Error estimates of an immersed finite element method for interface problems [*†‡]

Z. Li[§] T. Lin[¶] Y. Lin[‖] R.C. Rogers[**]

Abstract

This paper analyzes an immersed finite element method based on piewise linear polynomials introduced recently for solving interface problems. As an important feature, this method allows the interface to pass through triangles so that partitions with simple structures, such as the Cartesian partition, may be employed to solve a problem with an arbitrary interface. The main objectives here are to demonstrate the approximation capability of the immersed finite element space, and to derive error estimates for the related finite element solutions. Numerical results are also provided to illustrate some properties of this method.

1 Introduction

In [22], two (nonconforming and conforming) immersed finite element methods based on Cartesian grids are developed for boundary value problems of elliptic differential equations with discontinuities in the coefficients; singularities in the source terms; and arbitrary interfaces in the solution domains. These problems are referred as interface problems. In this paper, we investigate the nonconforming immersed finite element method, especially the related error estimates, in a more general framework.

The model interface problem of concern consists of an elliptic equation of the form

$$-\nabla \cdot (\beta \nabla u) + \kappa u = f, \tag{1.1}$$

where β has jumps across an interface Γ in the solution domain $\Omega \in R^2$, together with the Dirichlet boundary condition. In addition, we allow the source term f to have a Dirac delta function singularity on Γ of the form

$$f(\mathbf{x}) = f_c(\mathbf{x}) + \int_\Gamma Q(\mathbf{X})\, \delta(\mathbf{x} - \mathbf{X}) d\mathbf{X}, \tag{1.2}$$

where $Q(\mathbf{X})$ is the source strength; δ is the Dirac-delta function which is defined in the sense of distribution. Such a source function is an important feature of Peskin's immersed boundary

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[§]Center for Research in Scientific Computation & Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205

[¶]Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, tlin@math.vt.edu

[‖]Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

[**]Department of Mathematics, Virginia Tech, Blacksburg, VA 24061

method (IBM) [27, 28] that has been used for many problems in mathematical biology and computational fluid mechanics [3, 11, 12, 13, 29].

From equation (1.1) and (1.2), it is easy to obtained the jump conditions

$$[u] = 0, \qquad \text{continuity condition,} \tag{1.3}$$

$$[\beta u_n] = Q(s), \qquad \text{net flux across the interface,} \tag{1.4}$$

where the jump is defined as the difference of the limiting values from each side of the interface assuming the solution is piecewisely smooth on each side; u_n is the normal derivative of the solution. Many applications involve solving interface interface problems similar to the one defined by (1.1), (1.3), and (1.4), for example, the projection method for solving the Navier-Stokes equation [1, 2, 8], two phase flow [30], Hele-Shaw flow [14, 15], and many others.

It is well known (see [5, 7] and the references therein) that the standard Galerkin method with linear finite elements can be used to solve such elliptic interface problems. However, to achieve the optimal $O(h^2)$ accuracy in the numerical solutions, triangles are required to be aligned with the interface, i.e., the interface is allowed to pass a triangle only through its vertices. This restriction will obviously prevent the Galerkin method with linear finite elements to work efficiently for those applications in which an interface problem similar to the one defined by (1.1)-(1.4) has to be solved repeatedly, each time with a different interface Γ, because the partition has to be formed over and over again due to the variation (either the shape or the position) of the interfaces.

In the conventional Galerkin finite element method, the linear polynomial basis functions in each triangle are independent of the interface conditions, but the partition is formed according to the interface so that a better numerical result can be achieved. The immersed finite element method employs an opposite approach: the partition in the immersed finite element method can be independent of the interface, but the basis functions are constructed according to the jump conditions across interface. The immerse finite element method therefore has some advantages in solving interface problems. First, partitions with simple structure, for example, a Cartesian partition, can be used even if the interface Γ is arbitrary. The only constraint on a partition is that it must be fine enough to satisfactorily represent the interface. Secondly, it is possible to use one or just a few partitions to solve a sequence of interface problems so long as the interfaces do not vary too much. Also, the immersed finite element method may be used together with other Cartesian grid methods such as the level set method [26] and those implemented in Clawpack [17].

Since the immersed finite element method has no restriction on the partition, triangles in the partition can be separated into two groups; the first group consists of interface triangles, and the second contains noninterface triangles. A triangle is an interface triangle if it is separated into two subsets by the interface; otherwise, it is a non-interface triangle. Obviously, we can just use the usual linear basis functions to construct approximation in a non-interface triangle. However, in an interface triangle, we introduce basis functions with linear polynomials piecewisely defined in its two subsets formed by the interface such that these basis functions can satisfy the jump conditions (either exactly or approximately) on the interface and retain specified values at the vertices of this interface triangle. The idea here is similar to that used for the Hsieh-Clough-Tocher macro C^1 element [4] where each basis function consists of three cubic polynomials on the sub-triangles formed by connecting the vertices and the center of gravity so that the required continuity can be satisfied. We refer the readers to [10, 14, 18, 19, 20, 21, 23, 24, 25, 31] for more background materials about immersed interface and immersed finite element methods, as well as their applications. Our main effort here is devoted to error estimation for this method.

This paper is organized as follows. Section 2 contains some preliminary materials such as the specific interface problem to be considered, basic notations, an immersed finite element space for the interface problem, and some of its basic properties. Section 3 is devoted to an immersed finite element method for the interface problem and the related optimal error estimates. Section 4 presents a modified immersed finite element method whose implementation is easier from the point of view of numerical integration. Section 5 contains numerical results illustrating features of these immersed finite element methods.

2 The immersed finite element (IFE) space

2.1 Preliminaries

To be specific but without loss of generality, we consider the immersed finite element (IFE) method for the following boundary value problem:

$$-\nabla \cdot (\beta \nabla u) = f, \quad (x, y) \in \Omega, \tag{2.5}$$

$$u|_{\partial\Omega} = g, \tag{2.6}$$

together with the jump conditions on the interface Γ:

$$[u]\,|_\Gamma = 0, \tag{2.7}$$

$$[\beta u_n]\,|_\Gamma = 0. \tag{2.8}$$

Here, see the sketch in Figure 1, $\Omega \subset R^2$ is a convex polygonal domain, the interface Γ is a smooth curve separating Ω into two domains Ω^-, Ω^+ such that $\Omega = \Omega^- \cup \Omega^+$, and the coefficient $\beta(x, y)$ is a piecewise constant function defined by

$$\beta(x, y) = \begin{cases} \beta^-, & (x, y) \in \Omega^-, \\ \beta^+, & (x, y) \in \Omega^+. \end{cases}$$

We also assume that the boundition function g is zero, and the most of the results obtained in this paper can be extended by the usual way to the cases in which $g \neq 0$.

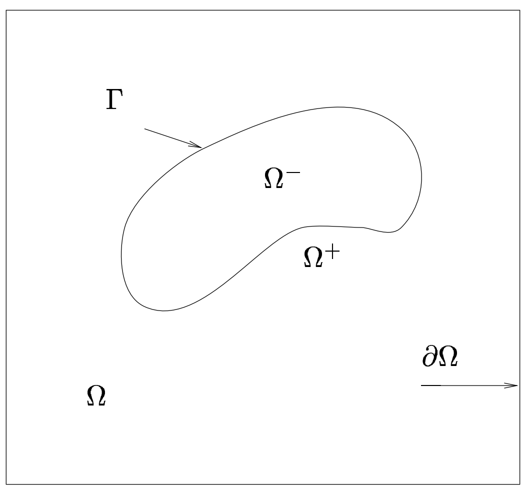

Figure 1: A sketch of the domain for the interface problem.

We let

$$\tilde{H}^m(\Omega) = \{u \mid u|_{\Omega^s} \in H^m(\Omega^s),\ s = -, +\}, \tag{2.9}$$

and we define the norm of $\tilde{H}^m(\Omega)$ to be

$$\|u\|^2_{m,\Omega} = \|u\|^2_{m,\Omega^-} + \|u\|^2_{m,\Omega^+}\,. \tag{2.10}$$

Semi-norms of $\tilde{H}^m(\Omega)$ can be defined accordingly by

$$|u|^2_{m,\Omega} = |u|^2_{m,\Omega^-} + |u|^2_{m,\Omega^+}\,. \tag{2.11}$$

For any subset T of Ω, we let

$$T^s = T \cap \Omega^s, \quad s = -, +.$$

For any function $f(x,y)$ defined in $T \subset \Omega$, we can restrict it to $T^s, s = -,+$ to obtain two functions as

$$f^s(x,y) = f(x,y), \quad \text{if } (x,y) \in T^s, s = -,+.$$

We use $\overline{DE}$ to denote the line segment between two points $D, E \in \Omega$. For any curve Γ, we use $\mathbf{n}_\Gamma$ to denote its unit normal vector pointing to a particular side of Γ. Also, for any measurable subset Λ of Ω, we use $|\Lambda|$ to denote its measure.

2.2 Finite element functions in interface triangles

Let $\mathcal{T}_h$ be a regular partition of Ω with a step size h. Without loss of generality, we assume that the triangles in this partition have the following features:

(H_1): If Γ meets one edge of a triangle at more than two points, then this edge is part of Γ.

(H_2): If Γ meets a triangle at two points, then these two points must be on different edges of this triangle.

For a typical triangle $T \in \mathcal{T}_h$, we use $A = (x_1, y_2), B = (x_2, y_2), C = (x_3, y_3)$ to denote its vertices, and use $D = (x_D, y_D)$ and $E = (x_E, y_E)$ to denote its interface points if T is an interface triangle, see the sketch in Figure 2.

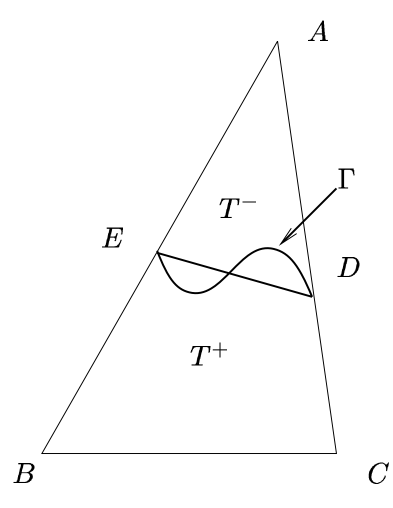

Figure 2: A typical interface triangle ΔABC. The curve between D and E is part of the interface Γ.

We now consider the finite element functions in an interface triangle. We follow an idea similar to that for the Hsieh-Clough-Tocher macro C^1 element [4] in which piecewise polynomials are used in a triangle to maintain certain desirable features. For our interface problem, we obviously would like the finite element functions to satisfy the jump conditions across the interface. Since the interface Γ separates an interface triangle T into two subsets T^- and T^+, we naturally can try to form a finite element function by two first degree polynomials defined in T^- and T^+, respectively. Note that each polynomial of degree one has three freedoms (coefficients). The values of the finite element function at the vertices of T provides three restrictions. The normal derivative jump condition provides another. Then we can have two more restrictions by requiring the continuity of the finite element function at interface points D and E. Intuitively, these six conditions can yield the desired piecewise linear polynomial in an interface triangle. This leads us to consider functions defined as follows:

$$
\phi(x,y) = \begin{cases} \phi^-(x,y) = a_1x + b_1y + c_1, & (x,y) \in T^-, \\ \phi^+(x,y) = a_2x + b_2y + c_2, & (x,y), \in T^+, \\ \phi^-(D) = \phi^+(D), \quad \phi^-(E) = \phi^+(E), & \\ (\beta^-\nabla\phi^- - \beta^+\nabla\phi^+) \cdot \mathbf{n}_{\overline{DE}} = 0, & \end{cases} \tag{2.12}
$$

where $\mathbf{n}_{\overline{DE}}$ is the unit vector perpendicular to the line $\overline{DE}$.

Lemma 2.1 *The function $\phi(x,y)$ defined by (2.12) in an interface triangle T is uniquely decided by its values at the three vertices of T.*

Proof. We need to prove this only in the reference triangle $\hat{T}$ whose vertices are $A_1 = (0,0)^T$, $A_2 = (1,0)^T$ and $A_3 = (0,1)^T$. Without loosing generality, we assume that $\phi(x,y)$ takes a value d_i at the vertex A_i for $i = 1,2,3$, respectively, and the interface Γ meet this triangle at $E = (a,0)^T$ and $D = (0,b)^T$ such that $A_1 \in \hat{T}^-$ with $0 < a \le 1, 0 < b \le 1$, and $ab < 1$. The values specified at the vertices force $\phi(x,y)$ to have the following form:

$$
\phi(x,y) = \begin{cases} \phi^-(x,y) = a_1x + b_1y + d_1, & (x,y) \in \hat{T}^-, \\ \phi^+(x,y) = d_3(1-x) + d_2(1-y) + c_3(1-x-y), & (x,y), \in \hat{T}^+, \end{cases}
$$

with a_1, b_1 and c_3 to be determined. The continuity condition at points E and D leads to

$$
\begin{cases} a_1a + d_1 = d_3(1-a) + d_2 + c_3(1-a), \\ b_1b + d_1 = d_3 + d_2(1-b) + c_3(1-b). \end{cases} \tag{2.13}
$$

The last equation in (2.12) requires

$$
\frac{\beta^-}{\beta^+}(a_1n_1 + b_1n_2) = -\big[(d_3 + c_3)n_1 + (d_2 + c_3)n_2\big], \tag{2.14}
$$

where $\mathbf{n} = (n_1, n_2)^T$ the unit vector perpendicular to the line $\overline{DE}$. Therefore, the coefficients a_1, b_1 and c_3 of ϕ satisfies the following linear system:

$$
\begin{pmatrix} a & 0 & a-1 \\ 0 & b & b-1 \\ Rn_1 & Rn_2 & n_1+n_2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_3 \end{pmatrix} = \begin{pmatrix} d_3(1-a) + d_2 - d_1 \\ d_3 + d_2(1-b) - d_1 \\ -d_3n_1 - d_2n_2 \end{pmatrix},
$$

with

$$
R = \frac{\beta^-}{\beta^+}. \tag{2.15}
$$

A simple calculation shows that the determinant of the matrix in this system is

$$ab\left\{n_1 + n_2 + R\left[(1-a)\frac{n_1}{a} + (1-b)\frac{n_2}{b}\right]\right\},$$

which can not be zero since $\mathbf{n} = (n_1, n_2)^T$ is not a zero vector and n_1, and n_2 do not have different signs. Thus, the above linear system must have a unique solution, and the function $\phi(x, y)$ here is uniquely determined by its values at the vertices of an interface triangle T.

■

We now let $S_h(T)$ be the linear space of all the functions defined by (2.12), and call it the immersed finite element space in an interface triangle T. The lemma above shows that the dimension of $S_h(T)$ is three. Furthermore, we notice:

- The proof of Lemma 2.1 itself provides a way to construct the nodal basis functions in an interface triangle.
- From the proof we can see that $\phi^-(x, y) = \phi^+(x, y)$ when $R = 1$, i.e., when the coefficient does not have the jump, the functions of $S_h(T)$ become the usual linear polynomials. In this case, $S_h(T)$ reduces to the standard linear finite element space.
- When $\Gamma \cap T$ is a straight line, the function $\phi(x, y)$ defined by (2.12) is continuous in T and therefore is in $H^1(T)$.

2.3 Interpolation errors

We now discuss the approximation capability of the space $S(T)$ when T is an interface triangle, and we will follow the usual framework for the theory of affine families. Consider a set $J(T)$ of functions such that every $u \in J(T)$ satisfies:

$$\begin{cases} u|_{T^i} = u^i \in H^2(T^i), \ i = -, +, \\ u^-(D) = u^+(D), \quad u^-(E) = u^+(E), \\ \int_{\Gamma\cap T} (\beta^- \nabla u^- - \beta^+ \nabla u^+) \cdot \mathbf{n}_\Gamma ds = 0. \end{cases} \tag{2.16}$$

The set $J(T)$ is a linear space in the usual sense, and for each non-negative integer $m \le 2$, we endow this space with a norm $\|\cdot\|_{m,T}$ defined in the same way as (2.10). Semi-norms of $J(T)$ are defined similarly.

Lemma 2.2 *For an interface triangle T, the space $S_h(T)$ is a subspace of $J(T)$.*

Proof. For any $\phi \in S_h(T)$, it is obvious that $\phi^s \in H^2(T^s), \quad s = -, +$. Also, because ϕ is a piecewise linear polynomial satisfying (2.12), Green's formula leads to

$$\int_{\Gamma\cap T} (\beta^- \nabla\phi^- - \beta^+ \nabla\phi^+) \cdot \mathbf{n}_\Gamma ds \quad = \quad -\int_{\overline{DE}} (\beta^- \nabla\phi^- - \beta^+ \nabla\phi^+) \cdot \mathbf{n}_{\overline{DE}} ds = 0.$$

Thus $\phi \in J(T)$ and $S_h(T)$ is a subspace of $J(T)$.

■

Remark: As a consequence of this lemma, any immersed finite element function $\phi \in S_h(T)$ can satisfy the derivative jump condition exactly in a weak sense as follows:

$$\int_{\Gamma\cap T} (\beta^- \nabla\phi^- - \beta^+ \nabla\phi^+) \cdot \mathbf{n}_\Gamma ds = 0.$$

Now, for any $u \in J(T)$ we let

$$|||u|||_{2,T} = |u|_{2,T} + |u(A)| + |u(B)| + |u(C)|. \tag{2.17}$$

Lemma 2.3 *$|||\cdot|||_{2,T}$ is a norm in the space $J(T)$, and this norm is equivalent to $\|\cdot\|_{2,T}$.*

Proof. We mainly need to show the equivalence. By the embedding theorem of Sobolev spaces, it is easy to see that for any $u \in J(T)$, there exists a constant $C_1 > 0$ such that

$$|||u|||_{2,T} \leq C_1 \|u\|_{2,T}\,. \tag{2.18}$$

Then we would like to show that there exists another constant $C_2 > 0$ such that

$$\|u\|_{2,T} \leq C_2 |||u|||_{2,T}. \tag{2.19}$$

Assume that this is not true, then there exists a sequence $\{v_k\} \subset J(T)$ such that

$$\|v_k\|_{2,T} = 1, \quad |||v_k|||_{2,T} \leq \frac{1}{k}, \tag{2.20}$$

and we can further assume that this sequence converges in $\|\cdot\|_{1,T}$ norm of $J(T)$. Since

$$\|v_l - v_m\|_{2,T} \leq C\big(\|v_l - v_m\|_{1,T} + |v_l|_{2,T} + |v_m|_{2,T}\big),$$

we conclude that $\{v_k\}$ is a Cauchy sequence in $J(T)$ with respect to the norm $\|\cdot\|_{2,T}$. Thus there exists a $v^* \in J(T)$ such that $\lim_{k\to\infty} v_k = v^*$. According to (2.20), v^* must be a piecewise polynomial of degree not more than 1. By the embedding theorem of the Sobolev space, v^* also satisfies the last two equations in (2.16), and from the proof of Lemma 2.2, we have

$$\begin{aligned}(\beta^-\nabla v^{*-} - \beta^+\nabla v^{*+})\cdot \mathbf{n}_{\overline{DE}} &= (\beta^-\nabla v^{*-} - \beta^+\nabla v^{*+})\cdot \mathbf{n}_{\overline{DE}} \frac{\int_{\overline{DE}} ds}{\int_{\overline{DE}} ds} \\ &= -\frac{1}{\int_{\overline{DE}} ds}\int_{\Gamma\cap T} (\beta^-\nabla v^{*-} - \beta^+\nabla v^{*+})\cdot \mathbf{n}_\Gamma ds = 0,\end{aligned}$$

Hence we have $v^* \in S_h(T)$ with

$$v^*(A) = v^*(B) = v^*(C) = 0.$$

By Lemma 2.1, $v^* = 0$ which contradicts with (2.20), and this implies that (2.19) should be valid. Both (2.18) and (2.19) plus some simple calculations show that $|||\cdot|||_{2,T}$ is a norm equivalent to $\|\cdot\|_{2,T}$.

■

For any $u \in J(T)$, we let $I_h u \in S_h(T)$ be its interpolant in $S_h(T)$ defined by

$$u(A) = I_h u(A),\ u(B) = I_h u(B),\ u(C) = I_h u(C),$$

To investigate the accuracy of $I_h u$, we will use the reference triangle $\hat{T}$ whose vertices are

$$\hat{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \hat{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \hat{C} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

If the interface triangle T has the following vertices

$$A = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, B = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, C = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix},$$

then T is the image of the $\hat{T}$ under the affine mapping below:

$$F(\hat{x},\hat{y}) \;=\; M\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad (\hat{x},\hat{y})^t \in \hat{T},$$

where the matrix M is defined as

$$M = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}.$$

Figure 3 gives a sketch of $\hat{T}$ and its image T under the affine mapping $F(\hat{x}, \hat{y})$. For each function $v(x, y)$ defined on T, we can have a function $\hat{v}(\hat{x}, \hat{y})$ defined on $\hat{T}$ by letting $\hat{v} = v \circ F$. The definition of $S_h(T)$ above can be extended to define $S_h(\hat{T})$. The following lemma states that the standard norm relationship between v and $\hat{v}$ hold even when T is an interface triangle.

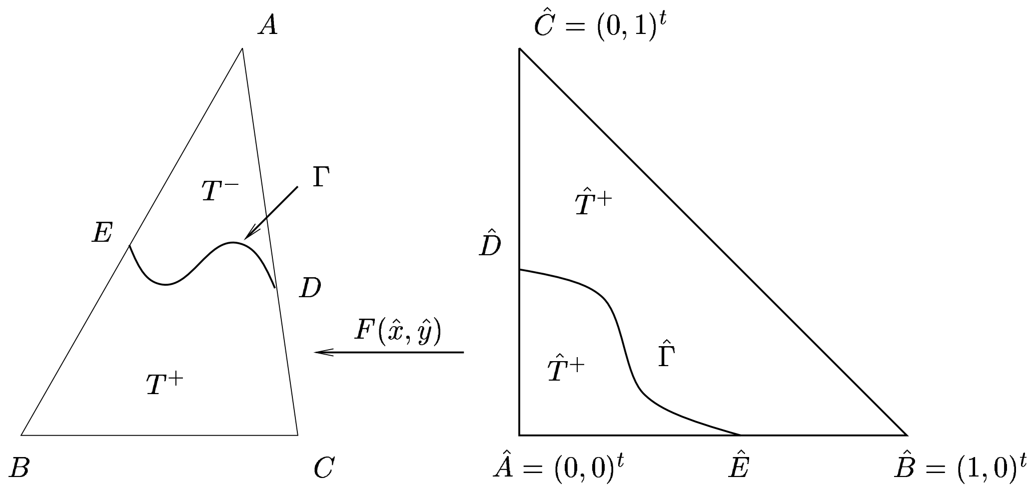

Figure 3: The reference element $\hat{T}$ and its image T under the affine mapping $F(\hat{x}, \hat{y})$.

Lemma 2.4 *For a given (either interface or non-interface) element T and non-negative integer $k \le 2$, there exists a constant C such that*

$$|v|_{k,T} \le C \left\|M^{-1}\right\|^k |\det M|^{1/2} |\hat{v}|_{k,\hat{T}},$$
$$|\hat{v}|_{k,\hat{T}} \le C \left\|M\right\|^k |\det M|^{-1/2} |v|_{k,T}$$

Proof. The results follow from arguments similar to those used in proving the same inequalities for functions in standard Sovolev spaces, see for example [9, 16].

■

The following theorem describes the approximation capability of $S_h(T)$.

Theorem 2.1 *Let T be an interface triangle, then for any $u \in J(T)$ we have*

(2.21) $$\|u - I_h u\|_{m,T} \le C h^{2-m} |u|_{2,T}, \quad 0 \le m < 2,$$

where h is the length of the longest edge of T.

Proof. First, we consider the case with the reference triangle $\hat{T}$. By Lemma 2.3,

$$\begin{aligned} \left\|\hat{u} - \hat{I_h u}\right\|_{2,\hat{T}} &\le |||\hat{u} - \hat{I_h u}|||_{2,\hat{T}} \\ &= C\left(\left|\hat{u} - \hat{I_h u}\right|_{2,\hat{T}} + \left|\hat{u}(\hat{A}) - \hat{I}_h u(\hat{A})\right| + \left|u(\hat{B}) - I_h u(\hat{B})\right| + \left|u(\hat{C}) - I_h u(\hat{C})\right| \right) \\ &= C \left|\hat{u} - \hat{I_h u}\right|_{2,\hat{T}} = C\,|\hat{u}|_{2,\hat{T}}. \end{aligned}$$

Then we can obtain (2.21) by Lemma 2.4 and the standard homogeneity argument [6, 16].

■

Now we use the partition $\mathcal{T}_h$ to define an immersed finite element (IFE) space $S_h(\Omega)$. We first define a nodal basis function $\phi(x, y)$ for each node $(x_N, y_N)^t$ of $\mathcal{T}_h$ piecewisely such that $\phi(x_N, y_N) = 1$ but zero at other nodes, and $\phi|_T \in S_h(T)$ for any triangle $T \in \mathcal{T}_h$. Here $S_h(T)$ is the usual space of linear polynomials when T is a non-interface triangle, or the immersed finite element space on T introduced in Section 2.2 when T is an interface triangle. Then we define $S_h(\Omega)$ as the span of these nodal basis functions, and it is easy to see that $S_h(\Omega)$ has the following properties:

- For a partition $\mathcal{T}_h$, the IFE space $S_h(\Omega)$ has the same number of nodal basis functions as that formed by the usual linear polynomials.
- For a partition $\mathcal{T}_h$ fine enough, most of its triangles are non-interface triangles, and most of the nodal basis functions of the IFE space $S_h(\Omega)$ are just the usual linear nodal basis functions except for few nodes in the vicinity of the interface Γ.
- For any $\phi \in S_h(\Omega)$, we have

  (2.22) $$\phi|_{\Omega\backslash\Omega'} \in H^1(\Omega\backslash\Omega'),$$

  where Ω' is the union of interface triangles.

Now we consider a function u satisfying

(2.23) $$u \in C(\Omega), \quad u|_{\Omega^s} \in H^2(\Omega^s), \ s = -, +$$

and

(2.24) $$(\beta^-\nabla u^- - \beta^+\nabla u^+) \cdot \mathbf{n} = 0,$$

on Γ, and we define its interpolant $I_h u$ in the IFE space $S_h(\Omega)$ by

$$I_h u(x, y) = u(x, y), \text{ if } (x, y) \text{ is a node of } \mathcal{T}_h.$$

From Theorem 2.1 we can easily obtain the following error estimate for $I_h u$.

Theorem 2.2 *Assume that u satisfies the conditions (2.23) and (2.24), then*

(2.25) $$\|u - I_h u\|_{0,\Omega} + h\,\|u - I_h u\|_{1,\Omega,h} \le Ch^2\,\|u\|_{2,\Omega}\,,$$

where

$$\|u\|_{m,\Omega,h} = \sum_{T\in\mathcal{T}_h} \|u\|_{m,T}\,.$$

■

3 A finite element solution and its error estimates

It is obvious that the finite element space $S_h(\Omega)$ introduced in the last section is not in the space to which the solution of the interface problem belongs. A function ϕ of $S_h(\Omega)$ is continuous in the set of the union of non-interface triangles. The possible discontinuous points of ϕ are on the interface Γ or on edges of an interface triangle which intersect with Γ somewhere between its vertices. Therefore the finite element method based on $S_h(\Omega)$ is nonconforming. To introduce the finite element method we consider the following bilinear form:

$$a_h(u,v) := \sum_{T\in\mathcal{T}_h} \int_{T\in\mathcal{T}_h} \beta\nabla u\cdot\nabla v dxdy, \text{ for all } u,v\in\tilde{H}^1(\Omega)\oplus S_h(\Omega), \tag{3.26}$$

and let

$$\|u\|_h := \sqrt{a_h(u,u)}, \quad \text{for all } u\in\tilde{H}^1(\Omega)\oplus S_h(\Omega), \tag{3.27}$$

which is a common quantity used in the error estimation of nonconforming finite elements, see for example [4]. Also it is obvious that

$$\|u\|_h \le C\,\|u\|_{1,\Omega,h}\,.$$

By simple calculations we can show that this bilinear form has the usual boundedness and coercivity as described in the lemma below.

Lemma 3.1 *There exists a positive constant C such that for any $u,v\in\tilde{H}^1(\Omega)\oplus S_h(\Omega)$, we have*

$$|a_h(u,v)| \le C\,\|u\|_h\,\|v\|_h\,, \tag{3.28}$$

$$a_h(u,v) \ge C\,\|u\|_h^2\,. \tag{3.29}$$

Also, $\|\cdot\|_h$ is a norm of the space $\tilde{H}_0^1(\Omega)\oplus S_{h0}(\Omega)$. Here $\tilde{H}_0^1(\Omega)\oplus S_{h0}(\Omega)$ consists of functions in $\tilde{H}^1(\Omega)\oplus S_h(\Omega)$ which are zero on $\partial\Omega$.

■

For the interface problem, we now define its immersed finite element solution as a function $u_h\in S_{h0}$ satisfying

$$a_h(u_h,v_h) = \left\langle f, v_h\right\rangle_0, \quad \forall v_h\in S_{h0}. \tag{3.30}$$

To aid the error estimation for u_h, we let $\mathcal{T}_h'$ be the collection of all the interface triangles, and let $\mathcal{E}'_h$ be set of all edges in $\mathcal{T}_h$ which meet the interface Γ between their vertices. Each element $T\in\mathcal{T}_h'$ is separated into two pieces T^- and T^+ by the interface Γ such that $T^-\subset\Omega^-$ and $T^+\subset\Omega^+$.

We first consider the consistency error in the weak form:

$$L_u(v_h) := a_h(u,v_h) - \left\langle f, v_h\right\rangle_0. \tag{3.31}$$

Here u is the solution of the boundary value problem, and v_h is a function in S_{h0}. From the Green's formula, we have

$$\begin{aligned} L_u(v_h) &= \sum_{e\in\mathcal{E}'_h}\int_e [\beta\frac{\partial u}{\partial n}v_h]ds + \sum_{T\in\mathcal{T}_h'}\int_{T\cap\Gamma}[\beta\frac{\partial u}{\partial n}v_h]ds, && (3.32)\\ &= \sum_{e\in\mathcal{E}'_h}\left(\int_{e^-}[\beta\frac{\partial u}{\partial n}v_h]ds + \int_{e^+}[\beta\frac{\partial u}{\partial n}v_h]ds\right) + \sum_{T\in\mathcal{T}_h'}\int_{T\cap\Gamma}[\beta\frac{\partial u}{\partial n}v_h]ds, && (3.33)\end{aligned}$$

where $\frac{\partial u}{\partial n}$ is the normal derivative. For the terms in the first summation, we assume that e is the common edge of two interface triangles T^l and T^r, see the sketch in Figure 4. Then we have

$$\int_{e^-}[\beta\frac{\partial u}{\partial n}v_h]ds = \int_{e^-}\beta\frac{\partial u}{\partial n}(v_h^l - v_h^r)ds \tag{3.34}$$

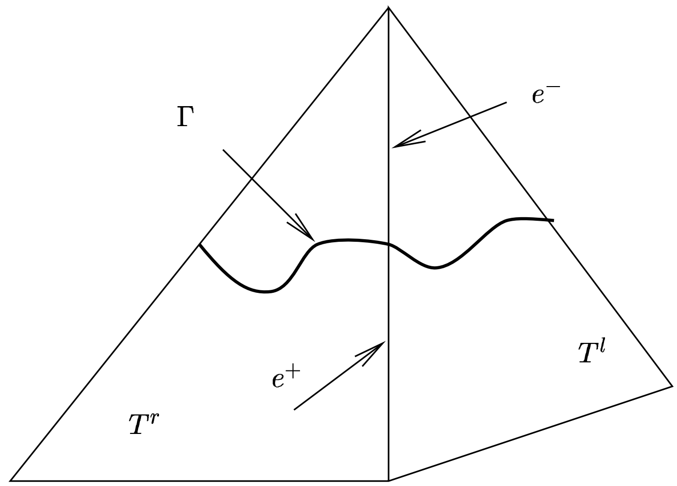

Figure 4: Two adjacent interface triangle whose common edge is cut into two pieces e^- and e^+ by the interface Γ.

because $\beta\dfrac{\partial u}{\partial n}$ has no jump in $(T^l \cup T^r) \cap \Omega^-$. Similarly,

$$\int_{e^+} [\beta \frac{\partial u}{\partial n} v_h] ds = \int_{e^+} \beta \frac{\partial u}{\partial n}(v_h^l - v_h^r) ds. \tag{3.35}$$

Here $v_h^i = v_h|_{T^i}, i = l, r$. For the second term, we have

$$\begin{aligned} \int_{T\cap\Gamma} [\beta \frac{\partial u}{\partial n} v_h] ds &= \int_{T\cap\Gamma} (\beta^- \frac{\partial u^-}{\partial n} v_h^- - \beta^+ \frac{\partial u^+}{\partial n} v_h^+) ds \\ &= \int_{T\cap\Gamma} \beta^- \frac{\partial u^-}{\partial n}(v_h^- - v_h^+) ds. \end{aligned} \tag{3.36}$$

With these preparations we can now derive an estimate for the consistency error in the following theorem.

Theorem 3.1 *Assume that the solution u of the interface problem is in $C(\Omega) \cap \tilde{H}^2(\Omega)$. Then*

$$\left| a_h(u, v_h) - \left\langle f, v_h \right\rangle_0 \right| \le Ch\, |u|_{2,\Omega}\, \|v_h\|_h\,, \tag{3.37}$$

for any $v_h \in S_h(\Omega)$.

Proof. According to the discussion above, we need to estimate the quantities in (3.34) - (3.36). For (3.34), we note that

$$v_h^l(A) = v_h^r(A)$$

where A is the common vertex of T^l and T^r in Ω^-. Then

$$\begin{aligned} \left| \int_{e^-} [\beta \frac{\partial u}{\partial n} v_h] ds \right| &= \left| \int_{e^-} \beta \frac{\partial u}{\partial n}(v_h^l - v_h^l(A) + v_h^r(A) - v_h^r) ds \right| \\ &\le \left| \int_{e^-} \beta \frac{\partial u}{\partial n}(v_h^l - v_h^l(A)) ds \right| + \left| \int_{e^-} \beta \frac{\partial u}{\partial n}(v_h^r(A) - v_h^r) ds \right|. \end{aligned} \tag{3.38}$$

Let $X_s = (x, y)$ be a point on e^-. The difference $v_h^l(X_s) - v_h^l(A)$ can be written in the following integral term

$$v_h^l(X_s) - v_h^l(A) = \int_{\overline{AX_s}} \frac{\partial v_h^l}{\partial t} d\eta,$$

where $\frac{\partial v_h^l}{\partial t}$ is the tangential derivative of v_h^l along the line $\overline{AX_s}$. Therefore, by the trace theorem and the fact that v_h^l is polynomial of degree 1,

$$\begin{aligned}
\left| \int_{e^-} \beta \frac{\partial u}{\partial n} (v_h^l - v_h^l(A)) ds \right| &\leq \left| \int_{e^-} \beta \frac{\partial u}{\partial n} \int_{\overline{AX_s}} \frac{\partial v_h^l}{\partial t} d\eta ds \right| \\
&\leq \beta^- \int_{e^-} \left| \frac{\partial u}{\partial n} \right| \left(\int_{\overline{AX_s}} \left| \frac{\partial v_h^l}{\partial t} \right| d\eta \right) ds \\
&\leq \beta^- h^{1/2} \int_{e^-} \left| \frac{\partial u}{\partial n} \right| \left(\int_{e^-} \left| \frac{\partial v_h^l}{\partial t} \right|^2 ds \right)^{1/2} ds \\
&\leq \beta^- h^{1/2} \left(\int_{e^-} \left| \frac{\partial v_h^l}{\partial t} \right|^2 ds \right)^{1/2} \int_{e^-} \left| \frac{\partial u}{\partial n} \right| ds \\
&\leq \beta^- h \left(\int_{e^-} \left| \frac{\partial v_h^l}{\partial t} \right|^2 ds \right)^{1/2} \left(\int_{e^-} \left| \frac{\partial u}{\partial n} \right|^2 ds \right)^{1/2} \\
&\leq C\beta^- h \left\| \nabla v_h^l \right\|_{1,T^l \cap \Omega^-} \left\| \nabla u \right\|_{1,T^l \cap \Omega^-} \\
&\leq C\beta^- h \left| v_h^l \right|_{1,T^l \cap \Omega^-} |u|_{2,T^l \cap \Omega^-} .
\end{aligned}$$

Similarly,

$$\left| \int_{e^-} \beta \frac{\partial u}{\partial n} (v_h^r(A) - v_h^r) ds \right| \leq C\beta^- h \, |v_h^r|_{1,T^r \cap \Omega^-} \, |u|_{2,T^r \cap \Omega^-} ,$$

and we have

$$\left| \int_{e^-} [\beta \frac{\partial u}{\partial n} v_h] ds \right| \leq C\beta^- h \big(\left| v_h^l \right|_{1,T^l \cap \Omega^-} |u|_{2,T^l \cap \Omega^-} + |v_h^r|_{1,T^r \cap \Omega^-} |u|_{2,T^r \cap \Omega^-} \big). \tag{3.39}$$

By the same argument, we can have

$$\left| \int_{e^+} [\beta \frac{\partial u}{\partial n} v_h] ds \right| \leq C\beta^+ h \big(\left| v_h^l \right|_{1,T^l \cap \Omega^+} |u|_{2,T^l \cap \Omega^+} + |v_h^r|_{1,T^r \cap \Omega^+} |u|_{2,T^r \cap \Omega^+} \big). \tag{3.40}$$

We now turn to the term on the interface. Assume that the interface Γ meets the edge of T at a point D. Then by the definition of the functions in $S_h(T)$ we have

$$v_h^-(D) = v_h^+(D).$$

Therefore we have

$$\begin{aligned}
\int_{T \cap \Gamma} [\beta \frac{\partial u}{\partial n} v_h] ds &= \int_{T \cap \Gamma} \beta^- \frac{\partial u^-}{\partial n} (v_h^- - v_h^-(D) + v_h^+(D) - v_h^+) ds \\
&= \int_{T \cap \Gamma} \beta^- \frac{\partial u^-}{\partial n} (v_h^- - v_h^-(D)) ds \\
&\quad + \int_{T \cap \Gamma} \beta^+ \frac{\partial u^+}{\partial n} (v_h^+(D) - v_h^+) ds \\
&= \int_{T \cap \Gamma} \beta^- \frac{\partial u^-}{\partial n} \left(\int_{\widehat{DX_s}} \frac{\partial v_h^-}{\partial t} d\eta \right) ds + \int_{T \cap \Gamma} \beta^+ \frac{\partial u^+}{\partial n} \left(\int_{\widehat{X_s D}} \frac{\partial v_h^+}{\partial t} d\eta \right) ds,
\end{aligned}$$

where $\widehat{X_s D}$ is the part of $T \cap \Gamma$ between $X_s = (x, y)$ and D. Repeating the same arguments as those used for (3.39), we have

$$\begin{aligned} &\left| \int_{T\cap\Gamma} [\beta \frac{\partial u}{\partial n} v_h] ds \right| \\ \leq\ & C \max\{\beta^-, \beta^+\} h \left(|v_h^-|_{1,T\cap\Omega^-} |u|_{2,T\cap\Omega^-} + |v_h^+|_{1,T\cap\Omega^+} |u|_{2,T\cap\Omega^+} \right). \end{aligned} \tag{3.41}$$

Finally, the estimate of (3.37) is obtained by putting estimates (3.39), (3.40) and (3.42) in (3.33).

■

Now, we are ready to derive an error estimate in the $\|\cdot\|_h$ norm.

Theorem 3.2 *Assume that the solution u of the interface problem is in $C(\Omega) \cap \tilde{H}^2(\Omega)$. Then the error of the immersed finite element solution u_h has the following estimate for a constant $C > 0$:*

$$\|u - u_h\|_h \leq Ch\, |u|_{2,\Omega}. \tag{3.42}$$

Proof. From the basic Lemma of Strang [4], we have

$$\|u - u_h\| \leq C \left(\inf_{v_h \in S_{h0}(\Omega)} \|u - v_h\|_h + \sup_{w_h \in S_{h0}(\Omega)} \frac{|L_u(w_h)|}{\|w_h\|_h} \right).$$

Then the estimate of (3.42) is obtained by applying Theorem 2.2 and 3.1 to the above.

■

We now turn to the error estimation in L^2 norm. Note that for any $g \in L^2(\Omega)$, the auxiliary problem

$$\int_\Omega \beta \nabla\phi \cdot \nabla v dxdy = \int_\Omega g\ v\ dxdy, \quad \text{for any } v \in H^1(\Omega) \tag{3.43}$$

has a unique solution in $C(\Omega) \cap \tilde{H}^2(\Omega)$ satisfying

$$\|\phi_g\|_{2,\Omega} \leq C\, \|g\|_{0,\Omega}. \tag{3.44}$$

We let ϕ_h be the immersed finite element solution of this auxiliary boundary value problem. Then, from Theorem 3.2 we have

$$\|\phi_g - \phi_h\|_h \leq Ch\, |\phi_g|_{2,\Omega} \leq Ch\, \|g\|_{0,\Omega}. \tag{3.45}$$

In addition, the estimate in Theorem 3.1 is still true if we substitute v_h by a function in $(C(\Omega) \cap \tilde{H}^2(\Omega)) \oplus S_{h0}$. Thus

$$\begin{aligned} \left| a_h(u - u_h, \phi_g) - \left\langle u - u_h, g \right\rangle_0 \right| &= \left| a_h(\phi_g, u - u_h) - \left\langle g, u - u_h \right\rangle_0 \right| \\ &\leq Ch\, |\phi_g|_{2,\Omega}\, \|u - u_h\|_h \\ &\leq Ch^2\, \|g\|_{0,\Omega}\, |u|_{2,\Omega}. \qquad (3.46) \\ \left| a_h(u, \phi_g - \phi_h) - \left\langle f, \phi_g - \phi_h \right\rangle_0 \right| &\leq Ch\, |u|_{2,\Omega}\, \|\phi_g - \phi_h\|_h \\ &\leq Ch^2\, |u|_{2,\Omega}\, \|\phi_g\|_{2,\Omega} \leq Ch^2\, |u|_{2,\Omega}\, \|g\|_{0,\Omega}. \qquad (3.47) \end{aligned}$$

These preparations lead to an error estimate in the $L2$ norm in the following theorem.

Theorem 3.3 *Assume that the conditions of Theorem 3.2 are satisfied. Then the error of the immersed finite element solution u_h has the following estimate for a constant $C > 0$:*

$$\|u - u_h\|_{0,\Omega} \leq Ch^2 |u|_{2,\Omega} . \tag{3.48}$$

Proof. By the generalize Aubin-Nitsche lemma [4], we have

$$\begin{aligned} \|u - u_h\|_{0,\Omega} \quad \leq \quad & \sup_{g \in L^2(\Omega)} \frac{1}{\|g\|_{0,\Omega}} \Big\{ \|u - u_h\|_h \|\phi_g - \phi_h\|_h + \Big| a_h(u - u_h, \phi_g) - \Big\langle u - u_h, g \Big\rangle_0 \Big| \\ & + \Big| a_h(u, \phi_g - \phi_h) - \Big\langle f, \phi_g - \phi_h \Big\rangle_0 \Big| \Big\}. \end{aligned}$$

The estimate (3.48) is then obtained by applying Theorem 3.2, (3.45), (3.46), and (3.47) to the above.

■

4 A modified scheme

The immersed finite element space introduced in Section 2 involves piecewise polynomials defined over subsets with curve boundaries in interface triangles. While this does not cause any difficulty in some of its applications such as the finite volume element method [10], we would like to have a finite element formulation in which the integrations are carried out over approximate triangle/quadrilaterals.

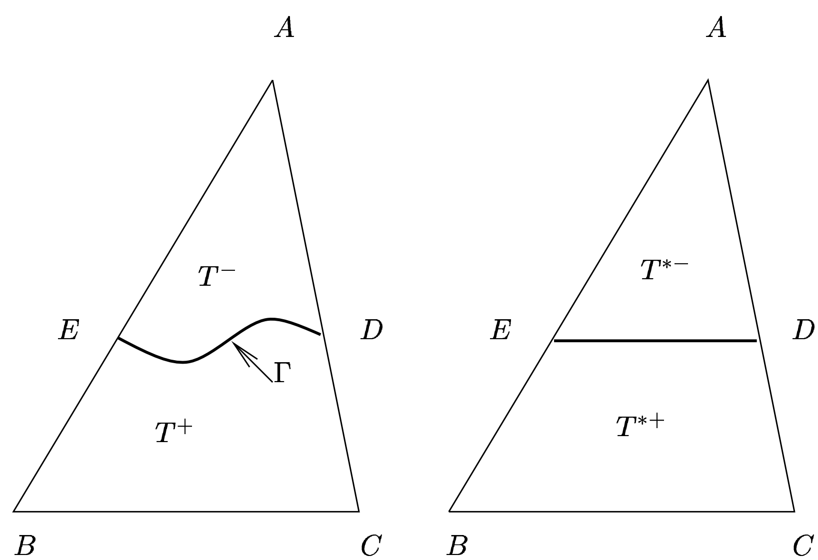

Figure 5: Subsets T^- and T^+ formed by the interface Γ in an interface triangle are modified to T^{*-} and T^{*+}

Let us consider a typical interface triangle $T \in \mathcal{T}_h$ such that

$$T = T^- \cup T^+, \quad T^- = T \cap \Omega^-, \quad T^+ = T \cap \Omega^+.$$

As before, we use A, B, C to denote the three vertices of T with $A \in \Omega^-$, $B, C \in \Omega^+$, and assume that the interface Γ meets two edges of T passing the vertex A at D and E. Then, T^- is a triangle with a curved edge $\overset{\frown}{ED}$, and T^+ is a quadrilateral with the same curved edge as T^-. On the other hand, this triangle can also be separated into T^{*-} and T^{*+} by the straight line between D and E, see the sketch in Figure 5. In the immersed finite element space $S_h(\Omega)$, we introduce a new bilinear form

$$\begin{aligned} a_h^*(v_h, w_h) \quad &:= \quad \sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_h'} \int_T \beta \nabla v_h \cdot \nabla w_h dxdy \\ &\quad + \sum_{T \in \mathcal{T}_h'} \Big[\int_{T^{*-}} \beta^- \nabla v_h^{*-} \cdot \nabla w_h^{*-} dxdy + \int_{T^{*+}} \beta^+ \nabla v_h^{*+} \cdot \nabla w_h^{*+} dxdy \Big], \end{aligned} \tag{4.49}$$

and a new linear functional

$$f_h^*(w_h) \quad := \quad \sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_h'} f w_h dxdy + \sum_{T \in \mathcal{T}_h'} \Big[\int_{T^{*-}} f w_h^{*-} dxdy + \int_{T^{*+}} f w_h^{*+} dxdy \Big],$$

where for any function $w_h \in S_h(\Omega)$, we let w_h^* be a piecewise linear function such that

$$w_h^*|_T(x,y) = \begin{cases} w_h^{*-}(x,y), & (x,y) \in T^{*-}, \\ w_h^{*+}(x,y), & (x,y) \in T^{*+}, \end{cases}$$

and

$$\begin{aligned} w_h^{*-}(x,y) = w_h^-, \quad (x,y) \in T^{*-} \cap T^-, \\ w_h^{*+}(x,y) = w_h^+, \quad (x,y) \in T^{*+} \cap T^+. \end{aligned} \tag{4.50}$$

Then we define the modified immersed finite element solution of the interface problem as a function $u_h^* \in S_{h0}(\Omega)$ satisfying

$$a_h^*(u_h^*, w_h) = f^*(w_h), \forall w_h \in S_{h0}(\Omega). \tag{4.51}$$

It is obvious that $a_h^*(v_h, w_h)$ is a continuous bilinear form in $S_h(\Omega)$ with respect to the norm $\|\cdot\|_h$. For further discusssions, we use $\tilde{T}$ to denote the subset in an interface triangle $T \in \mathcal{T}_h$ formed by the curve $\Gamma \cap T$ and the line $\overline{DE}$, and let

$$\begin{aligned} \tilde{T}^{*-} = \tilde{T} \cap T^{*-}, \\ \tilde{T}^{*+} = \tilde{T} \cap T^{*+}, \end{aligned} \tag{4.52}$$

see the sketch in Figure 6. Then, we can show that this bilinear form has the usual coerciveness as stated in the lemma below.

Lemma 4.1 *Assume*

$$\max \left\{ \frac{\left|\tilde{T}^{*+}\right|}{|T^-|}, \frac{\left|\tilde{T}^{*-}\right|}{|T^+|} \right\} \leq C, \quad i = -, +,$$

for a constant $0 < C < 1$. *Then, there exists a positive constant* $\tilde{C}$ *such*

$$a_h^*(v_h, v_h) \geq \tilde{C} \|v_h\|_h^2$$

for any $v_h \in S_h(\Omega)$.

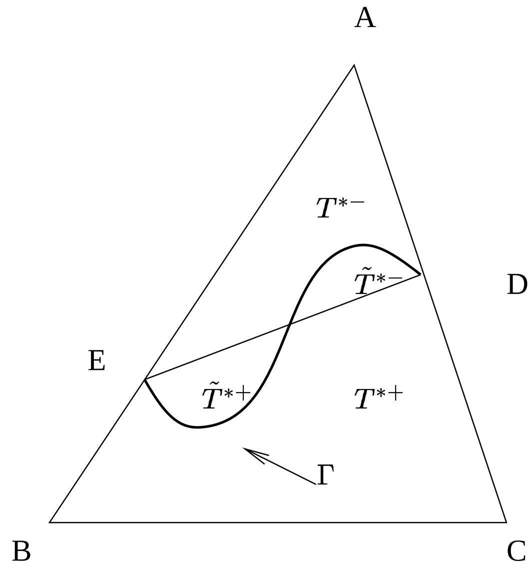

Figure 6: Sub-sets formed by the line $\bar{DE}$ and the interface between D and E.

Proof. For each interface triangle T, we have

$$\begin{aligned}
& \int_{T^{*-}} \beta^-\nabla v_h^{*-}\cdot\nabla v_h^{*-}dxdy + \int_{T^{*+}} \beta^+\nabla v_h^{*+}\cdot\nabla v_h^{*+}dxdy \\
= & \int_{T^-} \beta^-\nabla v_h^{*-}\cdot\nabla v_h^{*-}dxdy + \int_{\tilde{T}^{*-}} \beta^-\nabla v_h^{*-}\cdot\nabla v_h^{*-}dxdy - \int_{\tilde{T}^{*+}} \beta^-\nabla v_h^{*-}\cdot\nabla v_h^{*-}dxdy \\
& + \int_{T^+} \beta^+\nabla v_h^{*+}\cdot\nabla v_h^{*+}dxdy - \int_{\tilde{T}^{*-}} \beta^+\nabla v_h^{*+}\cdot\nabla v_h^{*+}dxdy + \int_{\tilde{T}^{*+}} \beta^+\nabla v_h^{*+}\cdot\nabla v_h^{*+}dxdy \\
\geq & \int_{T^-} \beta^-\nabla v_h^{-}\cdot\nabla v_h^{-}dxdy + \int_{T^+} \beta^+\nabla v_h^{+}\cdot\nabla v_h^{+}dxdy \\
& \quad - \int_{\tilde{T}^{*+}} \beta^-\nabla v_h^{-}\cdot\nabla v_h^{-}dxdy - \int_{\tilde{T}^{*-}} \beta^+\nabla v_h^{+}\cdot\nabla v_h^{+}dxdy \\
\geq & \int_{T^-} \beta^-\nabla v_h^{-}\cdot\nabla v_h^{-}dxdy + \int_{T^+} \beta^+\nabla v_h^{+}\cdot\nabla v_h^{+}dxdy \\
& \quad - C\int_{T^-} \beta^-\nabla v_h^{-}\cdot\nabla v_h^{-}dxdy - C\int_{T^+} \beta^+\nabla v_h^{+}\cdot\nabla v_h^{+}dxdy \\
= & (1-C)\left[\int_{T^-} \beta^-\nabla v_h^{-}\cdot\nabla v_h^{-}dxdy + \int_{T^+} \beta^+\nabla v_h^{+}\cdot\nabla v_h^{+}dxdy\right]
\end{aligned}$$

Therefore,

$$a_h^*(v_h, v_h) \quad \geq \quad (1-C)a_h(v_h, v_h) = \tilde{C}\,\|v_h\|_h^2\,.$$

■

The following lemmas are needed to derive an error estimate of the modified immersed finite element solution $u_h^* \in S_{h0}$.

Lemma 4.2 *There exists a constant C such that for any $w_h \in S_{h0}(\Omega)$ we have*

$$|f_h^*(w_h) - f_h(w_h)| \leq Ch\,\|f\|_{0,\Omega}\,\|w_h\|_h\,. \tag{4.53}$$

Proof. For each interface element $T \in \mathcal{T}_h'$,

$$\begin{aligned}
\int_{T^{*-}} f w_h^{*-} dxdy + \int_{T^{*+}} f w_h^{*+} dxdy &= \int_{T^-} f w_h^- dxdy + \int_{\tilde{T}^{*-}} f w_h^{*-} dxdy - \int_{\tilde{T}^{*+}} f w_h^- dxdy \\
&\quad + \int_{T^+} f w_h^+ dxdy - \int_{\tilde{T}^{*-}} f w_h^+ dxdy + \int_{\tilde{T}^{*+}} f w_h^{*+} dxdy \\
&= \int_{T^-} f w_h^- dxdy + \int_{T^+} f w_h^+ dxdy + \int_{\tilde{T}^{*-}} f\left(w_h^{*-} - w_h^+\right) dxdy \\
&\quad + \int_{\tilde{T}^{*+}} f\left(w_h^{*+} - w_h^-\right) dxdy.
\end{aligned}$$

Hence, by the definitions of $f_h(w_h)$ and $f_h^*(w_h)$ we have

$$(4.54) f_h^*(w_h) - f_h(w_h) = \sum_{T \in \mathcal{T}_h'} \left[\int_{\tilde{T}^{*-}} f\left(w_h^{*-} - w_h^+\right) dxdy + \int_{\tilde{T}^{*+}} f\left(w_h^{*+} - w_h^-\right) dxdy \right].$$

Take a point (x_n, y_n) on the line $\overline{DE}$ such that the line connecting $(x,y) \in \tilde{T}^{*-}$ and (x_n, y_n) is perpendicular to $\overline{DE}$. Then

$$\begin{aligned}
w_h^+(x,y) &= w_h^+(x_n,y_n) + \frac{\partial w_h^+}{\partial x}(x_n,y_n)(x - x_n) + \frac{\partial w_h^+}{\partial y}(x_n,y_n)(y - y_n) \\
&= w_h^-(x_n,y_n) + \frac{\partial w_h^+}{\partial x}(x_n,y_n)(x - x_n) + \frac{\partial w_h^+}{\partial y}(x_n,y_n)(y - y_n),
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\tilde{T}^{*-}} f\left(w_h^{*-} - w_h^+\right) dxdy \right| &\le h \int_{\tilde{T}^{*-}} |f| \left(\frac{\partial w_h^+}{\partial x} + \frac{\partial w_h^+}{\partial y} \right) dxdy \\
&\le h \int_{T^+} |f| \left(\frac{\partial w_h^+}{\partial x} + \frac{\partial w_h^+}{\partial y} \right) dxdy \\
&\le Ch \|f\|_{0,T^+} \left\| w_h^+ \right\|_{h,T^+}.
\end{aligned}$$

Similarly,

$$\left| \int_{\tilde{T}^{*+}} f\left(w_h^{*+} - w_h^-\right) dxdy \right| \le Ch \|f\|_{0,T^-} \left\| w_h^- \right\|_{h,T^-}.$$

The result of this lemma is then obtained by applying these estimates in (4.54).

■

Lemma 4.3 *There exists a constant C such that for any v_h, w_h of $S_{h0}(\Omega)$, we have*

$$|a_h^*(v_h, w_h) - a_h(v_h, w_h)| \le Ch \|v_h\|_h \|w_h\|_h.$$

Proof. Again, first consider a typical interface triangle $T \in \mathcal{T}_h'$, and we have

$$\begin{aligned}
&\int_{T^{*-}} \beta^- \nabla v_h^{*-} \cdot \nabla w_h^{*-} dxdy + \int_{T^{*+}} \beta^+ \nabla v_h^{*+} \cdot \nabla w_h^{*+} dxdy \\
= &\int_{T^-} \beta^- \nabla v_h^{*-} \cdot \nabla w_h^{*-} dxdy + \int_{\tilde{T}^{*-}} \beta^- \nabla v_h^{*-} \cdot \nabla w_h^{*-} dxdy - \int_{\tilde{T}^{*+}} \beta^- \nabla v_h^{*-} \cdot \nabla w_h^{*-} dxdy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{T^+} \beta^+ \nabla v_h^{*+} \cdot \nabla w_h^{*+} dxdy - \int_{\tilde{T}^{*-}} \beta^+ \nabla v_h^{*+} \cdot \nabla w_h^{*+} dxdy + \int_{\tilde{T}^{*+}} \beta^+ \nabla v_h^{*+} \cdot \nabla w_h^{*+} dxdy \\
= &\int_{T^-} \beta^- \nabla v_h^{*-} \cdot \nabla w_h^{*-} dxdy + \int_{T^+} \beta^+ \nabla v_h^{*+} \cdot \nabla w_h^{*+} dxdy \\
&+ \int_{\partial \tilde{T}^{*+}} \left(\beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} - \beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} \right) ds + \int_{\partial \tilde{T}^{*+}} \left(\beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} - \beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} \right) ds,
\end{aligned}
$$

and this gives

$$
\begin{aligned}
a_h^*(v_h, w_h) &= a_h(v_h, w_h) + \sum_{T \in \mathcal{T}_h'} \left[\int_{\partial \tilde{T}^{*+}} \left(\beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} - \beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} \right) ds \right. \\
&\qquad \left. + \int_{\partial \tilde{T}^{*+}} \left(\beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} - \beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} \right) ds \right]. \qquad (4.55)
\end{aligned}
$$

By definition,

$$
\beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} - \beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} = 0,
$$

on $\overline{DE}$. Thus, from the proof of Lemma 2.2,

$$
\begin{aligned}
\int_{\partial \tilde{T}^{*+}} \left(\beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} - \beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} \right) ds &= \int_{\partial \tilde{T}^{*+} \cap \Gamma} \left(\beta^- w_h^{*-} \frac{\partial v_h^{*-}}{\partial n} - \beta^+ w_h^{*+} \frac{\partial v_h^{*+}}{\partial n} \right) ds \\
&= \int_{\partial \tilde{T}^{*+} \cap \Gamma} \left[\beta^- (w_h^{*-} - w_h^{*-}(\tilde{D})) \frac{\partial v_h^{*-}}{\partial n} \right. \\
&\qquad \left. + \beta^+ (w_h^{*+}(\tilde{D}) - w_h^{*+}) \frac{\partial v_h^{*+}}{\partial n} \right] ds,
\end{aligned}
$$

where $\tilde{D}$ is a point on $\partial \tilde{T}^{*+} \cap \overline{DE}$. Following the same argument as that used for Theorem 3.1, we have

$$
\begin{aligned}
\left| \int_{\partial \tilde{T}^{*+} \cap \Gamma} \beta^- (w_h^{*-} - w_h^{*-}(\tilde{D})) \frac{\partial v_h^{*-}}{\partial n} ds \right| &= \left| \int_{\partial \tilde{T}^{*+} \cap \Gamma} \beta^- (w_h^- - w_h^-(\tilde{D})) \frac{\partial v_h^-}{\partial n} ds \right| \leq Ch \left| w_h^- \right|_{1,T^-} \left| v_h^- \right|_{1,T^-} \\
\left| \int_{\partial \tilde{T}^{*+} \cap \Gamma} \beta^+ (w_h^{*+} - w_h^{*+}(\tilde{D})) \frac{\partial v_h^{*+}}{\partial n} ds \right| &= \left| \int_{\partial \tilde{T}^{*+} \cap \Gamma} \beta^+ (w_h^+ - w_h^+(\tilde{D})) \frac{\partial v_h^+}{\partial n} ds \right| \leq Ch \left| w_h^+ \right|_{1,T^+} \left| v_h^+ \right|_{1,T^+}.
\end{aligned}
$$

Then the result of this lemma is obtained by applying these estimates in (4.55).

■

Finally, we can derive an error estimate for the modified immersed finite element solution $u_h^* \in S_{h0}$ in the following theorem.

Theorem 4.1 *Assume that the solution u of the interface problem is in $C(\Omega) \cap \tilde{H}^2(\Omega)$, and that the condition in Lemma 4.1 is satisfied. Then the modified immersed finite element solution u_h^* has the following estimate for a constant $C > 0$:*

$$
\|u_h^* - u\|_h \leq Ch\big(|u|_{2,\Omega} + \|f\|_{0,\Omega} \big).
$$

Proof. Let v_h be arbitrary function in S_{h0}, then, from Lemma 4.1, we have

$$
\begin{aligned}
\tilde{C} \|u_h^* - v_h\|_h^2 &\leq a_h^*(u_h^* - v_h, u_h^* - v_h) \\
&= a_h(u_h - v_h, u_h^* - v_h) + \big\{ a_h(v_h, u_h^* - v_h) - a_h^*(v_h, u_h^* - v_h) \big\} \\
&\qquad + \big\{ f_h^*(u_h^* - v_h) - f_h(u_h^* - v_h) \big\}.
\end{aligned}
$$

Thus,

$$\tilde{C}\left\|u_h^* - v_h\right\|_h \le C\left\|u_h - v_h\right\|_h + \sup_{w_h \in S_{h0}} \frac{|a_h(v_h, w_h) - a_h^*(v_h, w_h)|}{\|w_h\|_h} + \sup_{w_h \in S_{h0}} \frac{|f_h^*(w_h) - f_h(w_h)|}{\|w_h\|_h}.$$

Taking $v_h = u_h$ in the above and applying Lemma 4.2, 4.3 and Theorem 3.2 lead to

$$\begin{aligned}\tilde{C}\left\|u_h^* - u_h\right\|_h &\le Ch(\|u_h\|_h + \|f\|_{0,\Omega}) \\ &\le Ch(h\,|u_h|_{2,\Omega} + \|u\|_h + \|f\|_{0,\Omega}).\end{aligned}$$

The estimate above and the triangle inequality lead to the result of this theorem.

■

5 Numerical Examples

We now present some numerical results to illustrate features of the immerse finite element method. Because of its simplicity, we only present results obtained by using the modified IFE method based Cartesian partitions to solve the interface value problem defined by (2.5)-(2.8) in the rectangular domain $\Omega = (-1,1) \times (-1,1)$. The interface curve Γ is a circle with radius $r_0 = \pi/6.28$ which separates Ω into two subdomains Ω^- and Ω^+ with

$$\Omega^- = \left\{(x,y) \quad : \quad x^2 + y^2 \le r_0^2\right\}.$$

The boundary condition function $g(x,y)$ and the source term $f(x,y)$ are chosen such that for $\alpha = 3$,

$$u(x,y) = \begin{cases} \dfrac{r^\alpha}{\beta^-}, & \text{if } r \le r_0, \\ \dfrac{r^\alpha}{\beta^-} + \left(\dfrac{1}{\beta^-} - \dfrac{1}{\beta^+}\right) r_0^\alpha, & \text{otherwise}, \end{cases} \tag{5.56}$$

is the exact solution of this interface problem for a coefficient function $\beta(x,y)$ such that

$$\beta(x,y) = \begin{cases} \beta^-, (x,y) \in \Omega^-, \\ \beta^+, (x,y) \in \Omega^+. \end{cases}$$

Here

$$r = \sqrt{x^2 + y^2},$$

the domain Ω and the curve Γ are sketched in Figure 7 together with a typical partition for our numerical results. We will use the following quantities to describe the errors in an IFE solution:

$$e_s(h) = \begin{cases} \|u_h - u\|_0, & \text{when } s = 0, \\ \|u_h - u\|_h, & \text{when } s = 1, \\ \max_{j \in N_h} |u(\mathbf{x}_j) - u(\mathbf{x}_j)|, & \text{when } s = \infty. \end{cases}$$

where N_h is the set of nodal points of the partition. These quantities represent the error in the usual L^2, H^1, and the discrete L^∞ (at the nodal points) norms, respectively.

Table 1 contains actual errors of the IFE solutions with various partition size h for the boundary value problem with the coefficient function:

$$\beta(x,y) = \begin{cases} 1, & (x,y) \in \Omega^-, \\ 2, & (x,y) \in \Omega^+. \end{cases}$$

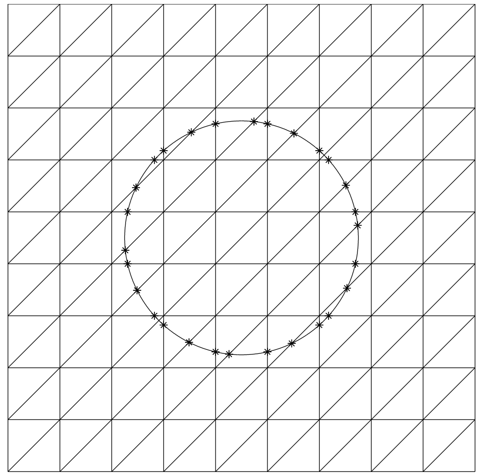

Figure 7: A typical grid for the boundary value problem.

By simple calculations, we can see that the data in this table obey

$$\|u_h - u\|_0 \approx Ch^2,$$
$$\|u_h - u\|_h \approx Ch$$

as predicted by the error estimates in the previous sections. Similar behavior of the IFE solutions can be seen in Table 2 for the boundary value problem with the coefficient function:

$$\beta(x,y) = \begin{cases} 2, & (x,y) \in \Omega^-, \\ 1, & (x,y) \in \Omega^+. \end{cases}$$

The IFE method also works well for the case in which the coefficient function has a large jump, see Table 3. The errors in this group of computations are also within the prediction of the error estimates.

The data in Tables 1 and 2 seem to suggest the second order convergence of the IFE method in the discrete L^∞ norm. However, results in Table 3 indicate that this can not be true for all the cases. The question under what conditions the IFE solution can have a second order convergence in the L^∞ norm is still open.

h	$e_1(h)$	$e_0(h)$	$e_\infty(h)$
1/8	1.2761×10^{-1}	1.0063×10^{-2}	4.6058×10^{-3}
1/16	6.3613×10^{-2}	2.5239×10^{-3}	1.0407×10^{-3}
1/32	3.1786×10^{-2}	6.3079×10^{-4}	2.7979×10^{-4}
1/64	1.5868×10^{-2}	1.5789×10^{-4}	6.4752×10^{-5}
1/128	7.9317×10^{-3}	3.9451×10^{-5}	1.6799×10^{-5}

Table 1: Numerical results for the case when $\beta^- = 1, \beta^+ = 2$.

h	$e_1(h)$	$e_0(h)$	$e_\infty(h)$
1/8	1.7544×10^{-1}	1.9584×10^{-2}	2.3746×10^{-3}
1/16	8.7530×10^{-2}	4.9074×10^{-3}	6.5468×10^{-4}
1/32	4.3729×10^{-2}	1.2292×10^{-3}	1.8158×10^{-4}
1/64	2.1851×10^{-2}	3.0733×10^{-4}	4.3849×10^{-5}
1/128	1.0922×10^{-3}	7.6875×10^{-5}	1.2411×10^{-5}

Table 2: Numerical results for the case when $\beta^- = 2, \beta^+ = 1$.

h	$e_1(h)$	$e_0(h)$	$e_\infty(h)$
1/8	3.1737×10^{-2}	2.7503×10^{-3}	4.1217×10^{-3}
1/16	1.6464×10^{-2}	7.3846×10^{-4}	9.5219×10^{-4}
1/32	8.6090×10^{-3}	1.4499×10^{-3}	5.7889×10^{-4}
1/64	4.2113×10^{-3}	4.3236×10^{-5}	1.0918×10^{-4}
1/128	2.1154×10^{-3}	9.3966×10^{-6}	4.2197×10^{-5}

Table 3: Numerical results for the case when $\beta^- = 1, \beta^+ = 1000$.

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