_H^2.
\]

Hence, setting \( \epsilon = \frac{c_1}{2} \), we have

\[
\text{Re} \sigma(\varphi, \varphi) \geq \frac{c_1}{2} |\varphi|_V^2 - \lambda_0 |\varphi|_H^2
\]

for \( \lambda_0 = \frac{|\nu|_\infty^2}{2c_1} + |\mu|_\infty \). Thus we see that \( \tilde{\sigma} \) given by

\[
\tilde{\sigma}(\varphi, \psi) = \sigma(\varphi, \psi) + \lambda_0 \langle \varphi, \psi \rangle
\]

\[
= \langle -A \varphi, \psi \rangle + \lambda_0 \langle \varphi, \psi \rangle
\]

\[
= \langle -(A - \lambda_0) \varphi, \psi \rangle
\]

is \( V \)-elliptic (indeed it is \( V \) coercive). We thus find that \( A - \lambda_0 \), and hence
\( A \), is the generator of an analytic semigroup in \( H = X = L^2(0,l) \).
12.3 Return to Example 6

We return to the beam equation. Recall the system is given by

\[ \rho y_{tt} + \gamma y_t + \partial^2 M = f \quad 0 < \xi < l, \]

with

\[ y(t,0) = 0 = \frac{\partial y}{\partial t}(t,0) \]

\[ M(t, l) = 0 = \partial M(t, l), \]

where \( M(t, \xi) = EI\partial^2 y + c_D I\partial^2 y_t \). We choose as our basic space \( H = L^2_2(0, l) \) with the weighted inner product \( \langle \varphi, \psi \rangle_H = \langle \rho \varphi, \psi \rangle_{L^2_2(0, l)} \). Then the weak form becomes

\[ \langle y_{tt} + \frac{\gamma}{\rho} y_t, \varphi \rangle_H + \langle EI \partial^2 y, \partial^2 \varphi \rangle_H + \langle c_D I \partial^2 y_t, \partial^2 \varphi \rangle_H = \langle f, \varphi \rangle_H \]

for all \( \varphi \in V = H^2_0(0, l) \). We choose the weighted inner product for \( V \) given by \( \langle \varphi, \psi \rangle_V = \int_0^l EI \varphi'' \psi'' \).

We define the sesquilinear forms \( \sigma_1 \) and \( \sigma_2 \) on \( V \times V \rightarrow \mathbb{C} \) by

\[ \sigma_1(\varphi, \psi) = \langle EI \varphi'', \psi'' \rangle_H = \int_0^l EI \varphi'' \psi'' \]

\[ \sigma_2(\varphi, \psi) = \langle c_D I \varphi'', \psi'' \rangle_H + \langle \frac{\gamma}{\rho} \varphi, \psi \rangle_H. \]

The weak form of the equation is then

\[ \langle y_{tt}, \varphi \rangle_H + \sigma_1(y, \varphi) + \sigma_2(y_t, \varphi) = \langle \frac{f}{\rho}, \varphi \rangle_H \]

for \( \varphi \in V \). To write this in first order vector form, we use the state space \( X_E = \mathcal{H} = V \times H \) with the space \( V = V \times V \), noting that \( V \rightarrow H \rightarrow V^* \) and \( V \hookrightarrow \mathcal{H} \hookrightarrow V^* \) form Gelfand triples, where \( V^* = V \times V^* \).

Homework Exercises

- Ex. 9: Explain why we have \( V^* = V \times V^* \) in the Gelfand triple instead of \( V^* = V^* \times V^* \).
We define the sesquilinear form \( \sigma : V \times V \to C \) by (for \( \chi = (\varphi, \psi), \zeta = (g, h) \) in \( V \))

\[
\sigma(\chi, \zeta) = \sigma((\varphi, \psi), (g, h)) = -\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h).
\]

Using the state variable \( w(t) = (y(t, \cdot), y_t(t, \cdot)) \) in \( X_E = \mathcal{H} \), we can rewrite the equation as

\[
\langle \dot{w}(t), \chi \rangle_{\mathcal{H}} + \sigma(w(t), \chi) = \langle F(t), \chi \rangle_{\mathcal{H}}
\]

for \( \chi \in V \), where \( F(t) = (0, \frac{1}{2} f(t)) \).

We readily argue that \( \sigma \) is bounded (continuous) and \( V \)-elliptic (actually, \( \sigma - \lambda_0 \cdot | \cdot |_{X_E}^2 \) is \( V \)-elliptic). Consider first the boundedness argument:

\[
|\sigma(\chi, \zeta)| = |\sigma((\varphi, \psi), (g, h))| = | -\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h)|
\]

\[
\leq |\psi|_V|g|_V + |\varphi|_V|h|_V + |\psi|_V|h|_V
\]

\[
\leq |\chi|_V|\zeta|_V + |\chi|_V|\zeta|_V + |\chi|_V|\zeta|_V
\]

\[
= (1 + \gamma_1 + \gamma_2)|\chi|_V|\zeta|_V
\]

for \( \chi, \zeta \in V \). The arguments for \( V \)-ellipticity are also simple: for \( \chi = (\varphi, \psi) \in V \) we find

\[
\text{Re } \sigma(\chi, \chi) = \text{Re } \{ -\langle \psi, \varphi \rangle_V + \sigma_1(\varphi, \psi) + \sigma_2(\psi, \psi) \}
\]

\[
= \text{Re } \{ -\langle \varphi, \psi \rangle_V + \langle \varphi, \psi \rangle_V + \sigma_2(\psi, \psi) \}
\]

\[
= \text{Re } \sigma_2(\psi, \psi)
\]

\[
\geq \delta_2|\psi|_V^2.
\]

\[
= \delta_2(|\varphi|_V^2 + |\psi|_V^2) - \delta_2|\varphi|_V^2
\]

\[
\geq \delta_2(|\varphi|_V^2 + |\psi|_V^2) - \delta_2(|\varphi|_H^2 + |\psi|_H^2)
\]

\[
= \delta_2|\chi|_H^2 - \delta_2|\chi|_H^2.
\]

We thus find that \( \sigma(\chi, \zeta) = \langle \tilde{A} \chi, \zeta \rangle_{V^*, V} \) gives rise to the infinitesimal generator \( \tilde{A} \) of a \( C_0 \) (indeed, analytic) semigroup on \( X_E = \mathcal{H} \). It is readily argued that \( \sigma(\chi, \zeta) = \langle -A \chi, \zeta \rangle_\mathcal{H} \) for \( \chi \in \mathcal{D}(A) = \{ \chi = (\varphi, \psi) \in \mathcal{H} | \psi \in V = \)

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\[ H^2_L(0,l), A_1\varphi + A_2\psi \in H, (EI\varphi'' + c_DI\psi'')(l) = 0, (EI\varphi'' + c_DI\psi'')(l) = 0 \] where

\[ A = \begin{pmatrix} 0 & I \\ -A_1 & -A_2 \end{pmatrix} \]

with \( A_1\varphi = \partial^2(EI\partial^2\varphi) \) and \( A_2\varphi = \partial^2(c_DI\partial^2\varphi) \).

**Homework Exercises**

- Ex. 10: Some books define \( \mathcal{D}(A) \) by

\[ \tilde{\mathcal{D}}(A) = (H^4(0,l) \cap H^2_L(0,l)) \times (H^4(0,l) \cap H^2_L(0,l)) \]

plus boundary conditions. We know \( A|_{\mathcal{D}(A)} \) is an infinitesimal generator of a \( C_0 \) semigroup which, in turn, implies \( A|_{\mathcal{D}(A)} \) is a closed operator. You can show \( A|_{\tilde{\mathcal{D}}(A)} \) is not closed. Therefore, we claim that \( \mathcal{D}(A) \neq \tilde{\mathcal{D}}(A) \). Is this true? Look at both the damped and undamped cases.
13 Summary of Results on Analytic Semigroup Generation by Sesquilinear Forms

We summarize results available for the special cases of analytic semigroupo
generation. For further details the reader can consult [BI, T].

Let $V$ and $H$ be complex Hilbert spaces with the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. Let $\langle \cdot, \cdot \rangle_{V^*,V}$ be the duality product, and $\sigma : V \times V \to C$ be a
sesquilinear form such that $\sigma$ is

1. $V$ continuous, i.e., $|\sigma(\varphi, \psi)| \leq \gamma|\varphi|_V|\psi|_V$.

2. $V$-elliptic, i.e., $\text{Re} \sigma(\varphi, \varphi) \geq \delta|\varphi|_V^2$. (We can, if necessary, replace this
   by a shift: $\text{Re} \sigma(\varphi, \varphi) + \lambda_0|\varphi|_H^2 \geq \delta|\varphi|_V^2$.)

As before, let $\hat{A} \in \mathcal{L}(V, V^*)$ (note that this is $-\mathcal{A}$ in our old notation)
and $A : D_A \subset H \to H$ be defined such that

$$
\sigma(\varphi, \psi) = \langle -\hat{A}\varphi, \psi \rangle_{V^*,V} \quad \text{for all } \varphi, \psi \in V
$$

$$
= \langle -A\varphi, \psi \rangle_H \quad \varphi \in D_A, \psi \in V.
$$

Then we have $\mathcal{R}(\hat{A}) = V^*$, $\mathcal{R}(A) = H$, and $0 \in \rho(\hat{A})$. We can also note

$$
\text{Re} \sigma(\varphi, \varphi) = \text{Re} \langle -\hat{A}\varphi, \varphi \rangle_{V^*,V} \geq \delta|\varphi|_V^2.
$$

for all $\varphi \in V$. In other words, $\text{Re} \langle \hat{A}\varphi, \varphi \rangle \leq -\delta|\varphi|_V^2 \leq 0$. Similarly, for
$\varphi \in D_A$, $\text{Re} \langle A\varphi, \varphi \rangle_H \leq 0$ which implies $A$ is dissipative. By Lumer Phillips,
because $A$ is dissipative and $\mathcal{R}(A) = H$, $A$ is the infinitesimal generator of
a $C_0$ semigroup of contractions $S(t)$ on $H$.

We recall the definition of dissipativeness in a Banach space $X$. An
operator $B \in \mathcal{D} \subset X \to X$ is dissipative if for each $x \in \mathcal{D}(B)$ there exists
$x^* \in F(x) \subset X^*$ such that $\text{Re} \langle x^*, Bx \rangle_{X^*,X} \leq 0$ where $F(x)$ is the duality
set. Let’s apply this definition to $X = V^*$, which is a reflexive Banach
space in its own right, with the operator $B = \hat{A}$, $\hat{A} : V \subset V^* \to V^*$. We
have $\hat{A}$ being dissipative in the Banach space $V^*$ means for $x \in V$ there
exists $x^* \in F(X) \subset X^* = V^{**} = V$ such that $\text{Re} \langle x^*, \hat{A}x \rangle_{V^*,V^*} \leq 0$ or
$\text{Re} \langle \hat{A}x, x^* \rangle_{V^*,V^*} \leq 0$. However, we have this holding for every $x^* \in V \subset V^*$.
(In particular, we can find such a $x^*$ in the duality set.) Therefore, $\hat{A} : V =
\mathcal{D}(\hat{A}) \subset V^* \to V^*$ is dissipative. Using Lumer Phillips again we have $\hat{A}$ is an
infinitesimal generator of a $C_0$ semigroup of contractions $\hat{S}(t)$ on $V^*$ where
$\hat{S}(t)|_H = S(t)$. 26
Recall $D_A = \{ x \in V | \hat{A}x \in H \}$. We define $\hat{D}_A = \{ x \in V | \hat{A}x \in V \}$ and define the operator $\hat{A} = A|_{\hat{D}_A}$ in $V$. We have $\mathcal{R}(\hat{A}) = V$; therefore the range statement needed for Lumer Phillips holds for $\hat{A}$. However, for $\hat{A}$ to be dissipative in $V$, we must have for each $x \in \hat{D}_A \subset V$ there exists $x^* \in F(x) \subset V^*$ such that $\text{Re} \langle x^*, \hat{A}x \rangle_{V^*,V} \leq 0$. We do not directly have that $\hat{A}$ is dissipative in $V$. To pursue this, we need to consider the Tanabe estimates.

### 13.1 Tanabe Estimates

(on “Regular Dissipative Operators”)

Suppose a sesquilinear form $\sigma$ (with associated operator $\hat{A}$) is $V$ continuous and $V$-elliptic. Then for $\text{Re} \lambda \geq 0$ and $\lambda \neq 0$, $R_\lambda(\hat{A}) = (\lambda I - \hat{A})^{-1} \in \mathcal{L}(V^*,V)$, and

1. $|R_\lambda(\hat{A})\varphi|_V \leq \frac{1}{\delta} |\varphi|_{V^*}$ for $\varphi \in V^*$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(V^*,V)} \leq \frac{1}{\delta}$.)

2. $|R_\lambda(\hat{A})\varphi|_H \leq \frac{M_0}{|\lambda|} |\varphi|_H$ for $\varphi \in H$ where $M_0 = 1 + \frac{1}{\delta}$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(H)} \leq \frac{M_0}{|\lambda|}$.)

3. $|R_\lambda(\hat{A})\varphi|_{V^*} \leq \frac{M_0}{|\lambda|} |\varphi|_{V^*}$ for $\varphi \in V^*$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(V^*)} \leq \frac{M_0}{|\lambda|}$.)

4. $|R_\lambda(\hat{A})\varphi|_V \leq \frac{M_0}{|\lambda|} |\varphi|_V$ for $\varphi \in V$. (In other words, $|R_\lambda(\hat{A})|_{\mathcal{L}(V)} \leq \frac{M_0}{|\lambda|}$.)

We give the arguments for 4. since it is a very strong and useful estimate. Define the dual or adjoint operator in the usual manner: define $\hat{A}^* \in \mathcal{L}(V, V^*)$ by $\sigma(\varphi, \psi) = \langle \varphi, -\hat{A}^*\psi \rangle_{V,V^*}$ for $\varphi, \psi \in V$. Then

$$\sigma^*(\varphi, \psi) = \overline{\sigma(\varphi, \psi)}$$

and

$$\sigma^*(\varphi, \psi) = \langle \varphi, -\hat{A}^*\psi \rangle_{V,V^*}$$

implies

$$\langle \varphi, -\hat{A}^*\psi \rangle_{V,V^*} = \overline{\langle \varphi, -\hat{A}^*\psi \rangle_{V,V^*}}.$$
Therefore, $\hat{A}^*$ also satisfies the estimates 1.-3. above because $\sigma^*$ is $V$ continuous and $V$-elliptic. Applying 3 to $\hat{A}^*$ gives us for $\Re \lambda \geq 0$, $\lambda \neq 0, \varphi, \psi \in V$

$$\left|\langle R_\lambda (\hat{A}) \varphi, \psi \rangle_{V, V^*} \right| = \left|\langle \varphi, R_\lambda (\hat{A}^*) \psi \rangle_{V, V^*} \right|$$

$$\leq |\varphi|_V |R_\lambda (\hat{A}^*) \psi|_{V^*}$$

$$\leq |\varphi|_V \frac{M_0}{|\lambda|} |\psi|_{V^*}.$$ 

Therefore, $|R_\lambda (\hat{A}) \varphi|_V \leq \frac{M_0}{|\lambda|} |\varphi|_V$.

We can show that the $C_0$ semigroups from above are actually analytic. The theorem below gives a useful condition for analyticity.

**Theorem 6** Let $T(t)$ be a $C_0$ semigroup on a Hilbert space $X$ with infinitesimal generator $A$, with $0 \in \rho(A)$. Then a semigroup is analytic on $X$ if there exists a constant $c$ such that

$$|R_{\mu + i\tau}(A)|_{\mathcal{L}(X)} \leq \frac{c}{|\tau|}$$

for $\mu > 0, \tau \neq 0$ where $\lambda = \mu + i\tau$.

See [P, Theorem II.5.2(b)].

From the Tanabe estimates, we have $|R_\lambda (\hat{A})|_{\mathcal{L}(X)} \leq \frac{c}{|\lambda|} = \frac{c}{\sqrt{\mu^2 + \tau^2}} \leq \frac{c}{|\tau|}$ for $X$ chosen as $V, H,$ or $V^*$. Therefore, our estimates suffice to provide analyticity of the associated semigroups. Thus we have the following theorem.

**Theorem 7** Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triple. Assume the sesquilinear form $\sigma$ is $V$ continuous and $V$-elliptic. Let $\hat{A}, A,$ and $\hat{A}$ be defined as above. Then

- $\hat{A}$ is an infinitesimal generator of an analytic semigroup $\hat{S}(t)$ of contractions on $V^*$.
- $A$ is an infinitesimal generator of an analytic semigroup $S(t)$ of contractions on $H$.
- $\hat{A}$ is an infinitesimal generator of an analytic semigroup $\hat{S}(t)$ of contractions on $V$. 

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We also have

- \( \text{dom}_{V^*}(\hat{A}) = V \).
- \( \text{dom}_{H}(\hat{A}) = D_A = \{ x \in V | \hat{A}x \in H \} \).
- \( \text{dom}_{V}(\hat{A}) = \hat{D}_A = \{ x \in V | \hat{A}x \in V \} \).

This is usually stated as \( A \) or \( \hat{A} \) generate an analytic semigroup of contractions on \( V, H, V^* \).

Remark: The results of this section will be of fundamental importance in subsequent discussion on feedback control problems for infinite dimensional systems such as parabolic and strongly damped hyperbolic partial differential equations as well as functional differential equations. These problems can be conveniently and profitably formulated in an abstract setting with the systems defined in terms of Gelfand triples and where the associated algebraic Riccati equations for the feedback gains are formulated in an appropriately defined space \( V \). In later sections we will discuss some of the results as developed in [BKcontrol, BI, 32] and summarized partially in [BSW].

13.2 Infinitesimal Generators in a General Banach Space

Recall that if \( A \) is an infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) in a Hilbert space \( X \), then \( S(t) = T^*(t) \) is a \( C_0 \) semigroup in \( X \) with infinitesimal generator \( A^* \). Thus, if \( A \) is an infinitesimal generator, \( \mathcal{D}(A) \) is dense in \( X \). Similarly, if \( A^* \) is an infinitesimal generator, \( \mathcal{D}(A^*) \) is also dense in \( X \). We can generalize this result in a general Banach space.

Theorem 8 If \( X \) is a reflexive Banach space and \( A \) is an infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) in \( X \), then \( A^* \) is an infinitesimal generator of a \( C_0 \) semigroup \( S(t) \) in \( X^* \) and \( S(t) = T^*(t) = (T(t))^* \). In other words, \( (e^{A^*t} \text{ on } X^*)^* = e^{At} \text{ on } X \).

Corollary 1 If \( \hat{A} \) is an infinitesimal generator of a \( C_0 \) semigroup on \( V^* \), then \( \hat{A}^* \) is an infinitesimal generator on \( V^{**} = V \) for any reflexive Banach space \( V \).
In the formulations of the previous sections, we know \( \hat{A} \in \mathcal{L}(V, V^*) \) and \( \hat{A}^* \in \mathcal{L}(V^{**}, V^*) = \mathcal{L}(V, V^*) \) are infinitesimal generators of \( C_0 \) semigroups of contractions on \( V^* \). In other words, \( \hat{S}^*(t) = e^{\hat{A}^* t} \) is a \( C_0 \) semigroup of contractions on \( V^* \). Applying the previous corollary, we have \( (\hat{S}^*(t))\ast = \hat{S}(t) \) on \( V \). However, \( \hat{S}^*(t) \in \mathcal{L}(V^*, V^*) \) implies \( (\hat{S}^*(t))\ast \in \mathcal{L}(V^{**}, V^{**}) = \mathcal{L}(V, V) \). Since \( V \) is a reflexive Hilbert space, \( \hat{S}(t) \) defined previously is exactly \( \hat{S}(t)|_V \) (here \( \hat{A} = A|_{\hat{D}_A} \) is the infinitesimal generator of \( \hat{S}(t) \) in \( V \).
14 General Second Order Systems

14.1 Introduction to Second Order Systems

The ideas in Example 6 can be used to treat general second order systems. Consider the general abstract second order system

\[ \ddot{y}(t) + A_2 \dot{y}(t) + A_1 y(t) = f(t) \]

or, in variational form

\[ \langle \ddot{y}(t), \varphi \rangle_{V^*,V} + \sigma_1(y(t), \varphi) + \sigma_2(\dot{y}(t), \varphi) = \langle f(t), \varphi \rangle_{V^*,V} \]  

(18)

where \( H \) is a complex Hilbert space. As usual, we assume that \( \sigma_1 \) and \( \sigma_2 \) are sesquilinear forms on \( V \) where \( V \hookrightarrow H \hookrightarrow V^* \) is a Gelfand triple. We also assume that \( \sigma_1 \) is continuous, \( V \)-elliptic and symmetric \( (\sigma_1(\varphi, \psi) = \sigma_1(\bar{\psi}, \bar{\varphi})) \). We assume that \( \sigma_2 \) is continuous and satisfies a weakened ellipticity condition which we formally call \( H \)-semiellipticity.

**Definition 7** A sesquilinear form \( \sigma \) on \( V \) is \( H \)-semielliptic if there is a constant \( b \geq 0 \) such that

\[ \text{Re} \sigma(\varphi, \varphi) \geq b |\varphi|_H^2 \quad \text{for all } \varphi \in V. \]

Note that \( b = 0 \) is allowed in this definition.

Since \( \sigma_1 \) and \( \sigma_2 \) are continuous, we have that there exists \( A_i \in \mathcal{L}(V, V^*) \), \( i = 1, 2 \), such that

\[ \sigma_i(\varphi, \psi) = \langle A_i \varphi, \psi \rangle_{V^*,V} \quad \text{for all } \varphi, \psi \in V, \quad i = 1, 2. \]

Following the ideas of Example 6, we define spaces \( \mathcal{V} = V \times V \) and \( \mathcal{H} = V \times H \) and rewrite our second order system as a first order vector system. Defining, for \( \chi = (\varphi, \psi), \zeta = (g, h) \in \mathcal{V} \), the sesquilinear form

\[ \sigma(\chi, \zeta) = \sigma((\varphi, \psi), (g, h)) = -\langle \psi, g \rangle_V + \sigma_1(\varphi, h) + \sigma_2(\psi, h), \]

we can write our system for \( x(t) = (y(t), \dot{y}(t)) \) as

\[ \langle \dot{x}(t), \chi \rangle_{\mathcal{H}} + \sigma(x(t), \chi) = \langle F(t), \chi \rangle_{\mathcal{H}} \quad \chi \in \mathcal{V} \]

where \( F(t) = (0, f(t)) \). This is formally equivalent to the system

\[ \dot{x}(t) = Ax(t) + F(t) \]
where $A$ is given by

$$D(A) = \{ x = (\varphi, \psi) \in \mathcal{H} | \psi \in V \text{ and } A_1 \varphi + A_2 \psi \in H \}$$

and

$$A = \begin{pmatrix} 0 & I \\ -A_1 & A_2 \end{pmatrix}.$$  

(19)

We first note that $\sigma$ is $V$ continuous. To see this, we observe that $\sigma_1$ and $\sigma_2$ being $V$ continuous implies

$$\sigma_1(\varphi, h) \leq \gamma_1 |\varphi|_V |h|_V$$

and

$$\sigma_2(\varphi, h) \leq \gamma_2 |\varphi|_V |h|_V.$$  

(20)

We also have $|\chi|_V^2 = |\varphi|_V^2 + |\psi|_V^2$ and $|\zeta|_V^2 = |g|_V^2 + |h|_V^2$. Putting all of this together, we have

$$|\sigma((\varphi, \psi), (g, h))| \leq |\psi|_V |g|_V + \gamma_1 |\varphi|_V |h|_V + \gamma_2 |\psi|_V |h|_V$$

$$\leq |\chi|_V |\zeta|_V + \gamma_1 |\chi|_V |\zeta|_V + \gamma_2 |\chi|_V |\zeta|_V$$

$$= (1 + \gamma_1 + \gamma_2) |\chi|_V |\zeta|_V.$$  

This indeed implies that $\sigma$ is $V$ continuous.

As $\sigma$ is $V$ continuous, $A$ is the negative of the restriction to $D(A)$ of the operator $\tilde{A} \in \mathcal{L}(V, V^\ast)$ defined by $\langle \sigma(\chi, \zeta) \rangle = \langle \tilde{A}\chi, \zeta \rangle_{V^\ast, V}$ so that $\sigma(\chi, \zeta) = \langle -A\chi, \zeta \rangle_{\mathcal{H}}$ for $\chi \in D(A), \zeta \in V$.

### 14.2 Results for $\sigma_2$ V-elliptic

If both $\sigma_1$ and $\sigma_2$ are $V$-elliptic and $\sigma_1$ is the same as the $V$ inner product, then we have exactly the case of Kelvin-Voigt damping in Example 6. We proved with these assumptions, $\sigma$ is $V$-elliptic. (Actually, we proved $\sigma(\cdot, \cdot) + \lambda_0 \langle \cdot, \cdot \rangle_{\mathcal{H}}$ is $V$-elliptic.) Therefore, we have $A$ is the infinitesimal generator of an analytic semigroup (not of contractions since $\lambda_0 > 0$) on $\mathcal{H}$.

Even if the $V$ inner product and $\sigma_1$ are not the same, this result is true. Since $\sigma_1$ is continuous, we have $|\sigma_1(\varphi, \psi)| \leq \gamma_1 |\varphi|_V^2$ while $\sigma_1$ is symmetric (i.e., $\sigma_1(\varphi, \psi) = \sigma_1(\psi, \varphi)$) implies $\text{Re } \sigma_1(\varphi, \varphi) = \sigma_1(\varphi, \varphi)$. Thus, $\sigma_1$ being $V$-elliptic is equivalent to $\sigma_1$ is $V$-coercive: $\sigma_1(\varphi, \varphi) \geq \delta |\varphi|_V^2$. Hence, $\sigma_1$ and the inner product are equivalent. We may thus define $V_1$ as the space $V$ with $\sigma_1$ as inner product, obtaining a space that is setwise equal and topologically equivalent to $V$. In the space $\mathcal{H}_1 = V_1 \times H$ the operator $A$ is now associated with the $V_1 = V_1 \times V_1$-elliptic form $\sigma^{(1)}(\chi, \zeta) = \langle -A\chi, \zeta \rangle_{\mathcal{H}_1}$ that (as we}
argued in Example 6) satisfies the conditions of our theorem. Hence, $A$ generates an analytic semigroup in $\mathcal{H}_1$ and hence an analytic semigroup in the equivalent space $\mathcal{H}$. Thus we have

**Theorem 9** Let $V \hookrightarrow H \hookrightarrow V^*$ and suppose that $\sigma_1$ and $\sigma_2$ of (18) are $V$ continuous and $V$-elliptic sesquilinear forms on $V$ and that $\sigma_1$ is symmetric. Then the operator $A$ defined in (19) and (20) is the infinitesimal generator of an analytic semigroup in $\mathcal{H} = V \times H$.

### 14.3 Results for $\sigma_2$ $H$-semielliptic

If $\sigma_2$ is not $V$-elliptic, then we will not, in general, obtain an analytic solution semigroup for our system. We will obtain a $C_0$ semigroup, but must work a little more to obtain such. So assume that $\sigma_2$ is only $H$-semielliptic. Then we have $A$ defined in (19) and (20) is dissipative in $\mathcal{H}_1$ since

$$\text{Re} \langle Ax, x \rangle_{\mathcal{H}_1} = \text{Re} \left\{ \sigma_1(\psi, \varphi) - \sigma_1(\varphi, \psi) - \sigma_2(\psi, \psi) \right\}$$

$$= \text{Re} \left\{ \sigma_1(\overline{\varphi}, \overline{\psi}) - \sigma_1(\varphi, \psi) - \sigma_2(\psi, \psi) \right\}$$

$$= -\text{Re} \sigma_2(\psi, \psi)$$

$$\leq -b|\psi|_H^2 \leq 0.$$  

To argue that $A$ is a generator, we use the Lumer Phillips theorem; thus we need to argue that for some $\lambda > 0$, the range of $\lambda I - A$ is $\mathcal{H}_1$. Thus, given $\zeta = (g, h) \in \mathcal{H}_1$, we wish to solve $(\lambda - A) \chi = \zeta$ for $\chi = (\varphi, \psi) \in \mathcal{D}(A)$.

So we consider the equation

$$(\lambda - A)(\varphi, \psi) = (g, h) \quad \text{for } (g, h) \in V_1 \times H.$$  

This is equivalent to

$$\begin{cases} 
\lambda \varphi - \psi = g \\
\lambda \psi + A_1 \varphi + A_2 \psi = h. 
\end{cases}$$  (21)

If we formally solve the first equation for $\psi = \lambda \varphi - g$ and substitute this into the second equation, we obtain

$$\lambda^2 \varphi - \lambda g + A_1 \varphi + A_2 (\lambda \psi - g) = h$$

or

$$\lambda^2 \varphi + A_1 \varphi + \lambda A_2 \psi = h + \lambda g + A_2 g.$$  (22)
This equation must be solved for \( \varphi \in V_1 \) (and then \( \psi \) defined by \( \psi = \lambda \varphi - g \) will also be in \( V_1 \)).

These formal calculations suggest that we define for \( \lambda > 0 \) the associated sesquilinear form on \( V \times V \to \mathbb{C} \)

\[
\sigma_\lambda(\varphi, \psi) = \lambda^2 \langle \varphi, \psi \rangle_H + \sigma_1(\varphi, \psi) + \lambda \sigma_2(\varphi, \psi).
\]

Since \( \sigma_1 \) is \( V \)-elliptic and \( \sigma_2 \) is \( H \)-semielliptic we have

\[
\text{Re } \sigma_\lambda(\varphi, \varphi) = \lambda^2 |\varphi|^2_H + \text{Re } \sigma_1(\varphi, \varphi) + \lambda \text{Re } \sigma_2(\varphi, \varphi)
\geq \lambda^2 |\varphi|^2_H + c_1 |\varphi|^2_V + \lambda b |\varphi|^2_H
= \lambda(\lambda + b) |\varphi|^2_H + c_1 |\varphi|^2_V
> c_1 |\varphi|^2_V
\]

for \( \tilde{\lambda} = \lambda(\lambda + b) > 0 \). Hence \( \sigma_\lambda \) is \( V \)-elliptic and \( (22) \) is solvable for \( \varphi \in V \) by Lax-Milgram. It follows that \( (21) \) is solvable for \( (\varphi, \psi) \in D(A) \), i.e., \( \mathcal{R}(\lambda - A) = \mathcal{H}_1 \). Thus we have that \( A \) generates a contraction semigroup in \( \mathcal{H}_1 \) and a \( C_0 \) semigroup in \( \mathcal{H} \).

**Theorem 10** Let \( V \hookrightarrow H \hookrightarrow V^* \) and suppose that \( \sigma_1 \) and \( \sigma_2 \) of (18) satisfy: \( \sigma_1 \) is \( V \)-elliptic, \( V \) continuous and symmetric, \( \sigma_2 \) is \( V \) continuous and \( H \)-semielliptic. Then \( A \) defined by (19) and (20) generates a \( C_0 \) semigroup in \( \mathcal{H} \).

14.4 Stronger Assumptions for \( \sigma_2 \)

If we strengthen the assumption on the damping form, we can obtain a stronger result.

**Theorem 11** Suppose \( \sigma_1 \) is \( V \)-elliptic, \( V \) continuous, and symmetric and \( \sigma_2 \) is \( H \)-elliptic, \( V \) continuous, and symmetric. Then \( A \) is the infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) in \( \mathcal{H} = V \times H \) that is exponentially stable, i.e., \( |T(t)\chi|_\mathcal{H} \leq Me^{-\omega t}|\chi|_\mathcal{H} \) for some \( \omega > 0 \).

To motivate the arguments used to establish this result, we consider for \( \omega > 0 \) the change of dependent variable \( y(t) = e^{-\omega t}r(t) \) in the equation

\[
\ddot{y} + A_2 \dot{y}(t) + A_1 y(t) = 0.
\]
Upon substitution, we obtain
\[ \ddot{r}(t) + \hat{A}_2 \dot{r}(t) + \hat{A}_1 r(t) = 0 \]  \tag{24}
where
\[
\hat{A}_1 = A_1 - \omega A_2 + \omega^2 I \\
\hat{A}_2 = A_2 - 2\omega I.
\]
This suggests that we define the sesquilinear forms
\[
\hat{\sigma}_1(\varphi, \psi) = \sigma_1(\varphi, \psi) - \omega \sigma_2(\varphi, \psi) + \omega^2 \langle \varphi, \psi \rangle_H
\]
\[
\hat{\sigma}_2(\varphi, \psi) = \sigma_2(\varphi, \psi) - 2\omega \langle \varphi, \psi \rangle_H
\]
so that \( \hat{\sigma}_i(\varphi, \psi) = \langle \hat{A}_i \varphi, \psi \rangle_{V^*, V}, i = 1, 2, \) and the transformed variational form of (23) is
\[
\langle \ddot{r}(t), \varphi \rangle_{V^*, V} + \hat{\sigma}_1(r(t), \varphi) + \hat{\sigma}_2(\dot{r}(t), \varphi) = 0
\]
for \( \varphi \in V. \)

We observe that \( \hat{\sigma}_1, \hat{\sigma}_2 \) are continuous and \( \hat{\sigma}_1 \) is symmetric since both \( \sigma_1 \) and \( \sigma_2 \) are. Since \( \sigma_2 \) is symmetric (hence \( \sigma_2(\varphi, \varphi) \) is real) and continuous with \( \sigma_2(\varphi, \varphi) \leq k_2|\varphi|^2_V, \) we have for \( \varphi \in V \)
\[
\text{Re} \hat{\sigma}_1(\varphi, \varphi) = \hat{\sigma}_1(\varphi, \varphi) \\
= \sigma_1(\varphi, \varphi) - \omega \sigma_2(\varphi, \varphi) + \omega^2 |\varphi|^2_H \\
\geq c_1|\varphi|^2_V - \omega \gamma_2 |\varphi|^2_H + \omega^2 |\varphi|^2_H \\
\geq (c_1 - \omega \gamma_2) |\varphi|^2_V.
\]

Hence \( \hat{\sigma}_1 \) is \( V \)-elliptic if \( \omega > 0 \) is chosen so that \( \omega < \frac{c_1}{\gamma_2}. \)

Moreover, we find that \( \hat{\sigma}_2 \) is \( H \)-semielliptic if \( \omega \) is chosen properly since
\[
\text{Re} \hat{\sigma}_2(\varphi, \varphi) = \text{Re} \sigma_2(\varphi, \varphi) - 2\omega |\varphi|^2_H \geq (b - 2\omega)|\varphi|^2_H.
\]

Therefore, \( \hat{\sigma}_2 \) is \( H \)-semielliptic if \( \omega < \frac{b}{2}. \)

Thus, if we choose \( \omega > 0 \) as \( \omega = \frac{1}{2} \min \left\{ \frac{b}{2}, \frac{c_1}{\gamma_2} \right\}, \) we find that \( \hat{\sigma}_1 \) and \( \hat{\sigma}_2 \) satisfy the assumptions of Theorem 10. By the arguments preceding that theorem, we see that
\[
\hat{A} = \begin{pmatrix} 0 & I \\ -\hat{A}_1 & -\hat{A}_2 \end{pmatrix}
\]
(see (19) and (20)) generates a contraction semigroup \( \hat{T}(t) \) on \( \hat{H}_1 = \hat{V}_1 \times H \) where \( \hat{V}_1 \) is \( V \) taken with \( \hat{\sigma}_1 \) as inner product (\( \hat{V}_1 \) is equivalent to \( V \)).
Now let $T(t)$ be the $C_0$-semigroup generated by $A$ (see (19), (20) and Theorem 10). If $x(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}$ and $w(t) = \begin{pmatrix} r(t) \\ \dot{r}(t) \end{pmatrix}$ are solutions of (23) and (24) respectively, we have $x(t) = T(t)x_0$ where $x_0 = \begin{pmatrix} y_0 \\ w_0 \end{pmatrix}$ and $w(t) = \hat{T}(t)w_0$. Since $y(t) = e^{-wt}r(t)$ and $\dot{y}(t) = -we^{wt}r(t) + e^{wt}\dot{r}(t)$, we see that $x(t) = e^{wt}\Gamma w(t)$ where

$$\Gamma = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}$$

and $w_0 = \Gamma^{-1}x_0$. It follows since $|\hat{T}(t)|_{\hat{H}_1} \leq 1$ that

$$|T(t)x_0|_{\hat{H}_1} \leq e^{-wt}|\Gamma\hat{T}(t)\Gamma^{-1}x_0|_{\hat{H}_1} \leq Me^{-wt}|x_0|_{\hat{H}_1}.$$

Since $\hat{H}_1$ and $\mathcal{H} = V \times H$ are norm equivalent, we thus find that the semigroup $T(t)$ is exponentially stable in $\mathcal{H}$. 

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15 Abstract Cauchy Problem

It is of great practical as well as theoretical interest to know when, and in what sense, solutions of the abstract equations

\[
\dot{x}(t) = Ax(t) + f(t) \quad \text{for } t \in [0, T]
\]

exist. Moreover, representations of such solutions in terms of a variation of parameters formula and the semigroup generated by \( A \) will play a fundamental role in control and estimation formulations. We begin by summarizing results available in the standard literature on linear semigroups and abstract Cauchy problems.

Consider the abstract Cauchy problem (ACP) given by (25) where \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup \( T(t) \) in a Hilbert space \( H \). We define a mild solution \( x_m \) of (25) as functions given by

\[
x_m(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds
\]

whenever this entity is well defined (i.e., \( f \) is sufficiently smooth).

We say that \( x : [0, T] \to H \) is a strong solution of (ACP) if \( x \in C([0, T], H) \cap C^1([0, T], H) \), \( x(t) \in \mathcal{D}(A) \) for \( t \in (0, T] \), and \( x \) satisfies (25) on \([0, T]\).

We have the following series of results from the literature.

**Theorem 12** If \( f \in L^1((0, T), H) \) and \( x_0 \in H \), there is at most one strong solution of (25). If a strong solution exists, it is given by (26).

**Theorem 13** If \( x_0 \in \mathcal{D}(A) \) and \( f \in C^1([0, T], H) \), then \( x_m \) given by (26) provides the unique strong solution of (25).

**Theorem 14** If \( x_0 \in \mathcal{D}(A), f \in C([0, T], H), f(t) \in \mathcal{D}(A) \) for each \( t \in [0, T] \) and \( Af \in C([0, T], H) \), then (26) provides the unique strong solution of (25).

**Theorem 15** Suppose \( A \) is the infinitesimal generator of an analytic semigroup \( T(t) \) on \( H \). Then if \( x_0 \in H \) and \( f \) is Hölder continuous (i.e. \( |f(t) - f(s)| \leq k|t-s|^\gamma \) for some \( \gamma \leq 1 \)), then \( x_m \) of (26) provides the unique strong solution of (25).
Unfortunately, all of these powerful results are too restrictive for use in many applications, including control theory, where typically \( f(t) = Bu(t) \) is not continuous, let alone Hölder continuous or \( C^1 \). For this reason, a weaker formulation is more appropriate. For this, we follow the presentations of Lions, Wolka, and Tanabe which are developed in the context of sesquilinear forms and Gelfand triples, \( V \hookrightarrow H \hookrightarrow V^* \), where \( V, H, V^* \) are Hilbert spaces.

We define the solution space \( W(0,T) \) by

\[
W(0,T) = \{ g \in L_2((0,T),V) : \frac{dg}{dt} \in L_2((0,T),V^*) \}
\]

with scalar product

\[
\langle g, h \rangle_W = \int_0^T \langle g(t), h(t) \rangle_V dt + \int_0^T \langle \frac{dg}{dt}(t), \frac{dh}{dt}(t) \rangle_{V^*} dt.
\]

Then it can be shown that \( W(0,T) \) is a Hilbert space which embeds continuously into \( C([0,T],H) \).

Assume \( \sigma : V \times V \to C \) satisfies for \( \varphi, \psi \in V \)

\[
\text{Re } \sigma(\varphi, \varphi) \geq c_1 |\varphi|_V^2 - \lambda_0 |\varphi|_H^2 \quad c_1 \geq 0, \lambda_0 \text{ real}, \quad \text{for all } \varphi \in V,
\]

\[
|\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V \quad \text{for all } \varphi, \psi \in V.
\]

Then, as already discussed, we have \( A \in \mathcal{L}(V,V^*) \) such that \( \sigma(\varphi, \psi) = \langle A\varphi, \psi \rangle_{V^*},V = \langle -A\varphi, \psi \rangle_H \) where \( A \) is the densely defined restriction of \(-A\) to the set \( D_A = \{ \varphi \in V | A\varphi \in H \} \). We have moreover, that \( A \) is the infinitesimal generator of an analytic semigroup \( T(t) \) on \( H \). In fact, from Theorem 8 we have that \(-A\) is the generator of an analytic semigroup \( T(t) \) in \( V, H \) and \( V^* \) and \( T(t) \) agrees with \( T(t) \) on \( V \) and \( H \).

We may consider solutions of (25) in the sense of \( V^* \), i.e., in the sense

\[
\langle \dot{x}(t), \psi \rangle_{V^*},V + \sigma(x(t), \psi) = \langle f(t), \psi \rangle_{V^*},V \quad \text{for } \psi \in V,
\]

\[
x(0) = x_0.
\]

By a strong solution of (25) in the \( V^* \) sense (also quite frequently called a weak or variational or distributional solution), we shall mean a function \( x \in L_2((0,T),V) \) such that \( \dot{x} \in L_2((0,T),V^*) \) and (27) (or equivalently \( \dot{x}(t) + Ax(t) = f(t) \)) holds almost everywhere on \( (0,T) \). Similarly, mild solutions \( x_m \in V^* \) are given by the analogue of (26)

\[
x_m(t) = T(t)x_0 + \int_0^t T(t-s)f(s) ds.
\]

We then have the fundamental existence and uniqueness theorem.
Theorem 16  Suppose $x_0 \in H$ and $f \in L_2((0,T),V^*)$. Then (27) has a unique strong solution in the $V^*$ or variational sense and this is given by the mild solution (28).

Proof

Let $\{\phi_i\}_{i=1}^\infty \subset V$ be a linearly independent total subset (i.e., a basis) of $V$. We define the “Galerkin” approximations by $x_k(t) = \sum_{i=1}^k w_i(t)\varphi_i$ where the coefficients $\{w_i\}$ are chosen so that

$$\langle \dot{x}_k(t), \varphi_j \rangle_H + \sigma(x_k(t), \varphi_j) = \langle f(t), \varphi_j \rangle_{V^*,V}$$

for $j = 1, \ldots, k$, satisfying the initial condition

$$x_k(0) = x_{k0}$$

where

$$x_{k0} = \sum_{i=1}^k w_{i0}\varphi_i \to x_0$$

in $H$ as $k \to \infty$. Equivalently, (29) can be written as

$$\sum_{i=1}^k \dot{w}_i(t)\langle \varphi_i, \varphi_j \rangle + \sum_{i=1}^k w_i(t)\sigma(\varphi_i, \varphi_j) = F_j(t)$$

where $F_j(t) = \langle f(t), \varphi_j \rangle_{V^*,V}$ for $j = 1, \ldots, k$. Therefore, $w_1, \ldots, w_k$ are unique solutions to a vector ordinary differential equation system.

Multiplying (29) by $w_j$ and summing over $j = 1, \ldots, k$, we obtain

$$\langle \dot{x}_k(t), x_k(t) \rangle_H + \sigma(x_k(t), x_k(t)) = \langle f(t), x_k(t) \rangle_{V^*,V}$$

with $x_k(0) = x_{k0} \to x_0$ in $H$. Therefore

$$\frac{1}{2} \frac{d}{dt} |x_k(t)|_H^2 + \sigma(x_k(t), x_k(t)) = \langle f(t), x_k(t) \rangle_{V^*,V}. \quad (30)$$

Integrating (30), we obtain

$$\frac{1}{2} |x_k(t)|_H^2 - \frac{1}{2} |x_k(0)|_H^2 + \int_0^t \sigma(x_k(s), x_k(s))ds = \int_0^t \langle f(s), x_k(s) \rangle_{V^*,V}ds.$$ 

Using the fact that $\sigma$ is $V$-elliptic, we have
\[ \frac{1}{2} |x_k(t)|_H^2 + c_1 \int_0^t |x_k(s)|_V^2 \, ds \leq \frac{1}{2} |x_k(0)|_H^2 + \int_0^t |\langle f(s), x_k(s) \rangle_{V^*, V}| \, ds \]
\[ \leq \frac{1}{2} |x_k(0)|_H^2 + \int_0^t \left( \frac{1}{4 \epsilon} |f(s)|_V^2 + \epsilon |x_k(s)|_V^2 \right) \, ds. \]

Therefore,
\[ \frac{1}{2} |x_k(t)|_H^2 + (c_1 - \epsilon) \int_0^t |x_k(s)|_V^2 \, ds \leq \frac{1}{2} |x_k(0)|_H^2 + \int_0^t \frac{1}{4 \epsilon} |f(s)|_V^2 \, ds \quad (31) \]

or
\[ \frac{1}{2} |x_k(t)|_H^2 + (c_1 - \epsilon) \int_0^t |x_k(s)|_V^2 \, ds \leq \frac{1}{2} |x_k(0)|_H^2 + \frac{1}{4 \epsilon} |f|^2_{L^2((0,t), V^*)}. \]

This implies we have \( \{x_k\} \) bounded in \( C((0, T), H) \) and in \( L_2((0, T), V) \). Since \( L_2((0, T), V) \) is a Hilbert space, we can choose \( \{x_{k_n}|x_{k_n} \rightharpoonup \tilde{x} \in L_2((0, T), V)\} \) to be a convergent subsequence of \( x_k \). Without loss of generality, we reindex and denote \( x_{k_n} \) by \( x_k \). Then the limit \( \tilde{x} \) is our candidate for a solution where \( x_k \rightharpoonup \tilde{x} \) in \( L_2((0, T), V) \).

Let \( \chi(t) \in C^1((0, T)) \) with \( \chi(T) = 0 \) and \( \chi(0) = 0 \) and define \( \Psi_j(t, \cdot) \) by \( \Psi_j(t, \cdot) = \chi(t) \varphi_j \). Multiplying (29) by \( \chi(t) \) and integrating, we have
\[ \int_0^T (\langle \dot{x}_k(t), \varphi_j \rangle_H \chi(t) + \sigma(x_k(t), \varphi_j) \chi(t) - \langle f(t), \varphi_j \rangle_{V^*, V} \chi(t)) \, dt = 0. \quad (32) \]

Integrating by parts, we find that (32) becomes
\[ -\int_0^T \langle x_k(t), \varphi_j \rangle \chi(t) \, dt + \int_0^T \sigma(x_k(t), \varphi_j) \chi(t) \, dt - \int_0^T \langle f(t), \varphi_j \rangle_{V^*, V} \chi(t) \, dt = 0. \]

We can now let \( k \to \infty \) and pass the limit through term by term to obtain
\[ \int_0^T -\langle \dot{x}(t), \varphi_j \rangle \chi(t) \, dt + \int_0^T \sigma(\dot{x}(t), \varphi_j) \chi(t) \, dt - \int_0^T \langle f(t), \varphi_j \rangle_{V^*, V} \chi(t) \, dt = 0, \quad (33) \]

holding for all \( \varphi_j \in V \). Recall that \( \{\varphi_j\} \) is a total subset of \( V \) and observe that the \( \chi_s \) such as above are dense in \( L_2(0, T) \). Thus we have (33) holding for all \( \Psi = \varphi \chi \) in \( L_2((0, T), V) \). We can rewrite \( \sigma(\dot{x}(t), \varphi) \chi(t) \) as \( \mathcal{A}\dot{x}(t)\Psi \) and \( \int_0^T \langle f(t), \varphi \rangle_{V^*, V} \chi(t) \, dt = f(\Psi) \).
Therefore, (33) becomes
\[
\langle \frac{d}{dt} \tilde{x}, \Psi \rangle_{V^*, V} + (A\tilde{x} - f)\Psi = 0
\]
where \( \Psi \in L_2((0, T), V) \), or \( \tilde{x} \) satisfies \( \frac{d}{dt} \tilde{x} + A\tilde{x} - f = 0 \) in the \( L_2((0, T), V^*) \) sense. However, we have the following needed theorem (details can be found in [E]).

**Theorem 17** Let \( X \) be a reflexive Banach space. Then
\[
L_p((0, T), X)^* \cong L_q((0, T), X^*)
\]
where \( \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty \).

Returning to our arguments we thus have that the solution to the equation exists in the \( L_2((0, T), V^*) \) sense and is given by \( \tilde{x} \). To obtain \( \tilde{x}(0) = x_0 \), we may use the same arguments with arbitrary \( \chi \in C^1(0, T), \chi(T) = 0, \) but \( \chi(0) \neq 0 \).

To prove uniqueness of the solution, it suffices to argue that the solution corresponding to \( x_0 = 0, f = 0 \) is identically zero. With these specific values for \( f \) and \( x_0 \), (27) can be written as
\[
\langle \dot{x}(t), \varphi \rangle_{V^*, V} + \sigma(x(t), \varphi) = 0, \tag{34}
\]
\[
x(0) = 0.
\]
Let \( \varphi = x(t) \). Then (34) becomes
\[
\frac{1}{2} \frac{d}{dt} |x(t)|_H^2 + \sigma(x(t), x(t)) = 0.
\]
Integrating by parts and using the \( V \)-ellipticity of \( \sigma \), we obtain
\[
\frac{1}{2} |x(t)|_H^2 + \int_0^t c_1 |x(s)|_V^2 ds \leq 0.
\]
Therefore, \( x(t) = 0 \) and thus the solution is unique.

To establish continuous dependence of the solution, define
\[
x(\cdot; x_0, f) : (x_0, f) \in H \times L_2((0, T), V^*) \to x \in L_2((0, T), V) \cap C((0, T), H).
\]
Therefore, \( x \in L_2((0, T), V^*) \). Taking the limits in (31) and using the property that the norms are weakly lower semi-continuous, we obtain the following relation:
\[
\frac{1}{2} |x(t)|_H^2 + (c_1 - \epsilon) \int_0^t |x(s)|_V^2 ds \leq \frac{1}{2} |x_0|^2 + \frac{1}{4\epsilon} \int_0^t |f(s)|_{V^*}^2 ds.
\]
and hence
\[
\sup_{t \in [0,T]} \frac{1}{2} |x(t)|_H^2 + \left( c_1 - \epsilon \right) \int_0^T |x(s)|_{V^*}^2 \, ds \leq \frac{1}{2} |x_0|_H^2 + \frac{1}{4\epsilon} \int_0^T |f(s)|_{V^*}^2 \, ds.
\]

Since the map \((x_0, f) \rightarrow x\) is a linear map on \(H \times L_2((0,T), V^*)\), we have immediately that \(x\) is continuous on \(H \times L_2((0,T), V^*)\) into \(C((0,T), H)\), \(L_2((0,T), V^*)\) and \(L_2((0,T), V)\).

Finally, we need to prove the equivalence between this solution and the mild solution given by (28). From (28) we have that the map \((x_0, f) \rightarrow x(\cdot, x_0, f)\) is continuous from \(H \times L_2((0,T), V^*)\) to \(L_2((0,T), V^*)\). We have just argued that the variational solution \(x_{\text{var}}(\cdot, x_0, f)\) is continuous in the same sense. Thus \(x_{\text{var}}\) and \(x_m\) are both continuous in the above sense. Recall that if two functions agree on a dense subset of some set, then the solutions must agree on the entire set. Therefore, if there is a dense subset of \(H \times L_2((0,T), V^*)\) on which \(x_{\text{var}}\) and \(x_m\) agree, then they will agree on the entire set.

Choose \(x_0 \in D_A\) and \(f \in C^1((0,T), H)\). Then Theorem 13 guarantees that \(x_m\) is the unique solution in the \(H\) sense. However, if \(x_m\) is a strong solution in the \(H\) sense, then it must also be a variational solution (i.e., a strong solution in the \(V^*\) sense). However, the mild solution being unique means \(x_m(\cdot, x_0, f) = x_{\text{var}}(\cdot, x_0, f)\) for \((x_0, f) \in D_A \times C^1((0,T), H)\). But \(D_A \times C^1((0,T), H)\) is dense in \(H \times L_2((0,T), V^*)\). Hence, we have the equivalence between the solutions corresponding to data \((x_0, f)\) in \(H \times L_2((0,T), V^*)\).
16 Weak Formulations for Second Order Systems

16.1 Model Formulation

We return to the general second order systems (18) of Section 14 to illustrate well-posedness ideas in the context of the abstract hyperbolic model with time dependent stiffness and damping given by

\[ \langle \ddot{y}(t), \psi \rangle_{V^*,V} + d(t; \dot{y}(t), \psi) + a(t; y(t), \psi) = \langle f(t), \psi \rangle_{V^*,V} \]

where \( V \subset V_D \subset H \subset V_D^* \subset V^* \) are Hilbert spaces with continuous and dense injections, where \( H \) is identified with its dual and \( \langle \cdot, \cdot \rangle \) denotes the associated duality product. We first show under reasonable assumptions on the time-dependent sesquilinear forms \( a(t; \cdot, \cdot) : V \times V \to \mathbb{C} \) and \( d(t; \cdot, \cdot) : V_D \times V_D \to \mathbb{C} \) that this model possesses a unique solution and that the solution depends continuously on the data of the problem. We also consider well-posedness as well as finite element type approximations in associated inverse problems. The model is a weak formulation that includes models in abstract differential operator form that include plate, beam and shell equations with several important kinds of damping.

Applications for such systems are abundant and range from systems with periodic or structured pattern time dependence that occurs for example in thermally dependent systems orbiting in space (periodic exposure to sunlight) to earth bound structures (bridges, buildings) and aircraft/space structure components with extreme temperature exposures (winter vs. summer). On a longer time scale, such systems are important in time dependent health of elastic structures where long term (slowly varying) time dependence of parameters may be used in detecting aging/fatiguing as represented by changes in stiffness and/or damping. Moreover, applications may be found in modern “smart material structures” [BSW, RCS] where one “controls” damping and/or elasticity via piezoceramic patches, electrically active polymers, etc.

As before let \( V \) and \( H \) be complex Hilbert spaces forming a “Gelfand triple” \( V \subset H = H^* \subset V^* \) with duality product \( \langle \cdot, \cdot \rangle_{V^*,V} \). The injections are assumed to be dense and continuous and the spaces are assumed to be separable. Moreover, we assume that there exists a Hilbert space \( V_D \) (the damping space), such that \( V \subset V_D \subset H = H^* \subset V_D^* \subset V^* \), allowing for a wide class of damping models. Thus the duality products \( \langle \cdot, \cdot \rangle_{V^*,V} \) and \( \langle \cdot, \cdot \rangle_{V_D^*,V_D} \) are the natural extensions by continuity of the inner product \( \langle \cdot, \cdot \rangle \) in \( H \) to \( V^* \times V \) and \( V_D^* \times V_D \), respectively. The \( H \)-norm is denoted by \( | \cdot | \) or \( | \cdot |_H \) for clarity if needed and the \( V \) and \( V_D \)-norms are denoted \( | \cdot |_V \) and
\[ \ddot{y}(t) + D(t)\dot{y}(t) + A(t)y(t) = f(t), \quad \text{in } V^*, \quad t \in (0, T); \tag{35} \]

with \( y(0) = y^0 \in V \) and \( \dot{y}(0) = y^1 \in H \) which is equivalent to

\[ \langle \ddot{y}(t), \psi \rangle_{V^*, V} + \langle D(t)\dot{y}(t), \psi \rangle_{V^*, V^*} + \langle A(t)y(t), \psi \rangle_{V^*, V} = \langle f(t), \psi \rangle_{V^*, V}, \tag{36} \]

for all \( \psi \in V \) and \( t \in (0, T) \), with \( y(0) = y^0 \in V \) and \( \dot{y}(0) = y^1 \in H \).

Following standard terminology we will call a function \( y \in L^2(0, T; V) \), with \( \dot{y} \in L^2(0, T; H) \) and \( \ddot{y} \in L^2(0, T; V^*) \) a weak solution of the initial-value problem (35) if it solves the equation (36) in the strong \( V^* \) sense of the previous section, or, equivalently, solves the equation (45) below, with \( y(0) = y^0 \in V \) and \( \dot{y}(0) = y^1 \in H \) given.

We will assume that the operators \( A \) and \( D \) arise (as defined precisely in (42)-(43) below) from time-dependent sesquilinear forms \( a \) and \( d \) satisfying the following natural ellipticity, coercivity and differentiability conditions.

First, we assume hermitian symmetry, that is

\[ (H1) \quad a(t, \phi, \psi) = a(t, \psi, \phi) \quad \text{for all } \phi, \psi \in V. \]

\[ (H2) \quad |a(t; \phi, \psi)| \leq c_1 |\phi|_V |\psi|_V, \quad \phi, \psi \in V \text{ where } c_1 \text{ is independent of } t. \]

We assume, further, that

\[ (H3) \quad a(t; \phi, \psi) \text{ for } \phi, \psi \in V \text{ fixed is continuously differentiable with respect to } t \text{ for } t \in [0, T] \text{ (} T \text{ finite) and} \]

\[ |a(t; \phi, \psi)| \leq c_2 |\phi|_V |\psi|_V, \quad \text{for all } t \in [0, T], \tag{37} \]

\[ c_2 \text{ once again independent of } t. \]

We also assume that the sesquilinear form \( a(t; \phi, \psi) \) is \( V \)-elliptic, so that

\[ (H4) \quad \text{There exist a constant } \alpha > 0 \text{ such that} \]

\[ |a(t; \phi, \phi)| \geq \alpha |\phi|^2_V \text{ for all } t \in [0, T], \text{ and for all } \phi \in V. \tag{38} \]

For the sesquilinear form \( d \) we assume similarly.
\((H5)\) \(|d(t; \phi, \psi)| \leq c_3|\phi|_{V_D}|\psi|_{V_D}, \ \phi, \psi \in V_D\) where \(c_3\) is independent of \(t\).

We assume, further, that

\((H6)\) \(d(t; \phi, \psi)\) for \(\phi, \psi \in V_D\) fixed is \textit{continuously differentiable} with respect to \(t\) for \(t \in [0, T]\) (\(T\) finite) and

\[
|\dot{d}(t; \phi, \psi)| \leq c_4|\phi|_{V_D}|\psi|_{V_D}, \ \text{for all} \ t \in [0, T],
\]

\((39)\)

\(c_4\) once again independent of \(t\).

Then \(t \to d(t; \phi, \psi)\) and \(t \to a(t; \phi, \psi)\) are \(C^1[0, T]\) for all \(\phi, \psi \in V_D\), and \(\phi, \psi \in V\), respectively, which implies that \(d(t; \phi, \psi)\) and \(a(t; \phi, \psi)\) are sufficiently well-behaved in order to have existence for \((35)\) or \((36)\). We also assume that the sesquilinear form \(d(t; \phi, \psi)\) is \(V_D\)-coercive. That is,

\((H7)\) There exist constants \(\lambda_d\) and \(\alpha_d > 0\), such that

\[
\text{Re} \ d(t; \phi) + \lambda_d|\phi|^2 \geq \alpha_d|\phi|^2_{V_D} \ \text{for all} \ t \in [0, T], \ \text{and for all} \ \phi \in V_D.
\]

\((40)\)

We know then from our earlier considerations that there exist representation operators \(A(t)\) and \(D(t)\)

\[
A(t) : V \to V^*, \ D(t) : V_D \to V_D^*,
\]

\((41)\)

which for each fixed \(t\) are continuous and linear, with

\[
a(t; \phi, \psi) = \langle A(t)\phi, \psi \rangle_{V^*, V}, \ \text{for all} \ \phi, \psi \in V,
\]

\((42)\)

and

\[
d(t; \phi, \psi) = \langle D(t)\phi, \psi \rangle_{V_D^*, V_D}, \ \text{for all} \ \phi, \psi \in V_D.
\]

\((43)\)

We will now consider the following problem: Given finite \(T\) and \(f \in L_2(0, T; V_D^*)\) along with initial conditions

\[
y^0 \in V, \ y^1 \in H,
\]

we wish to find a function \(y \in L_2(0, T; V), \dot{y} \in L_2(0, T; V_D)\) such that in \(V^*\) we have

\[
\begin{aligned}
\dot{y}(t) + D(t)\dot{y}(t) + A(t)y(t) &= f(t), \ t \in (0, T), \\
y(0) &= y^0, \ \dot{y}(0) = y^1.
\end{aligned}
\]

\((44)\)
That is, for $f \in L^2(0,T; V_D^*)$

$$\langle \dot{y}(t), \psi \rangle_{V^*,V} + d(t; \dot{y}(t), \psi) + a(t; y(t), \psi) = \langle f(t), \psi \rangle_{V^*,V} \quad \text{for all } \psi \in V.$$  \hfill (45)

We remark that (45) makes sense since $f(t) \in V_D^* \subset V^*$.

This formulation covers linear beam (i.e., Example 6 discussed earlier), plate and shell models with numerous damping models (Kelvin-Voigt, viscous, square-root, structural and spatial hysteresis) frequently studied in the literature. The formulation above is non-standard in the sense that the damping sesquilinear form is incorporated in the variational model and is time-dependent. The problem without damping ($d = 0$) and $f \in L^2(0,T; H)$ was treated by Lions in [Li], and subsequently in [W]. The less general case without damping and $V = H_0^1(\Omega)$ is treated for example in [Ev]. The model above with $d : V_D \to C$ independent of time is treated in [BSW]. The following theorem is a time-dependent extension of the previous results.

**Theorem 18** Assume that $(f, y^0, y^1) \in L^2(0,T; V_D^*) \times V \times H$. Then there exists a unique solution $y$ to (45) with $(y, \dot{y}) \in L^2(0,T; V) \times L^2(0,T; V_D)$, and the mapping

$$(f, y^0, y^1) \to (y, \dot{y}),$$

is continuous and linear on

$$L^2(0,T; V_D^*) \times V \times H \to L^2(0,T; V) \times L^2(0,T; V_D).$$ \hfill (47)

As we will see, Theorem 18 can then be extended to stronger smoothness results on solutions.

**Theorem 19** Assume that $(f, y^0, y^1) \in L^2(0,T; V_D^*) \times V \times H$. Then there exists (perhaps after modifications on a set of measure zero) a unique solution $y$ to (45) with $(y, \dot{y}) \in C(0,T; V) \times (C(0,T; H) \cap L^2(0,T; V_D))$, and the mapping

$$(f, y^0, y^1) \to (y, \dot{y}),$$

is continuous and linear on

$$L^2(0,T; V_D^*) \times V \times H \to C(0,T; V) \times (C(0,T; H) \cap L^2(0,T; V_D)).$$ \hfill (49)

**Remark:** If we only have that the inequality (40) for the damping sesquilinear form $d$ is satisfied with $\alpha_d = 0$, the results are still true with modifications. Then it will be necessary that $f \in L^2(0,T; H)$ and one obtains only that $\dot{y} \in L^2(0,T; H)$; that is, we have the same results as if
there was no damping. (See, e.g., [Li].) As an added comment, we note that J. L. Lions was one of the early and most prolific contributors to functional analytic formulations of partial differential equations and control theory. His many contributions include the seminal texts [Li, LiMag].

16.2 Discussion of the Model

It is well known from the literature that the strong form of the operator formulation (35) of the problem in general causes computational problems due to irregularities stemming from non-smooth terms - typically in the force/moment terms in, for example, elasticity problems. The weak formulation has proven advantageous both for theoretical and practical purposes, specifically in the effort to estimate parameters or for control purposes [BSW]. To give a particular example illustrating and motivating our discussions here, we consider a version of the beam of Example 6 but with both ends fixed. For an Euler-Bernoulli beam of length \( l \), width \( b \), thickness \( h \) and linear density \( \rho \), where the parameters \( b, h, \rho \) may be functions that depend on time and/or spatial position along the beam, the equation for transverse displacements \( y = y(t, \xi) \) (in strong form [BSW]) is given by

\[
\rho \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial \xi^2} \left( \widetilde{C}_D I \frac{\partial^3 y}{\partial t \partial \xi^2} + \widetilde{E} I \frac{\partial^2 y}{\partial \xi^2} \right) = f \quad 0 < \xi < l,
\]

with fixed end boundary conditions

\[
y(t, 0) = \frac{\partial y}{\partial \xi}(t, 0) = y(t, l) = \frac{\partial y}{\partial \xi}(t, l).
\]

Here we assume Kevin-Voigt structural damping with damping coefficient \( \widetilde{C}_D I = \widetilde{C}_D I(t, \xi) \) and the possibly time and spatially dependent stiffness coefficient given by \( \widetilde{E} I = \widetilde{E} I(t, \xi) \). For simplicity we assume \( \rho \) is constant and scale the system by taking \( \rho = 1 \). One can readily compute that \( \widetilde{E} I = Eh^3b/12, \widetilde{C}_D I = C_D h^3b/12 \), where the Young’s modulus \( E \), the damping coefficient \( C_D \) and the geometric parameters \( h, b \) may in general all be time and/or spatially dependent. As we have seen, in weak form this can be written

\[
\langle \dot{y}(t), \psi \rangle_{V^*, V} + \langle \widetilde{C}_D I \frac{\partial^2 y(t)}{\partial \xi^2} + \widetilde{E} I \frac{\partial^2 y(t)}{\partial \xi^2}, \frac{\partial^2 \psi}{\partial \xi^2} \rangle_H = \langle f(t), \psi \rangle_{V^*, V},
\]

for all \( \psi \in V \), where \( H = L_2(0, l) \) and \( V = V_D = H^2_0(0, l) \) with

\[
H^2_0(0, l) \equiv \{ \psi \in H^2(0, l) | \psi(0) = \psi'(0) = \psi(l) = \psi'(l) = 0 \}.
\]
Here we have adopted the usual notation $\psi' = \frac{\partial \psi}{\partial \xi}$.

This has the form (36) or (45) with

$$a(t; \phi, \psi) = \langle A(t)\phi, \psi \rangle_{V^*, V} = \int_0^l \widetilde{EI}(t, \xi)\phi''(\xi)\psi''(\xi)\,d\xi$$  \hfill (54)

$$d(t; \phi, \psi) = \langle D(t)\phi, \psi \rangle_{V^*, V} = \int_0^l \widetilde{C_D}I(t, \xi)\phi''(\xi)\psi''(\xi)\,d\xi.$$  \hfill (55)

Models such as this can be generalized to higher dimensions (in $\mathbb{R}^n$ for $n = 2, 3$) to treat more general beams, plates, shells, and solid bodies [BSW, RCS]. Of great interest are a number of useful damping models that can be readily used with these equations and treated using the abstract formulation developed here. We discuss briefly some of these damping models without going into much detail, as the purpose of this section is to establish well-posedness and approximation properties of the abstract model. We consider briefly several damping models of interest in practice.

Time-dependent Kelvin-Voigt damping: Let $\omega \subset [0, l]$ with $1_\omega$ denoting the characteristic function of $\omega$. Let $\gamma, \delta > 0$ denote material parameters and let $t \rightarrow k(t)$ denote a sufficiently smooth function. Then a time-dependent damping sesquilinear form is given by

$$d(t; \phi, \psi) = \int_0^l (\gamma + \delta k(t))1_\omega(\xi)\phi''(\xi)\psi''(\xi)\,d\xi,$$  \hfill (56)

for $\phi, \psi \in V_D = V = H_0^2(0, l)$. This gives a model for a mechanical structure damped by a time-varying actuator, localized somewhere inside the structure. This could be piezoceramic actuators or other “smart” devices, with the possibility of them varying in time.

Time-dependent viscous damping: This is a velocity-proportional damping, given (with the notation from above) by the sesquilinear form

$$d(t; \phi, \psi) = \int_0^l k(t, \xi)\phi(\xi)\psi(\xi)\,d\xi,$$  \hfill (57)

with $k \in C^1(0, T; L_\infty(0, l))$ denoting the damping coefficient. One can take $V_D = L_2(0, l)$ here.

Time-dependent spatial hysteresis damping: This model, without time-dependence, is discussed in [DLR], and, as noted in [BSW], it has been
shown to be appropriate for composite material models where graphite fibers are embedded in an epoxy matrix. The time-dependent sesquilinear form that we consider here can now be constructed with the following compact operator $K(t)$ on $L^2(0, l)$:

$$(K(t)\phi)(\xi) = \int_0^t k(t, \xi, \zeta)\phi(\zeta)d\zeta,$$

where the nonnegative integral kernel $k$ belongs to $C^1(0, T; L_\infty(0, l) \times L_\infty(0, l))$.

Letting $d(t; \phi, \psi) = \int_0^t (\nu(\xi) - K(t))\phi'(\xi)\psi(\xi)d\xi$

with $V_D = H^1(0, l)$.

### 16.3 Proof of Theorems 18 and 19

As with the first order systems investigated earlier, we will follow a standard Galerkin approximation method (see for example [BSW, LiMag, W]) with necessary, non-trivial modifications as given in [BaPed] due to the presence of the time-dependent forms. So, let $\{w_j\}_{j=1}^\infty$ denote a basis (i.e., a linearly independent total set) in $V$ that is also a basis in $V$. This is possible since $V$ is dense in $H$. For a fixed $m$ we denote by $V_m$ the finite dimensional subspace spanned by $\{w_j\}_{j=1}^m$, and we let $y_m^0$ and $y_m^1$ be chosen in $V_m$ such that

$$y_m^0 \to y^0 \text{ in } V, \quad y_m^1 \to y^1 \text{ in } H, \text{ for } m \to \infty.$$  

We now define the approximate solution $y_m(t)$ of order $m$ of our problem in the following way:

$$y_m(t) = \sum_{j=1}^m g_{jm}(t)w_j,$$

where the $g_{jm}(t)$ are determined uniquely from the $m$-dimensional linear system:

$$(\ddot{y}_m(t), w_j) + d(t; \dot{y}_m(t), w_j) + a(t; y_m(t), w_j) = \langle f(t), w_j \rangle_{V^*, V}, \quad j = 1, 2, ..., m;$$
with \( y_m(0) = y_m^0 \) and \( \dot{y}_m(0) = y_m^1 \). Multiplying (63) with \( \tilde{g}_{jm}(t) \) and summing over \( j \) yields

\[
(\dot{y}_m(t), \dot{y}_m(t)) + d(t; \dot{y}_m(t), \dot{y}_m(t)) + a(t; y_m(t), \dot{y}_m(t)) = \langle f(t), \dot{y}_m(t) \rangle_{V^*, V}.
\]

(64)

Now, since

\[
\frac{d}{dt} a(t; y_m(t), y_m(t)) = 2 \Re a(t; y_m(t), \dot{y}_m(t)) + \dot{a}(t; y_m(t), y_m(t)),
\]

(65)

we see that

\[
\frac{d}{dt} \{ |y_m(t)|^2 + a(t; y_m(t), y_m(t)) \} + 2 \Re d(t; \dot{y}_m(t), \dot{y}_m(t)) = \dot{a}(t; y_m(t), y_m(t)) + 2 \Re \langle f(t), \dot{y}_m(t) \rangle_{V^*, V}
\]

and by integrating this equality we find

\[
|\dot{y}_m(t)|^2 + a(t; y_m(t), y_m(t)) = \int_0^t 2 \Re d(t; \dot{y}_m(s), \dot{y}_m(s)) ds =
\]

\[
|\dot{y}_m(0)|^2 + a(0; y_m^0, y_m^0) + \int_0^t \dot{a}(s; y_m(s), y_m(s)) ds + \int_0^t 2 \Re \langle f(s), \dot{y}_m(s) \rangle_{V^*, V} ds.
\]

Using the coercivity conditions for \( a \) and \( d \), together with the inequality (recall that \( f(s) \in V_D^* \))

\[
|\langle f(s), \dot{y}_m(s) \rangle_{V^*, V}| \leq \frac{1}{4\epsilon} |f(s)|^2_{V_D^*} + \epsilon |\dot{y}_m(s)|^2_{V_D}
\]

(66)

we obtain, for all \( \epsilon > 0 \)

\[
|\dot{y}_m(t)|^2 + a|y_m(t)|^2_V + \int_0^t 2(\alpha_d - \epsilon)|\dot{y}_m(s)|^2_{V_D} ds \leq |y_m^1|^2 + c_1 |y_m^0|_V^2 + c_2 \int_0^t |y_m(s)|_V^2 ds + 2\lambda_d \int_0^t |\dot{y}_m(s)|_V^2 ds + \int_0^t \frac{1}{2\epsilon} |f(s)|_D^2 ds.
\]

Since \( y_m^0 \to y^0 \) in \( V \), \( y_m^1 \to y^1 \) in \( H \) and \( f \in L_2(0, T; V_D^* \) \), we have that, for \( \epsilon > 0 \) fixed and \( m \) large, there exist a constant \( C > 0 \), such that

\[
|y_m^1|^2 + c_1 |y_m^0|_V^2 + \int_0^t \frac{1}{2\epsilon} |f(s)|_D^2 ds \leq C,
\]

(67)

hence

\[
|\dot{y}_m(t)|^2 + a|y_m(t)|_V^2 + \int_0^t 2(\alpha_d - \epsilon)|\dot{y}_m(s)|^2_{V_D} ds \leq C + c_2 \int_0^t |y_m(s)|_V^2 ds + 2\lambda_d \int_0^t |\dot{y}_m(s)|^2 ds.
\]
Then, in particular
\[
|\dot{y}_m(t)|^2 + \alpha |y_m(t)|_V^2 \leq C + c_2 \int_0^t |y_m(s)|_V^2 \, ds + 2\lambda_d \int_0^t |\dot{y}_m(s)|_V^2 \, ds.
\] (68)

By Gronwall’s inequality we then see that the sequence \(\{\dot{y}_m\}\) is bounded in \(C(0,T;H)\) and that the sequence \(\{y_m\}\) is bounded in \(C(0,T;V)\). From this fact together with the inequality (68) we conclude that \(\{\dot{y}_m\}\) is also bounded in \(L_2(0,T;V_D)\). Then it is possible to extract a subsequence \(\{y_{m_k}\} \subset \{y_m\}\) and functions \(y \in L_2(0,T;V)\) and \(\tilde{y} \in L_2(0,T;V_D)\), such that \(y_{m_k} \rightharpoonup y\), weakly in \(L_2(0,T;V)\) and \(\dot{y}_{m_k} \rightharpoonup \dot{y}\), weakly in \(L_2(0,T;V_D)\). But for \(0 \leq t < T\) we have in \(V\), hence in \(V_D\) and \(H\), that
\[
y_{m_k}(t) = y_{m_k}(0) + \int_0^t \dot{y}_{m_k}(s) \, ds.
\] (69)

But \(y_{m_k}(0) \to y^0\) in \(V\) and hence in \(V_D\), while, for \(t\) fixed, \(\int_0^t \dot{y}_{m_k}(s) \, ds \rightharpoonup \int_0^t \dot{y}(s) \, ds\), weakly in \(V_D\). So, by taking the weak limit in \(V_D\) in (69), we obtain in \(V_D\) the equality
\[
y(t) = y^0 + \int_0^t \tilde{y}(s) \, ds,
\] (70)
from which we conclude that \(\dot{y}(t)\) is in \(V_D\) a.e., with \(\dot{y} = \tilde{y}\) and \(y(0) = y^0\).

We need now to show that \(y\) is actually a solution to the problem (45), with \(\dot{y}(0) = y^1\). To see this, take a function \(\varphi \in C^1([0,T])\), satisfying \(\varphi(T) = 0\), and define, for \(j < m\), the function \(\varphi_j\) by \(\varphi_j(t) = \varphi(t)w_j\), where \(\{w_j\}_1^m\) was the basis spanning \(V_m\). Now, for a fixed \(j < m\), we multiply (63) with \(\overline{\varphi}(t)\) and integrate to obtain
\[
\int_0^T (\langle \ddot{y}_m(s), \varphi_j(s) \rangle + d(s; \dot{y}_m(s), \varphi_j(s)) + a(s; y_m(s), \varphi_j(s))) \, ds =
\int_0^T \langle f(s), \varphi_j(s) \rangle_{V_D^*,V_D} \, ds.
\]

Noticing that, for each \(t\), we have that \(d(t; \cdot, \varphi_j(t)) \in V_D^*\) and \(a(t; \cdot, \varphi_j(t)) \in V^*\), we find, using the weak convergence above that for \(m = m_k \to \infty\) and integration by parts in the first term, that
\[
\int_0^T (-(\dot{y}(s), \varphi_j(s)) + d(s; \dot{y}(s), \varphi_j(s)) + a(s; y(s), \varphi_j(s))) \, ds =
\int_0^T \langle f(s), \varphi_j(s) \rangle_{V_D^*,V_D} \, ds + (y^1, \varphi_j(0)),
\] (71)
for every $j$. Now further restrict $\varphi$ to also satisfy $\varphi \in C^\infty_0(0, T)$ and write (71) as

$$\int_0^T \dot{\varphi}(s)(-\ddot{y}(s), w_j) + \int_0^T \varphi(s)(d(s; \dot{y}(s), w_j) + a(s; y(s), w_j) - (f(s), w_j)_{V^*_D, V_D})ds = 0, \quad (72)$$

for each $j$. But by (72), we have

$$\frac{d}{dt}(\dot{y}(t), w_j) + d(t; \dot{y}(t), w_j) + a(t; y(t), w_j) = (f(t), w_j)_{V^*_D, V_D}, \quad (73)$$

for all $w_j$. By of density of $\bigcup_m^\infty V_m$ in $V$ we conclude that $\ddot{y} \in L^2(0, T; V^*)$ and that for all $\psi \in V$

$$\langle \ddot{y}(t), \psi \rangle + d(t; \dot{y}(t), \psi) + a(t; y(t), \psi) = \langle f(t), \psi \rangle_{V^*_D, V_D}, \quad (74)$$

which was (45). Hence the $y$ we have constructed is indeed a solution to the equation and by (70) we have that $y(0) = y^0$. In order to verify that $\dot{y}(0) = y^1$ we integrate by parts in (71), and by application of (73) we find that, for all $j$:

$$-(\dot{y}(s), \varphi_j(s))|_{s=0}^{s=T} = (y^1, \varphi_j(0)), \quad (75)$$

or, equivalently

$$\langle \ddot{y}(0), w_j \rangle_{\mathbb{P}}(0) = \langle y^1, w_j \rangle_{\mathbb{P}}(0). \quad (76)$$

Hence $\ddot{y}(0) = y^1$.

In order to prove uniqueness, let $y$ be a solution of our problem (45) corresponding to $(y^0, y^1, f) = (0, 0, 0)$, and define for a fixed $t_1 \in (0, T)$ (arbitrarily chosen) the function $\psi$ by

$$\psi(t) = \begin{cases} -\int_t^{t_1} y(s)ds & \text{for } t < t_1, \\ 0 & \text{for } t \geq t_1, \end{cases} \quad (77)$$

so $\psi(T) = 0$. Obviously $\psi(t) \in V$ for all $t$, so we can take $\psi(t) = \psi$ in (45) which yields

$$\langle \ddot{y}(t), \psi(t) \rangle_{V^*, V} + d(t; \dot{y}(t), \psi(t)) + a(t; y(t), \psi(t)) = \langle f(t), \psi(t) \rangle_{V^*, V}. \quad (78)$$

Because $\dot{\psi}(t) = y(t)$ for $t < t_1$ (a.e), we have that

$$\int_0^{t_1} (\langle \ddot{y}(t), \psi(t) \rangle_{V^*, V} + \langle \dot{y}(t), y(t) \rangle_{V^*, V})dt = \int_0^{t_1} \frac{d}{dt}(\langle \dot{y}(t), \psi(t) \rangle_{V^*, V})dt = 0, \quad (79)$$

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due to \( \psi(t_1) = 0 \) and the initial conditions. Using this and by integration of (78) we find

\[
\int_0^{t_1} ((\dot{y}(t), y(t))_{V^*, V} - d(t; \dot{y}(t), \psi(t)) - a(t; y(t), \psi(t)))\,dt = 0; \tag{80}
\]

hence

\[
\int_0^{t_1} \frac{d}{dt}(|y(t)|^2 - a(t; \psi(t), \psi(t)))\,dt =
2 \int_0^{t_1} (\dot{a}(t; \psi(t), \psi(t)) + \text{Re} \, d(t; \dot{y}(t), \psi(t)))\,dt. \tag{81}
\]

Because \( \psi(t_1) = 0 \) and \( y(0) = y_0 = 0 \) this yields

\[
|y(t_1)|^2 + a(0; \psi(0), \psi(0)) = 2 \int_0^{t_1} (\dot{a}(t; \psi(t), \psi(t)) + \text{Re} \, d(t; \dot{y}(t), \psi(t)))\,dt. \tag{82}
\]

From the assumptions on \( a \) and \( \dot{a} \) we arrive at

\[
|y(t_1)|^2 + \alpha |\psi(0)|^2_V \leq 2 \int_0^{t_1} (c_2|\psi(t)|^2_V + \text{Re} \, d(t; \dot{y}(t), \psi(t)))\,dt. \tag{83}
\]

Now notice that

\[
d(t; \dot{y}(t), \psi(t)) = \frac{d}{dt}(d(t; y(t), \psi(t))) - \dot{d}(t; y(t), \psi(t)) - d(t; y(t), y(t)), \tag{84}
\]

so (from the initial conditions)

\[
\int_0^{t_1} d(t; \dot{y}(t), \psi(t))\,dt = \int_0^{t_1} (-\dot{d}(t; y(t), \psi(t)) - d(t; y(t), y(t)))\,dt. \tag{85}
\]

Because

\[
-\text{Re} \, d(t; y(t), y(t)) \leq \lambda_D |y(t)|^2 - \alpha_d |y(t)|^2_{V_D} \tag{86}
\]

we have that

\[
|y(t_1)|^2 + \alpha |\psi(0)|^2_V \leq
2 \int_0^{t_1} (c_2|\psi(t)|^2_V + \lambda_D|y(t)|^2 - \alpha_d |y(t)|^2_{V_D} + \text{Re} \, \dot{d}(t; y(t), \psi(t)))\,dt. \tag{87}
\]

Now we introduce the function \( \omega(t) = \int_0^t y(s)\,ds \) and use

\[
|\psi(t)|^2_V = |\omega(t) - \omega(t_1)|^2_V \leq 2|\omega(t)|^2_V + 2|\omega(t_1)|^2_V \tag{88}
\]

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and from Gronwall’s inequality we see that $y = 0$ in the interval $[0, t_1]$. Since the length of $t_1$ is independent of the choice of origin, we conclude that $y = 0$ on $[t_1, 2t_1]$, etc. Hence $y = 0$ and uniqueness is proved. That the
solution depends continuously on the data is obvious from the inequalities used to show existence; indeed, from (67) and (68) and the weak lower semicontinuity of norms we conclude that the constructed solution satisfies

\[ |y(t)|^2_V + |\dot{y}(t)|^2 + \delta \int_0^t |\dot{y}(t)|^2_{V_D} \leq K (|y^0|^2_V + |y^1|^2 + \int_0^t |f(s)|^2_{V_D} ds) \] (93)

for some positive constants \( \delta \) and \( K \). Integrating from 0 to \( T \) yields the desired result (since \((y^0, y^1, f) \to (y, \dot{y})\) is linear). This completes the proof of Theorem 18.

The proof of Theorem 19 follows from the inequality (92) and the original proof in the case \( d = 0 \) from [LiMag, p. 275–279], because we do not gain any additional regularity from the form \( d \) in this case.
17 Inverse or Parameter Estimation Problems

In the generic abstract parameter estimation problem, we consider a dynamic model of the form (35) where the operators $A$ and $D$ and possibly the input $f$ depend on some unknown (i.e., to be estimated) functional parameters $q$ in an admissible family $Q \subset C^1(0,T;L_\infty(\Omega;Q))$ of parameters. Here $\Omega$ is the underlying set on which the functions of $H,V,V_D$ are defined (e.g., the spatial set $\Omega = (0,l)$ in the heat, transport, and beam examples). We assume that the time dependence of the operators $A$ and $D$ are through the time dependence of the parameters $q(t) \in L_\infty(\Omega;Q)$ where $Q \subset R^p$ is a given constraint set for the values of the parameters. That is, $A(t) = A_1(q(t)), D(t) = A_2(q(t))$ so that we have

$$\ddot{y}(t) + A_2(q(t))\dot{y}(t) + A_1(q(t))y(t) = f(t,q) \quad \text{in } V^*$$
$$y(0) = y^0, \quad \dot{y}(0) = y^1. \quad (94)$$

Thus we introduce the sesquilinear forms $\sigma_1, \sigma_2$ by

$$a(t;\phi,\psi) \equiv \sigma_1(q(t))(\phi,\psi) = \langle A_1(q(t))\phi,\psi \rangle_{V^*,V}, \quad (95)$$
$$d(t;\phi,\psi) \equiv \sigma_2(q(t))(\phi,\psi) = \langle A_2(q(t))\phi,\psi \rangle_{V^*_D,V_D}. \quad (96)$$

It will be convenient in subsequent arguments to use the notation

$$\dot{\sigma}_i(q)(\phi,\psi) \equiv \frac{d}{dt}\sigma_i(q)(\phi,\psi) \quad (97)$$

which in the event that $\sigma_i$ is linear in $q$ becomes $\dot{\sigma}_i(q)(\phi,\psi) = \sigma_i(\dot{q})(\phi,\psi)$. For example, in the Euler Bernoulli example of (52), we would have

$$\sigma_1(q(t))(\phi,\psi) = \int_0^l \widetilde{EI}(t,\xi)\phi''(\xi)\psi''(\xi) d\xi$$
$$\sigma_2(q(t))(\phi,\psi) = \int_0^l \widetilde{C_D}I(t,\xi)\phi''(\xi)\psi''(\xi) d\xi, \quad (98)$$

where $q = (\widetilde{EI},\widetilde{C_D}I) \in C^1(0,T;L_\infty(0,l;R^2_2))$. Note that in this case we do have $\dot{\sigma}_i(q)(\phi,\psi) = \sigma_i(\dot{q})(\phi,\psi)$.

In terms of parameter dependent sesquilinear forms we thus will write (94) as

$$\langle \dot{y}(t),\phi \rangle + \sigma_2(q(t))\langle \dot{y}(t),\phi \rangle + \sigma_1(q(t))\langle y(t),\phi \rangle = \langle f(t,q),\phi \rangle$$
$$y(0) = y^0, \quad \dot{y}(0) = y^1. \quad (100)$$
for all $\phi \in V$. As in (36), $\langle \cdot, \cdot \rangle$ denotes the duality product $\langle \cdot, \cdot \rangle_{V^*, V}$. In some problems the initial data $y^0, y^1$ may also depend on parameters $\tilde{q}$ to be estimated, i.e., $y^0 = y^0(\tilde{q}), y^1 = y^1(\tilde{q})$. We shall not discuss such problems here, although the ideas we present can be used to effectively treat such problems. We instead refer readers to [13] for discussions of general estimation problems where not only the initial data but even the underlying spaces $V$ and $H$ themselves may depend on unknown parameters.

It is assumed that the parameter-dependent sesquilinear forms $\sigma_1(q), \sigma_2(q)$ of (100) satisfy the continuity and ellipticity conditions (H2)-(H7) of Section 1 uniformly in $q \in Q$; that is, the constants $c_1, c_2, \alpha, c_3, c_4, \lambda, \alpha_d$ of (H2)-(H7) can be found independently of $q \in Q$.

In general inverse problems, one must estimate the functional parameters $q$ from dynamic observations of the system (94) or (100). A fundamental consideration in problem formulation involves what will be measured in the dynamic experiments producing the observations. To discuss these measurements in a specific setting, we consider the transverse vibrations (for example, equation (52)) of our beam example where $y(t) = y(t, \cdot)$. Measurements, of course, depend on the sensors available. If one considers a truly smart material structure as in [BSW, RCS], it contains both sensors and actuators which may or may not rely on the same physical device or material. In usual mechanical experiments, there are several popular measurement devices [BSW, BT], some of which could possibly be used in a smart material configuration. If one uses an accelerometer placed at the point $\xi \in (0, \ell)$ along the beam, then one obtains observations $\ddot{y}(t, \xi)$ of beam acceleration. A laser vibrometer will yield data of velocity $\dot{y}(t, \xi)$ while proximity probes including displacement solenoids produce measurements of displacement $y(t, \xi)$. In the case of a beam (or structure) with piezoceramic patches, the patches may be used as sensors as well as actuators. In this case one obtains observations of voltages which are proportional to the accumulated strain; this is discussed fully in [BSW, RCS].

Whatever the measuring devices, the resulting observations can be used in a maximum likelihood or one of several least squares formulations of the parameter estimation problem, depending on assumptions about the statistical model [7, BT] for errors in the observation process. In the least squares formulations, the problems are stated in terms of finding parameters which give the best fit of the parameter-dependent solutions of the partial differential equation to dynamic system response data collected after various excitations, while the maximum likelihood estimator results from maximizing a given (assumed) likelihood function for the parameters, given the data.
In the beam example, the parameters to be estimated include the stiffness coefficient $EI(t, \xi)$, the Kelvin-Voigt damping parameter $c_D(t, \xi)$, and any control related parameters that arise in the actuator input $f$. Details regarding the estimation of these parameters in the time independent case for an experimental beam are given in Section 5.4 of [BSW], while similar experimental results for a plate are summarized in Section 5.5 of the same reference.

The general ordinary least squares parameter estimation problem can be formulated as follows. For a given discrete set of measured observations $z = \{z_i\}_{i=1}^{N_t}$ corresponding to model observations $z_{ob}(t_i)$ at times $t_i$ as obtained in most practical cases, we consider the problem of minimizing over $q \in Q$ the least squares output functional

$$J(q, z) = \| \tilde{C}_2 \left\{ \tilde{C}_1 \{ y(t_i, \cdot; q) \} - \{ z_i \} \right\} \|_2^2,$$

(101)

where $\{ y(t_i, \cdot; q) \}$ are the parameter dependent solutions of (94) or (100) evaluated at each time $t_i$, $i = 1, 2, \ldots, N_t$ and $\| \cdot \|$ is an appropriately chosen Euclidean norm. Here the operators $\tilde{C}_1$ and $\tilde{C}_2$ are observation operators that depend on the type of observed or measured data available. The operator $\tilde{C}_1$ may have several forms depending on the type of sensors being used. When the collected data $z_i$ consists of time domain displacement, velocity, or acceleration values at a point $\bar{\xi}$ on the beam as discussed above, the functional takes the form

$$J_\nu(q, z) = \sum_{i=1}^{N_t} \left| \frac{\partial^\nu w}{\partial t^\nu} (t_i, \bar{\xi}; q) - z_i \right|^2,$$

(102)

for $\nu = 0, 1, 2$, respectively. In this case the operator $\tilde{C}_1$ involves differentiation (either $\nu = 0, 1$ or 2 times, respectively) with respect to time followed by pointwise evaluation in $t$ and $\bar{\xi}$. We shall in our presentation of the next section adopt the ordinary least squares functional (101) to formulate and develop our results.

17.1 Approximation and Convergence

In this section we present a corrected version (Theorem 5.2 of [BSW] contains error in statement and proof) and extension (to treat time dependent coefficients) of arguments for approximation and convergence in inverse problems found in Section 5.2 of [BSW]. For more details on general inverse problem methodology in the context of abstract structural systems, the reader may

The minimization in our general abstract parameter estimation problems for (101) involves an infinite dimensional state space $H$ and an infinite dimensional admissible parameter set $Q$ (of functions). To obtain computationally tractable methods, we thus consider Galerkin type approximations in the context of the variational formulation (100). Let $H^N$ be a sequence of finite dimensional subspaces of $H$, and $Q^M$ be a sequence of finite dimensional sets approximating the parameter set $Q$. We denote by $P_N$ the orthogonal projections of $H$ onto $H^N$.

A family of approximating estimation problems with finite dimensional state spaces and parameter sets can be formulated by seeking $q \in Q^M$ which minimizes

$$J^N(q, z) = \left| \tilde{C}_2 \left\{ \tilde{C}_1 \{ y^N(t_i, \cdot ; q) \} - \{ z_i \} \right\} \right|^2,$$

where $y^N(t; q) \in H^N$ is the solution to the finite dimensional approximation of (100) given by

$$\langle \ddot{y}^N(t), \phi \rangle + \sigma_2(q(t))\dot{y}^N(t), \phi \rangle + \sigma_1(q(t))(y^N(t), \phi) = \langle f(t, q), \phi \rangle$$

$$y^N(0) = P^N y^0, \quad \dot{y}^N(0) = P^N y^1,$$

for $\phi \in H^N$. For the parameter sets $Q$ and $Q^M$, and state spaces $H^N$, we make the following hypotheses.

(A1M) The sets $Q$ and $Q^M$ lie in a metric space $\tilde{Q}$ with metric $d$. It is assumed that $Q$ and $Q^M$ are compact in this metric and there is a mapping $i^M : Q \rightarrow \tilde{Q}^M$ so that $Q^M = i^M(Q)$. Furthermore, for each $q \in Q$, $i^M(q) \rightarrow q$ in $\tilde{Q}$ with the convergence uniform in $q \in Q$.

(A2N) The finite dimensional subspaces $H^N$ satisfy $H^N \subset V$ as well as the approximation properties of the next two statements.

(A3N) For each $\psi \in V$, $|\psi - P^N\psi|_V \rightarrow 0$ as $N \rightarrow \infty$.

(A4N) For each $\psi \in V_D$, $|\psi - P^N\psi|_{V_D} \rightarrow 0$ as $N \rightarrow \infty$.

The reader is referred to Chapter 4 of [BSW] for a complete discussion motivating the spaces $H^N$ and $V_D$.

We also need some regularity with respect to the parameters $q$ in the parameter dependent sesquilinear forms $\sigma_1, \sigma_2$. In addition to (uniform in $Q$)
ellipticity/coercivity and continuity conditions (H2)-(H7), the sesquilinear forms \( \sigma_1 = \sigma_1(q), \sigma_2 = \sigma_2(q) \) and \( \dot{\sigma}_1 = \dot{\sigma}_1(q) \) are assumed to be defined on \( Q \) and satisfy the continuity-with-respect-to-parameter conditions

(H8) \( |\sigma_1(q)(\phi, \psi) - \sigma_1(\tilde{q})(\phi, \psi)| \leq \gamma_1 d(q, \tilde{q})|\phi|_V|\psi|_V, \) for \( \phi, \psi \in V \)

(H9) \( |\dot{\sigma}_1(q)(\phi, \psi) - \dot{\sigma}_1(\tilde{q})(\phi, \psi)| \leq \gamma_3 d(q, \tilde{q})|\phi|_V|\psi|_V, \) for \( \phi, \psi \in V \)

(H10) \( |\sigma_2(q)(\xi, \eta) - \sigma_2(\tilde{q})(\xi, \eta)| \leq \gamma_2 d(q, \tilde{q})|\xi|_{V_D}|\eta|_{V_D}, \) for \( \xi, \eta \in V_D \)

for \( q, \tilde{q} \in Q \) where the constants \( \gamma_1, \gamma_2, \gamma_3 \) depend only on \( Q \).

Solving the approximate estimation problems involving (103),(104), we obtain a sequence of parameter estimates \( \{\bar{q}_{N,M}\} \). It is of paramount importance to establish conditions under which \( \{\bar{q}_{N,M}\} \) (or some subsequence) converges to a solution for the original infinite dimensional estimation problem involving (100),(101). Toward this goal we have the following results.

**Theorem 20** To obtain convergence of at least a subsequence of \( \{\tilde{q}^{N,M}\} \) to a solution \( \tilde{q} \) of minimizing (101) subject to (100), it suffices, under assumption (A1M), to argue that for arbitrary sequences \( \{q^{N,M}\} \) in \( Q^M \) with \( q^{N,M} \to q \) in \( Q \), we have

\[
\tilde{C} \tilde{C} y^N(t; q^{N,M}) \to \tilde{C} \tilde{C} y(t; q).
\]

(105)

**Proof:** Under the assumptions (A1M), let \( \{\tilde{q}^{N,M}\} \) be solutions minimizing (103) subject to the finite dimensional system (104) and let \( q^{N,M} \in Q \) be such that \( i^M(q^{N,M}) = \tilde{q}^{N,M} \). From the compactness of \( Q \), we may select subsequences, again denoted by \( \{\tilde{q}^{N,M}\} \) and \( \{q^{N,M}\} \), so that \( q^{N,M} \to \tilde{q} \in Q \) and \( \tilde{q}^{N,M} \to \tilde{q} \) (the latter follows the last statement of (A1M)). The optimality of \( \{\tilde{q}^{N,M}\} \) guarantees that for every \( q \in Q \)

\[
J^N(\tilde{q}^{N,M}, z) \leq J^N(i^{M}(q), z).
\]

(106)

Using (105), the last statement of (A1M) and taking the limit as \( N, M \to \infty \) in the inequality (106), we obtain \( J(\tilde{q}, z) \leq J(q, z) \) for every \( q \in Q \), or that \( \tilde{q} \) is a solution of the problem for (100),(101). We note that under uniqueness assumptions on the problems (a situation that we hasten to add is not often realized in practice), one can actually guarantee convergence of the entire sequence \( \{\tilde{q}^{N,M}\} \) in place of subsequential convergence to solutions.

We note that the essential aspects in the arguments given above involve compactness assumptions on the sets \( Q^M \) and \( Q \). Such compactness ideas play a fundamental role in other theoretical and computational aspects of
these problems. For example, one can formulate distinct concepts of *problem stability* and *method stability* as in [13] involving some type of continuous dependence of solutions on the observations \( z \), and use conditions similar to those of (105) and (A1M), with compactness again playing a critical role, to guarantee stability. We illustrate with a simple form of *method stability* (other stronger forms are also amenable to this approach–see [13]).

We might say that an *approximation method*, such as that formulated above involving \( Q^M, H^N \) and (103), is *stable* if

\[
dist(\tilde{q}^{N,M}(z^k), \tilde{q}(z^*)) \to 0
\]

as \( N, M, k \to \infty \) for any \( z^k \to z^* \) (in this case in the appropriate Euclidean space), where \( \tilde{q}(z) \) denotes the set of all solutions of the problem for (101) and \( \tilde{q}^{N,M}(z) \) denotes the set of all solutions of the problem for (103). Here “dist” represents the usual distance set function. Under (105) and (A1M), one can use arguments very similar to those sketched above to establish that one has this method stability. If the sets \( Q^M \) are not defined through a mapping \( i^M \) as supposed above, one can still obtain this method stability if one replaces the last statement of (A1M) by the assumptions:

(i) If \( \{q^M\} \) is *any* sequence with \( q^M \in Q^M \), then there exist \( q^* \) in \( Q \) and subsequence \( \{q^{M_k}\} \) with \( q^{M_k} \to q^* \) in the \( \hat{Q} \) topology.

(ii) For *any* \( q \in Q \), there exists a sequence \( \{q^M\} \) with \( q^M \in Q^M \) such that \( q^M \to q \) in \( \hat{Q} \).

Similar ideas may be employed to discuss the question of *problem stability* for the problem of minimizing (101) over \( Q \) (i.e., the original problem) and again compactness of the admissible parameter set plays a critical role.

Compactness of parameter sets also plays an important role in computational considerations. In certain problems, the formulation outlined above (involving \( Q^M = i^M(Q) \)) results in a computational framework wherein the \( Q^M \) and \( Q \) all lie in some uniform set possessing compactness properties. The compactness criteria can then be reduced to uniform constraints on the derivatives of the admissible parameter functions. There are numerical examples (for example, see [BI86]) which demonstrate that imposition of these constraints is necessary (and sufficient) for convergence of the resulting algorithms. (This offers a possible explanation for some of the numerical failures [YY] of such methods reported in the engineering literature.)

Thus we have that compactness of admissible parameter sets play a fundamental role in a number of aspects, both theoretical and computational,
in parameter estimation problems. This compactness may be assumed (and
imposed) explicitly as we have outlined here, or it may be included implicitly
in the problem formulation through Tikhonov regularization as discussed for
example by Kravaris and Seinfeld [KS], Vogel [Vog] and widely by many
others. In the regularization approach one restricts consideration to a sub-
set $Q_1$ of parameters which has compact embedding in $Q$ and modifies the
least-squares criterion to include a term which insures that minimizing se-
quences will be $Q_1$ bounded and hence compact in the original parameter
set $Q$.

After this short digression on general inverse problem concepts, we return
to the condition (105). To demonstrate that this condition can be readily
established in many problems of interest to us here, we give the following
general convergence results.

**Theorem 21** Suppose that $H^N$ satisfies (A2N),(A3N),(A4N) and assume
that the sesquilinear forms $\sigma_1(q), \hat{\sigma}_1(q)$ and $\sigma_2(q)$ satisfy (H8),(H9),(H10),
respectively, as well as (H1)-(H7) of Section 1 (uniformly in $q \in Q$). Fur-
thermore, assume that $q \to f(\cdot; q)$ is continuous from $Q$ to $L_2(0, T; V^*_D)$.

Let $q^N$ be arbitrary in $Q$ such that $q^N \to q$ in $Q$. Then if in addition $\dot{y} \in
L_2(0, T; V)$, we have as $N \to \infty$,

\[
\dot{y}^N(t; q^N) \to \dot{y}(t; q) \quad \text{in } V \text{ norm for each } t > 0
\]

\[
y^N(t; q^N) \to y(t; q) \quad \text{in } L_2(0, T; V_D) \cap C(0, T; H),
\]

where $(y^N, \dot{y}^N)$ are the solutions to (104) and $(y, \dot{y})$ are the solutions to
(100).

**Proof:** From Theorem 19 of Section 1 we find that the solution of (100)
satisfies $y(t) \in V$ for each $t$, $\dot{y}(t) \in V_D$ for almost every $t > 0$. Because

\[
|y^N(t; q^N) - y(t; q)|_V \leq |y^N(t; q^N) - P^N y(t; q)|_V + |P^N y(t; q) - y(t; q)|_V,
\]

and (A3N) implies that second term on the right side converges to 0 as
$N \to \infty$, it suffices for the first convergence statement to show that

\[
|y^N(t; q^N) - P^N y(t; q)|_V \to 0 \quad \text{as } N \to \infty.
\]

Similarly, we note that this same inequality with $y^N, y$ replaced by $\dot{y}^N, \dot{y}$
and the $V$-norm replaced by the $V_D$-norm along with (A4N) permits us to
claim that the convergence

\[
|\dot{y}^N(t; q^N) - P^N \dot{y}(t; q)|_{V_D} \to 0 \quad \text{as } N \to \infty,
\]

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is sufficient to establish the second convergence statement of the theorem. We shall, in fact, establish the convergence of $\dot{y}^N - P^Ny$ in the stronger $V$ norm.

Let $y^N = y^N(t; q^N), \ y = y(t; q)$, and $\Delta^N = \Delta^N(t) \equiv y^N(t; q^N) - P^Ny(t; q)$. Then

$$\dot{\Delta}^N = \dot{y}^N - \frac{d}{dt}P^Ny = \dot{y}^N - P\dot{y}$$

and

$$\ddot{\Delta}^N = \ddot{y}^N - \frac{d^2}{dt^2}P^Ny$$

because $\dot{y} \in L_2((0, T), V_D), \ \ddot{y} \in L_2((0, T), V^*)$. We suppress the dependence on $t$ in the arguments below when no confusion will result. From (100) and (104), we have for $\psi \in H^N$

$$\langle \ddot{\Delta}^N, \psi \rangle_{V^*, V} = \langle \ddot{y}^N - \ddot{y} - \frac{d^2}{dt^2}P^Ny, \psi \rangle_{V^*, V}$$

$$= \langle f(q^N), \psi \rangle_{V^*, V} - \sigma_2(q^N)(\dot{y}^N, \psi) - \sigma_1(q^N)(y^N, \psi)$$

$$- \langle f(q), \psi \rangle_{V^*, V} + \sigma_2(q)(\dot{y}, \psi) + \sigma_1(q)(y, \psi)$$

$$+ \langle \ddot{y} - \frac{d^2}{dt^2}P^Ny, \psi \rangle_{V^*, V}.$$  (108)

This can be written as

$$\langle \ddot{\Delta}^N, \psi \rangle_{V^*, V} + \sigma_1(q^N)(\Delta^N, \psi)$$

$$= \langle \ddot{y} - \frac{d^2}{dt^2}P^Ny, \psi \rangle_{V^*, V} - \langle f(q) - f(q^N), \psi \rangle_{V^*, V}$$

$$+ \sigma_2(q^N)(\dot{y} - P^Ny, \psi) + \sigma_2(q)(\dot{y}, \psi) - \sigma_2(q^N)(\dot{y}, \psi)$$

$$+ \sigma_1(q^N)(y - P^Ny, \psi) + \sigma_1(q)(y, \psi) - \sigma_1(q^N)(y, \psi)$$

$$- \sigma_2(q^N)(\dot{\Delta}^N, \psi).$$

Choosing $\ddot{\Delta}^N$ as the test function $\psi$ in (108) and employing the equality $\langle \ddot{\Delta}^N, \ddot{\Delta}^N \rangle_{V^*, V} = \frac{1}{2} \frac{d}{dt} |\dot{\Delta}^N|_H^2$ (this follows using definitions of the duality mapping - see [? and the hypothesis (A2N)), we have using the symmetry
We observe that \( \langle \ddot{y} - \frac{d^2}{dt^2} P^N y, \dot{\Delta}^N \rangle_{V^*, V} \equiv 0 \) because \( P^N \) is an orthogonal projection. Thus, we find

\[
\frac{1}{2} \frac{d}{dt} \left\{ |\Delta^N|^2_H + \sigma_1(q^N)(\Delta^N, \Delta^N) \right\} = \text{Re} \left\{ \langle \dot{y} - \frac{d^2}{dt^2} P^N y, \dot{\Delta}^N \rangle_{V^*, V} - \langle f(q) - f(q^N), \dot{\Delta}^N \rangle_{V^*_D, V^*_D} + \sigma_2(q^N)(\dot{y} - P^N \dot{y}, \dot{\Delta}^N) + \sigma_2(q)(\dot{y}, \dot{\Delta}^N) - \sigma_2(q^N)(y - P^N y, \dot{\Delta}^N) + \sigma_1(q)(\Delta^N) - \sigma_1(q^N)(\Delta^N, \Delta^N) \right\},
\]

where

\[
T_1^N = \Delta \sigma_1^{(N)}(y, \dot{\Delta}^N) \equiv \sigma_1(q)(y, \Delta^N) - \sigma_1(q^N)(y, \Delta^N),
\]

\[
T_2^N = \Delta \sigma_2^{(N)}(\dot{y}, \dot{\Delta}^N) \equiv \sigma_2(q)(\dot{y}, \dot{\Delta}^N) - \sigma_2(q^N)(\dot{y}, \dot{\Delta}^N),
\]

\[
T_3^N = \langle \Delta f^N, \dot{\Delta}^N \rangle_{V^*_D, V^*_D} \equiv \langle f(q) - f(q^N), \dot{\Delta}^N \rangle_{V^*_D, V^*_D}.
\]

Note that here we have used that \( \dot{y} \in L_2(0, T; V) \). Integrating the terms in \( (110) \) from 0 to \( t \) and using the initial conditions

\[
\Delta^N(0) = y^N(0) - P^N y(0) = y^N(0) - P^N y^0 = 0,
\]

\[
\dot{\Delta}^N(0) = \dot{y}^N(0) - P^N \dot{y}(0) = \dot{y}^N(0) - P^N y^1 = 0,
\]

we obtain (here we do include arguments \( (t) \) and \( (s) \) in our calculations and
estimates to avoid confusion)

\[
\frac{1}{2} |\Delta^N(t)|^2_H + \sigma_1(q^N(t))(\Delta^N(t), \Delta^N(t)) = \int_0^t \left\{ \text{Re} \left[ \sigma_2(q^N(s))(\dot{y}(s) - P^N \dot{y}(s), \dot{\Delta}^N(s)) \right] \\
- \sigma_1(q^N(s))(\dot{y}(s) - P^N \dot{y}(s), \Delta^N(s)) - \dot{\sigma}_1(q^N(s))(\dot{y}(s) - P^N \dot{y}(s), \Delta^N(s)) \\
- \sigma_2(q^N(s))(\Delta^N(s), \Delta^N(s)) + T_1^N(s) \\
+ T_2^N(s) + T_3^N(s) + \dot{\sigma}_1(q^N(s))(\Delta^N(s), \Delta^N(s)) \right\} ds \\
+ \text{Re} \left\{ \sigma_1(q(t))(y(t) - P^N y(t), \Delta^N(t)) \right\}. \quad (112)
\]

We consider the \(T_i^N\) terms in this equation. We have

\[
T_1^N = \frac{d}{dt} \Delta \sigma_1^N(y, \Delta^N) - \Delta \dot{\sigma}_1^N(y, \Delta^N) - \Delta \dot{\sigma}_1^N(y, \Delta^N)
\]

so that

\[
\int_0^t T_1^N(s) ds = \Delta \sigma_1^N(y(t), \Delta^N(t)) \\
- \int_0^t \left\{ \Delta \sigma_1^N(\dot{y}(s), \Delta^N(s)) - \Delta \dot{\sigma}_1^N(y(s), \Delta^N(s)) \right\} ds.
\]

Using (H8),(H9) we thus obtain

\[
\text{Re} \int_0^t T_1^N(s) ds \leq \frac{\gamma_2^2}{4\epsilon} d(q^N, q)^2 |y(t)|^2_V + |\Delta^N(t)|^2_V \\
+ \int_0^t \left\{ \frac{\gamma_1^2}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|^2_V + |\Delta^N(s)|^2_V \\
+ \frac{\gamma_3^2}{4\epsilon} d(q^N, q)^2 |y(s)|^2_V + |\Delta f^N(s)|^2_V \right\} ds. \quad (113)
\]

Similarly, using (H10) we find

\[
\text{Re} \int_0^t (T_2^N(s) + T_3^N(s)) ds \leq \\
\int_0^t \left\{ \frac{\gamma_2^2}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|^2_V + 2\epsilon |\Delta^N(s)|^2_{V_D} + \frac{1}{4\epsilon} |\Delta f^N(s)|^2_V \right\} ds. \quad (114)
\]
Applying Gronwall’s inequality, we find that
\begin{align*}
\frac{1}{2} & |\Delta^N(t)|_H^2 + \alpha|\Delta^N(t)|_V^2 + \int_0^t \alpha_d |\Delta^N(s)|_{V_D}^2 ds \\
& \leq \int_0^t \left\{ \lambda_d |\Delta^N(s)|_H^2 + \frac{c_2}{4q} |\dot{y}(s) - P^N \dot{y}(s)|_V^2 + 3\epsilon |\dot{\Delta}^N(s)|_{V_D}^2 \\
& \quad + \frac{c_4}{4q} |\dot{y}(s) - P^N \dot{y}(s)|_V^2 + \frac{c_2}{2} |y(s) - P^N y(s)|_V^2 + (1 + \epsilon + c_2)|\Delta^N(s)|_V^2 \\
& \quad + \frac{\gamma_1}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|_V^2 + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |y(s)|_V^2 + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|_V^2 \\
& \quad + \frac{1}{4\epsilon} |\Delta f^N(s)|_V^2 \right\} ds \\
& \quad + \frac{c_1}{4\epsilon} |y(t) - P^N y(t)|_V^2 + \int_0^t \left\{ \frac{c_3}{4\epsilon} |\dot{y}(s) - P^N \dot{y}(s)|_V^2 + \frac{c_2}{2} |\dot{y}(s) - P^N \dot{y}(s)|_V^2 \right\} ds \\
& \quad + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |y(t)|_V^2 + \int_0^t \left\{ \frac{\gamma_3}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|_V^2 + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |y(s)|_V^2 + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|_V^2 + \right\} ds.
\end{align*}

This finally reduces to
\begin{align*}
\frac{1}{2} |\Delta^N(t)|_H^2 + (\alpha - 2\epsilon)|\Delta^N(t)|_V^2 + \int_0^t (\alpha_d - 3\epsilon)|\dot{\Delta}^N(s)|_{V_D}^2 ds \\
& \leq \int_0^t \left\{ \lambda_d |\Delta^N(s)|_H^2 + (1 + \epsilon + c_2)|\Delta^N(s)|_V^2 \\
& \quad + \frac{\gamma_1}{4\epsilon} d(q^N, q)^2 |y(t)|_V^2 + \int_0^t \left\{ \frac{c_3}{4\epsilon} |\dot{y}(s) - P^N \dot{y}(s)|_V^2 + \frac{c_2}{2} |\dot{y}(s) - P^N \dot{y}(s)|_V^2 \right\} ds \\
& \quad + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |y(t)|_V^2 + \int_0^t \left\{ \frac{\gamma_3}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|_V^2 + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |y(s)|_V^2 + \frac{\gamma_2}{4\epsilon} d(q^N, q)^2 |\dot{y}(s)|_V^2 + \right\} ds.
\end{align*}

Therefore, under the assumptions we have as \( N \to \infty \) and \( q^N \to q \)
\[
\sup_{t \in (0,T)} [\delta_1^N(t) + \delta_2^N(t) + \delta_3^N(t)] \to 0.
\]

Applying Gronwall’s inequality, we find that
\[
\dot{\Delta}^N \to 0 \text{ in } C(0,T;H) \\
\Delta^N \to 0 \text{ in } C(0,T;V) \\
\dot{\Delta}^N \to 0 \text{ in } L_2(0,T;V_D),
\]
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and hence the convergence statement of the theorem holds.

Remark: The condition \( \dot{y} \in L_2(0,T;V) \) routinely occurs if the structure under investigation has sufficiently strong damping so that \( V_D \) is equivalent to \( V \).

17.2 Some Further Remarks

We have provided sufficient conditions and detailed arguments for existence and uniqueness of solutions to abstract second order non-autonomous hyperbolic systems such as those with time dependent “stiffness”, “damping” and input parameters. In addition, we have considered a class of corresponding inverse problems for these systems and argued convergence results for approximating problems that yield a type of method stability as well as a framework for finite element type computational techniques. The efficacy of such methods have been demonstrated for autonomous systems in earlier efforts [BSW]. The approaches developed here can readily be extended (albeit with considerable tedium) for higher dimensional spatial systems such as plates, shells, etc.
18 “Weak” or “Variational Form”

We consider the origin of the terms “weak or variational form” as opposed to strong or closed form of PDE’s. We use the beam equation to illustrate ideas.

Recall Example 6, the cantilever beam. This example, given in classical form (which can be derived in a straightforward manner using force and moment balance—see [BT]) is

\[
\rho \frac{\partial^2 y}{\partial t^2} + \gamma \frac{\partial y}{\partial t} + \frac{\partial^2}{\partial \xi^2} \left( EI \frac{\partial^2 y}{\partial \xi^2} + c_D I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) = f(t, \xi)
\]

with boundary conditions

\[
\begin{align*}
y(t, 0) &= 0 \\
\frac{\partial y}{\partial \xi}(t, 0) &= 0
\end{align*}
\]

(115)

\[
\left. \left( EI \frac{\partial^2 y}{\partial \xi^2} + c_D I \frac{\partial^3 y}{\partial \xi^2 \partial t} \right) \right|_{\xi=l} = 0
\]

(116)

and initial conditions

\[
\begin{align*}
y(0, \xi) &= \Phi(\xi) \\
\frac{\partial y}{\partial t}(0, \xi) &= \Psi(\xi)
\end{align*}
\]

To facilitate our discussions, we consider an undamped and unforced version (i.e., \(\gamma = c_D I = 0, f = 0\)) of the above system. Rather than force and moment balance, we consider energy formulations for the beam. For a segment of the beam in \([\xi, \xi + \Delta \xi]\), one can argue that the kinetic energy (at a given time \(t\)) is given by

\[
KE(t) = T(t) = \frac{1}{2} \int_{\xi}^{\xi+\Delta\xi} \rho \left( \frac{\partial y}{\partial t}(t, \xi) \right)^2 d\xi,
\]

and hence the kinetic energy of the entire beam is given by

\[
T(t) = \frac{1}{2} \int_0^l \rho y^2(t, \xi) d\xi.
\]

Similarly, the potential (or strain) energy \(U\) of the beam at any given time \(t\) is given by

\[
PE(t) = U(t) = \frac{1}{2} \int_0^l EI \left( \frac{\partial^2 y}{\partial \xi^2} \right)^2 d\xi.
\]
A fundamental tenant of the mechanics of rigid or elastic bodies is Hamilton’s “Principle of Stationary Action” (often, in a misnomer, referred to as Hamilton’s principle of “least action”) which postulates that any system undergoing motion during a period \([t_0, t_1]\) will exhibit motion \(y(t, \xi)\) that provides the “least action” for the system with a stationary value. The “Action” is defined by

\[
A = \int_{t_0}^{t_1} (KE - PE)dt
\]

\[
= \int_{t_0}^{t_1} [T(t) - U(t)]dt.
\]

For the beam of Example 6, this means that the motion or vibrations \(y(t, \xi)\) must provide a stationary value to the action

\[
A[y] = \int_{t_0}^{t_1} \int_0^l \left[\frac{1}{2} \rho \dot{y}^2 - \frac{1}{2} EI \left(\frac{\dot{y}}{\xi}\right)^2\right] d\xi dt
\]

integrated over any time interval \([t_0, t_1]\). Through the calculus of variations (a field of mathematics that was the precursor to modern control theory), this leads to an equation of motion for the vibrations \(y\) that the beam motion must satisfy.

To further explore this, we consider \(y(t, \xi)\) as the motion of the beam and consider a family of variations \(y(t, \xi) + \epsilon \eta(t, \xi)\) where \(\eta\) is chosen so that \(y + \epsilon \eta\) is an “admissible variation”, i.e., \(y + \epsilon \eta\) must satisfy the essential boundary conditions (115).

We define \(V = H^2_L(0, l) = \{\varphi \in H^2(0, l) | \varphi(0) = \varphi'(0) = 0\}\). Let \(\psi \in C^2(t_0, t_1)\) with \(\psi(t_0) = \psi(t_1) = 0\). Then \(\eta \in \mathcal{N} \equiv \{\eta \in H^2(0, l) | \eta(t_0, \xi) = \eta(t_1, \xi) = 0\}\) satisfies \(\eta\) is \(C^2\) in \(t\), \(H^2\) in \(\xi\) with \(\eta(t_0, \xi) = \eta(t_1, \xi) = 0\) and \(\eta(t, 0) = \eta(t, l) = 0\). Then by Hamilton’s principle, we must have that \(A[y + \epsilon \eta]\) for \(\epsilon > 0, \eta \in \mathcal{N}\), must have a stationary value at \(\epsilon = 0\). That is,

\[
\frac{d}{d\epsilon} A[y + \epsilon \eta]|_{\epsilon = 0} = 0.
\]  

(117)

Since

\[
A[y + \epsilon \eta] = \int_{t_0}^{t_1} \int_0^l \left[\frac{1}{2} \rho (\dot{y} + c \dot{\eta})^2 - \frac{1}{2} EI (y'' + c \eta'')^2\right] d\xi dt,
\]

we find from (117) the weak (in time \(t\) and space \(\xi\)) form of the equation given by

\[
0 = \int_{t_0}^{t_1} \int_0^l [\rho \ddot{y} - EI y'''] d\xi dt
\]  

(118)

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for all $\eta \in \mathcal{N}$.

To explore further, we formally integrate by parts in the first term (with respect to $t$) to obtain

$$
\int_{t_0}^{t_1} \int_0^t \rho \dddot{y} \eta \, dt = - \int_{t_0}^{t_1} \int_0^t \rho \dddot{y} \eta \, dt + \int_0^t \rho \dddot{y} \eta \big|_{t=t_0}^t
$$

since $\eta(t_0, \xi) = \eta(t_1, \xi) = 0$. Since $\eta$ has the form $\eta = \psi \varphi$, equation (118) has the form

$$
\int_{t_0}^{t_1} \int_0^t [\rho \dddot{y} + EI \dddot{y}'] \psi \, dt = 0
$$

for all $\psi \in C^2[t_0, t_1]$ with $\psi(t_0) = \psi(t_1) = 0$, and all $\varphi \in V$. Since this holds for arbitrary $\psi$, we must have in the $L_2(t_0, t_1)$ sense

$$
\int_0^t [\rho \dddot{y} + EI \dddot{y}'] \psi \, dt = 0 \quad \text{for all} \quad \varphi \in V.
$$

In our former notation of Gelfand triples with $V = H^2_0(0, l)$ and $H = L_2(0, l)$, this may be written

$$
\langle \rho \dddot{y}, \varphi \rangle_{V^*, V} + \langle EI \dddot{y}', \varphi' \rangle_H = 0 \quad \text{for all} \quad \varphi \in V
$$

in the $L_2(t_0, t_1)$ sense, which is exactly the “weak” or “variational” form of the beam equation we have encountered previously. Note that in fact the true variational form was given in (118); that is,

$$
\int_{t_0}^{t_1} [-\langle \rho \dddot{y}, \varphi \rangle \psi + \langle EI \dddot{y}', \varphi' \rangle \psi] \, dt = 0
$$

for all $\varphi \in V$ and $\psi \in C^2[t_0, t_1]$ with $\psi(t_0) = \psi(t_1) = 0$. (See the proofs and our remarks concerning solutions in the $L_2(t_0, t_1; V^*) \cong L_2(t_0, t_1; V^*)$ sense in the well-posedness (existence) results for second order systems discussed earlier in Section 16.

We note that if the variational solution $y$ has additional smoothness so that $y \in V \cap H^4(0, l)$ (more precisely $EIy'' \in H^2(0, l)$), then we can integrate by parts twice (with respect to $\xi$) in the second term of (119) to obtain in place of (119):

$$
\int_{t_0}^{t_1} \int_0^t [\rho \dddot{y} + EI \dddot{y}'] \psi \, d\xi \, dt + \int_{t_0}^{t_1} -(EIy'')' \varphi' \big|_{\xi=0}^\xi \psi \, dt
$$

$$
+ \int_{t_0}^{t_1} (EIy'')' \varphi' \big|_{\xi=0}^\xi \psi \, dt = 0
$$
for $\varphi \in V, \psi \in C^2[t_0, t_1]$ with $\psi(t_0) = \psi(t_1) = 0$. This can be written

$$\int_{t_0}^{t_1} \left[ \int_0^l [\rho \ddot{y} \varphi + (EIy'')' \varphi] d\xi + -EIy'' \varphi|_{\xi=l} + (EIy'')' \varphi|_{\xi=l} \right] \psi dt = 0$$

for arbitrary $\varphi \in V$. We note once again that this results in the strong or classical form of the equations

$$\rho \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial \xi^2} (EI \frac{\partial^2 y}{\partial \xi^2}) = 0$$

with the essential boundary conditions

$$y(t, 0) = y'(t, 0) = 0$$

as well as the natural boundary conditions

$$EI \frac{\partial^2 y}{\partial \xi^2}(t, l) = \frac{\partial}{\partial \xi} (EI \frac{\partial^2 y}{\partial \xi^2})(t, l) = 0$$

holding.

We remark on an additional perspective of the weak or variational or energy formulation of dynamical systems such as the beam equation. The weak or variational form may be thought of as Euler’s equations in the calculus of variations. If

$$J(y) = \int F(t, y, \dot{y}) dt = A[y],$$

then the condition of stationarity

$$\frac{d}{d\epsilon} J(y + \epsilon \eta)|_{\epsilon=0} = 0$$

implies

$$\int (F_y \eta + F_{\dot{y}} \dot{\eta}) dt = 0$$

which is the “true” Euler’s equation. We may then invoke the duBois Raymond lemma below to obtain

$$\frac{\partial F}{\partial y} = \frac{d}{dt} \frac{\partial F}{\partial \dot{y}}$$

in a distributional sense. Specifically, we have
Lemma 1 (duBois Raymond’s Lemma)

If

\[ \int (G_1 \dot{\eta} + G_2 \ddot{\eta}) = 0 \]

for all \( \eta \), then

\[ \frac{d}{dt} G_2 = G_1 \]

in a weak sense. In other words, \( G_1 \) is the distributional derivative of \( G_2 \).

If we assume enough smoothness and integrate by parts, we obtain the strong form of Euler’s equation:

\[ - \frac{d}{dt} F_y(t, y, \dot{y}) + F_y(t, y, \ddot{y}) = 0. \]

In the derivation above, we considered the undamped and unforced version of the equation. In the case of the forced beam, we can add a conservative force term \( W = f y \) in our derivation, and we will obtain the desired \( \langle f, \varphi \rangle \) term in our result. However, there is no known way to derive the weak form with damping. In other words, Hamilton’s principle is essentially valid for conservative forces, but it doesn’t conveniently handle nonconservative (dissipative) forces (damping).
19 Finite Element Approximations

We will now consider finite element approximations or Galerkin approximations for general first order systems for which parabolic systems are special cases. Consider

\[
\begin{aligned}
\dot{x}(t) &= Ax + F \quad \text{in } V^* \ (H \text{ if possible}) \\
x(0) &= x_0
\end{aligned}
\]  

(120)

where \( V \hookrightarrow H \hookrightarrow V^* \) is the usual Gelfand triple. We can write the above system in the weak or variational form (i.e., in \( V^* \)) as

\[
\begin{aligned}
\langle \dot{x}(t), \varphi \rangle_{V^*,V} + \sigma(x(t), \varphi) &= \langle F(t), \varphi \rangle_{V^*,V} \\
x(0) &= x_0
\end{aligned}
\]

for \( \varphi \in V \). If \( \sigma \) is \( V \) continuous and \( V \)-elliptic, and \( S(t) \sim e^{At} \) (i.e., \( A \) is the infinitesimal generator of a \( C_0 \) semigroup \( S(t) \)), we can write

\[
x(t) = S(t)x_0 + \int_0^t S(t - \xi)F(\xi)d\xi
\]

(121)

where \( x \in L_2(0,T;V) \bigcap C(0,T;H) \) and \( \dot{x} \in L_2(0,T;V^*) \). We can use this formulation to give a nice treatment of finite element approximations of Galerkin type.

In general, this is an infinite dimensional space; therefore, we want to project the system into a finite dimensional space in which we can compute. Let \( H^N = \text{span}\{B^N_1, B^N_2, ..., B^N_N\} \subset V \) be the approximation of \( H \). Typical choices for the \( B^N_j \) are piecewise linear or piecewise cubic splines (e.g., see \cite{[13]}). The idea is to replace (120) by

\[
\begin{aligned}
\dot{x}^N(t) &= A^N x^N(t) + F^N(t) \quad \text{in } H^N \\
x^N(0) &= x^N_0,
\end{aligned}
\]

or equivalently, replace (121) by

\[
x^N(t) = S^N(t)x^N_0 + \int_0^t S^N(t - \xi)F^N(\xi)d\xi
\]

where \( S^N(t) \sim e^{A^Nt} \).

One of the key constructs we need is \( P^N : H \rightarrow H^N \) which is called the \textit{orthogonal projection} of \( H \) onto \( H^N \). In other words, \( P^N \) is defined by

\[
\langle P^N \varphi - \varphi, \psi \rangle = 0 \quad \text{for all } \psi \in H^N
\]
We would like $F^N \to F$ and $x_0^N \to x_0$, so we take $x_0^N = P^N x_0$ and $F^N(t) = P^N F(t)$. We also want $A^N \in \mathcal{L}(H^N)$ and $A^N \approx A$. However, we have defined $S^N(t) = e^{A^N t}$ and $S(t) = e^{At}$; therefore, if we had $S^N(t) \to S(t)$, then we would be able to argue $x^N(t) \to x(t)$. This is considered in the Trotter-Kato theorem which will be discussed later.

Now to relate this to the computational aspects of finite elements, we first restrict the equations

$$(\dot{x}(t), \varphi) + \sigma_1(x(t), \varphi) = \langle F(t), \varphi \rangle \quad \text{for all } \varphi \in V \quad (122)$$

to $H^N \times H^N$. In other words, let

$$x^N(t) = \sum_{j=1}^N w_j^N(t)B_j^N$$

be a trial solution with

$$x^N(0) = \sum_{j=1}^N w_0^N_j B_j^N.$$ 

Substituting this into (122), we have

$$\langle \sum_{j=1}^N \dot{w}_j^N(t)B_j^N, \varphi \rangle + \sigma_1(\sum_{j=1}^N w_j^N(t)B_j^N, \varphi) = \langle F(t), \varphi \rangle \quad (123)$$

for $\varphi \in H^N$. Successively choose $\varphi = B_1^N, B_2^N, \ldots, B_N^N$ in (123). From this we obtain an $N \times N$ vector system for $w^N(t) = (w_1^N(y), \ldots, w_N^N(t))^T$ given by

$$\sum_{j=1}^N \dot{w}_j^N(t)(B_j^N, B_i^N) + \sum_{j=1}^N w_j^N(t)\sigma(B_j^N, B_i^N) = \langle F(t), B_i^N \rangle \quad (124)$$

for $i = 1, 2, \ldots, N$.

Using standard engineering and applied mathematics terminology, we define the mass matrix $M^N = ((B_j^N, B_i^N))$, the stiffness matrix $K^N = (\sigma(B_j^N, B_i^N))$, and the column vector $F^N(t) = (\langle F(t), B_i^N \rangle)$. Then (124) can be written

$$\begin{cases} 
M^N \dot{w}^N(t) + K^N w^N(t) = F^N(t) \\
w^N(0) = w_0^N 
\end{cases} \quad (125)$$
or
\[
\begin{align*}
\dot{w}^N(t) &= -(M^N)^{-1}K^N w^N(t) + (M^N)^{-1} F^N(t) \\
 w^N(0) &= w_0^N.
\end{align*}
\]

Considering \( w_0^N \), we have \( x^N(0) = P^N x_0 \) which implies \( \langle P^N x_0 - x_0, B_i^N \rangle = 0 \) for \( i = 1, \ldots, N \). However, \( x_0^N = \sum_{j=1}^N w_{0j}^N B_j^N \). Therefore,
\[
\sum_{j=1}^N w_{0j}^N B_j^N - x_0, B_i^N \rangle = 0
\]
for \( i = 1, \ldots, N \) which gives
\[
\sum_{j=1}^N w_{0j}^N \langle B_j^N, B_i^N \rangle = \langle x_0, B_i^N \rangle.
\]

Defining \( w_0^N = \text{col}(w_{01}^N, \ldots, w_{0N}^N) \), we have
\[
w_0^N = (M^N)^{-1} \text{col}(\langle x_0, B_i^N \rangle).
\]

From this, our system for \( w \) becomes
\[
\begin{align*}
\dot{w}^N(t) &= -(M^N)^{-1}K^N w^N(t) + (M^N)^{-1} F^N(t) \\
w_0^N &= (M^N)^{-1} \text{col}(\langle x_0, B_i^N \rangle).
\end{align*}
\]

However, we normally do not solve the system in this form. If \( \langle B_i, B_j \rangle = 0 \) for \( i \neq j \), then \( M^N \) is diagonal and the system of the form (125) is an easier system with which to work. More generally, the (finite element) system is solved in the form
\[
\begin{align*}
M^N \dot{w}^N(t) &= -K^N w^N(t) + F^N(t) \\
M^N w_0^N &= \text{col}(\langle x_0, B_i^N \rangle),
\end{align*}
\]
rather than inverting the matrix \( M^N \) which is, if not diagonal, usually a banded matrix (tri-banded for piecewise linear splines, seven banded for cubic splines, etc, –see [13]), lending itself to fast algebraic solvers (e.g., those based on LU decompositions).
20 Trotter-Kato Approximation Theorem

The Trotter-Kato Approximation Theorem is the functional analysis version of the Lax Equivalence Principle used in finite difference approximation for PDE’s which dates back to the 1960’s. The fundamental ideas underlying the Lax Equivalence Principle is that “consistency” and “stability” are achieved if and only if we have “convergence” of our system. If we have a PDE

\[ u_{tt} = Au \]

and an approximation

\[ u_{tt}^N = A^N u^N \]

then consistency refers to \( A^N \to A \) in some sense. Stability refers to \( |e^{A^N t}| \leq Me^{\omega t} \), and convergence means \( e^{A^N t} \to e^{At} \) in some sense. For relevant and much more detailed discussions, see [RM].

There are two different versions of the Trotter-Kato theorem which we will discuss here. We will first consider the operator convergence form of the Trotter-Kato theorem.

**Theorem 22** Let \( X \) and \( X^N \) be Hilbert spaces such that \( X^N \subset X \). Let \( P^N : X \to X^N \) be an orthogonal projection of \( X \) onto \( X^N \). Assume \( P^N x \to x \) for all \( x \in X \). Let \( A^N, A \) be infinitesimal generators of \( C_0 \) semigroups \( S^N(t), S(t) \) on \( X^N, X \) respectively satisfying

(i) there exists \( M, \omega \) such that \( |S^N(t)| \leq Me^{\omega t} \) for each \( N \)

(ii) there exists \( D \) dense in \( X \) such that for some \( \lambda, (\lambda I - A)D \) is dense in \( X \) and \( A^N P^N x \to Ax \) for all \( x \in D \).

Then for each \( x \in X \), \( S^N(t)P^N x \to S(t)x \) uniformly in \( t \) on compact intervals \([0, T]\).

The arguments for this result are given in [Pa, Chapter 3, Theorem 4.5]. We next will examine the resolvent convergence version of the Trotter-Kato theorem. This form is a modification of the previous version.

**Theorem 23** Replace (ii) in the above theorem by (\( \tilde{\text{ii}} \)) defined as

(\( \tilde{\text{ii}} \)) There exists \( \lambda \in \rho(A) \) \( \bigcap_{N=1}^{\infty} \rho(A^N) \) with \( \text{Re}(\lambda) > \omega \) so that \( R_\lambda(A^N)P^N x \to R_\lambda(A)x \) for each \( x \in X \).
Then under this and the remaining conditions in the above theorem, the conclusions also hold.

For proofs, see [Pa, Chap. 3, Theorems 4.2,4.3,4.4]. See also [13, Theorem 1.14]. Convergence rates are given in [13, Theorem 1.16].

For certain problems, it is not necessary for \( H^N \subset \text{dom}(A) \) which carries both the essential and natural boundary conditions. We may need to only choose an appropriate approximation \( H^N \) such that \( H^N \subset V \) which carries just the essential boundary conditions. If we restrict ourselves to first order systems in the context of the Gelfand triple \( V \hookrightarrow H \hookrightarrow V^* \) with \( H^N \) approximating \( V \) as \( N \to \infty \) then we have a special case of the Trotter-Kato theorem.

Let the condition (C1) be denoted by \((C1)\) For each \( z \in V \), there exists \( \hat{z}^N \in H^N \) such that \( |z - \hat{z}^N|_V \to 0 \) as \( N \to \infty \).

Suppose \( \sigma \) is \( V \)-elliptic, i.e., \( \text{Re}\sigma(\varphi, \varphi) \geq \delta |\varphi|_V^2 \) for \( \delta > 0 \). Also assume \( \sigma \) is \( V \) continuous, i.e., \( |\sigma(\varphi, \psi)| \leq \gamma |\varphi|_V |\psi|_V \). Let \( P^N : H \to H^N \) be an orthogonal projection. Then

\[
|P^N z - z| = \inf \{|z^N - z|_H \mid z^N \in H^N\}.
\]

Under (C1), we have \( |P^N z - z|_H \leq |\hat{z}^N - z|_H \leq |\hat{z}^N - z|_V \to 0 \) as \( N \to \infty \). Therefore, under (C1), \( P^N z \to z \) for \( z \in H \).

Next we define \( A^N \). We have \( \sigma(\varphi, \psi) = \langle -A\varphi, \psi \rangle_H \) for \( \varphi \in \text{dom}(A) = \{ \psi \in V \mid A\psi \in H \} \), \( \psi \in V \) where \( A \) is an infinitesimal generator of a \( C_0 \) semigroup of contractions on \( H \), i.e., \( |e^{At}| \leq 1 \). We define \( A^N \) through the restriction of \( \sigma \) to \( H^N \times H^N \). Therefore, \( A^N : H^N \to H^N \) is defined by

\[
\sigma(\varphi^N, \psi^N) = \langle -A^N\varphi^N, \psi^N \rangle_H \quad \sigma(\varphi^N, \psi^N) = \langle -A^N\varphi^N, \psi^N \rangle_{V^*,V}.
\]

By \( V \)-ellipticity, we obtain \( A^N \) is an infinitesimal generator of a contraction semigroup on \( H^N \), i.e., \( |e^{A^Nt}| \leq 1 \).

**Theorem 24** If \( \sigma \) is \( V \)-elliptic, \( V \) continuous and (C1) holds, then

\[
R_\lambda(A^N)P^N z \to R_\lambda(A)z
\]

in the \( V \) norm for \( z \in H \) and \( \lambda = 0 \).

See [BI] for additional discussion.
Proof

Let \( z \in H \) and take \( \lambda = 0 \). Now define \( w^N = R_\lambda(A^N)P^Nz \) and \( w = R_\lambda(A)z \) where \( \sigma(\varphi, \psi) = \langle -A\varphi, \psi \rangle_H, \varphi \in \text{dom}(A), \psi \in V \). By definition, we have \( w \in \text{dom}(A) \).

By (C1), there exists \( \hat{w}^N \in H^N \) such that \( |\hat{w}^N - w|_V \to 0 \) as \( N \to \infty \).

Let \( z^N = w^N - \hat{w}^N \). We need to show \( z^N \to 0 \) in \( V \).

Since \( R_\lambda(A) = (\lambda I - A)^{-1} \), we have

\[
\sigma(w, z^N) = \langle -AR_\lambda(A)z, z^N \rangle_H = \langle z, z^N \rangle
\]

while

\[
\sigma(w^N, z^N) = \langle -A^N R_\lambda(A^N)z, z^N \rangle_H = \langle z, z^N \rangle,
\]

and hence these are equal. Thus

\[
\delta|z^N|^2_V \leq \sigma(z^N, z^N)
\]

\[
= \sigma(w^N, z^N) - \sigma(\hat{w}^N, z^N)
\]

\[
= \sigma(w, z^N) - \sigma(\hat{w}^N, z^N)
\]

\[
= \sigma(w - \hat{w}^N, z^N)
\]

\[
\leq \gamma|w - \hat{w}^N|_V|z^N|_V.
\]

Therefore, we have \( \delta|z^N|_V \leq \gamma|w - \hat{w}^N|_V \). However, \( |\hat{w}^N - w|_V \to 0 \) implies \( |z^N|_V \to 0 \) and thus \( |w - \hat{w}^N|_V \leq |w - \hat{w}^N|_V + |\hat{w}^N - w^N|_V \to 0 \).

Remark: Theorem 2.2 of [BI] is a parameter dependent version of this, i.e., \( A = A(q), A^N = A^N(q^N), q, q^N \in Q \), which can be used in inverse problems.

Theorem 25 Suppose \( \sigma \) is \( V \)-elliptic, \( V \) continuous, and (C1) holds. Then \( S^N(t)P^Nz \to S(t)z \) in the \( V \) norm for each \( z \in H \) uniformly in \( t \) on compact intervals.

Proof

The arguments involve using Tanabe type estimates after considering an appropriate setting and using the Trotter-Kato, first in \( H \) and then in \( V \).

Let \( X = H, X^N = H^N, P^N : H \to H^N \) be an orthogonal projection. Then \( P^Nz \to z \) for all \( z \in H \) by (C1). Convergence in \( H \) norm follows immediately from application of the Trotter-Kato in \( H \). To obtain \( V \) convergence is somewhat more work and more delicate; for these details we refer the reader to [BI, Theorem 2.3].
References


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[50] I. Lasiecka, D. Lukes and L. Pandolfi, Input dynamics and nonstandard Riccati equations with applications to boundary control of damped


