Consider the chemostat model developed in class:

\[ \frac{dN}{dt} = r(c)N - qN \]  
\[ \frac{dc}{dt} = q(c_0 - c) - \frac{1}{y} r(c)N, \]

where \( N(0) = N_0, c(0) = 0 \). Moreover,

\[ r(c) = \frac{R_{\text{max}} c}{K_m + c}, \quad q = \frac{Q}{V}. \]

\[ \theta = (V, Q, R_{\text{max}}, K_m, c_0, y)^T, \]

(Note here that the symbol “\( y \)” represents a parameter and not a realization of an observation.) Typical values for the parameters in this model are

\[ V = 20 \text{ l}, \quad c_0 = 2.5 \text{ g/l}, \quad K_m = 12.3 \mu g/ml, \quad R_{\text{max}} = 0.85/\text{hr}, \quad y = 10.6 \text{ l/g} \]

Take \( N_0 = 0.025 \text{ g} \).

1. Simulate and plot graphs of the forward solution to the system (1) using the parameter values and initial conditions given above for values of \( Q \) ranging from 14 to 16 l/hr.

2. Now take several of your simulation results from part 1. to use as data for observations at times \( t_j, j = 1, \ldots, n \), and, for each, use a least squares inverse problem formulation to estimate:
   (a) \( R_{\text{max}} \), holding all other model parameters fixed at the given values.
   (b) \( K_m \), holding all other model parameters fixed at the given values.
   (c) \( R_{\text{max}} \) and \( K_m \) (simultaneously), holding all other model parameters fixed at the given values.

Each of (a)–(c) should be done using (i) observations of \( c(t) \) only, (ii) observations of \( N(t) \) only, and (iii) observations of both \( c(t) \) and \( N(t) \). Carry out each combination of (a)–(c) and (i)–(iii) for several different sets of time points \( (t_1, \ldots, t_n) \) and discuss how the solution depends on the choice of \( n \), the values of \( t_j, j = 1, \ldots, n \), and the availability of observations of \( c(t), N(t) \), or both.

3. Take a typical forward simulation result from part 1. and add random noise as follows.
   Let \( f(t_j, \theta) = \{ f^{(1)}(t_j, \theta), f^{(2)}(t_j, \theta) \}^T \) be the value of the forward solution from part 1 [observations of \( c(t) \) and \( N(t) \) at \( t_j \) at a particular value of \( \theta \)] and obtain realizations of “data” \( (y_j, t_j), j = 1, \ldots, n \), according to the statistical model

\[ Y_j = f(t_j, \theta) + \sigma \begin{pmatrix} f^{(1)}(t_j, \theta) & 0 \\ 0 & f^{(2)}(t_j, \theta) \end{pmatrix} \epsilon_j. \]
Create four data sets by taking $\sigma = 0.05$ and $\sigma = 0.10$ with $\epsilon_j = (\epsilon_j^{(1)}, \epsilon_j^{(2)})^T$ generated independently for each $j$ according to:

(a) $\epsilon_j^{(k)}$ independent for $k = 1, 2$, each uniformly distributed on $(-\sqrt{3}, \sqrt{3})$

(b) $\epsilon_j^{(k)}$ independent for $k = 1, 2$, each having a standard normal distribution (mean 0, variance 1).

For each of these four data sets, repeat part 2.