**On a Nonlinear Beam Equation**

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**ABSTRACT**

Existence and uniqueness of weak solutions to a nonlinear beam equation is established under relaxed assumptions (locally Lipschitz plus affine domination) on the nonlinearity in the stiffness constitutive law. The results provide alternatives to previous theories requiring rather stringent monotonicity assumptions. The techniques and arguments are applicable to a large class of nonlinear second order (in time) partial differential equation systems.

**Keywords:** well posedness, locally Lipschitz nonlinear elastic systems

# 1 Introduction

In this note we consider the nonlinear partial differential equation

$$w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{txxxx} = [g(w_{xx})]_{xx} + f, \tag{1.1}$$

with boundary and initial conditions given by

$$\begin{aligned} &w_x(t,0) = w(t,0) = 0, \quad && w_x(t,1) = w(t,1) = 0, \\ &w(0,\cdot) = w_0 \in H_0^2(0,1), && w_t(0,\cdot) = w_1 \in L^2(0,1). \end{aligned} \tag{1.2}$$

The problem (1.1)-(1.2) with a monotone function $g$ was studied in [2] where the existence and uniqueness of global weak solutions were established. These results were extended in [3] to a general second order evolution system with a monotone nonlinearity. The goal of this paper is to prove existence and uniqueness of weak solutions for (1.1)-(1.2) where the nonlinear function $g$ satisfies only a local Lipschitz condition.

This paper is organized as follows. In Section 2 a local existence-uniqueness result is established for a locally Lipschitz continuous function $g$. Section 3 is devoted to the global existence of weak solutions. Concluding remarks are presented in Section 4.

## 2 Existence and Uniqueness

We begin this section by letting $H = L^2(0,1)$ and $V = H_0^2(0,1)$, so we have the Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ with $V^* = H^{-2}(0,1)$. Denote by $\langle \cdot, \cdot \rangle$ the inner product in $H$, while $\langle \cdot, \cdot \rangle_{V^*,V}$ stands for the usual duality product. Let $||\cdot||, ||\cdot||_V$, and $||\cdot||_{V^*}$ denote the norms of the spaces $H$, $V$, and $V^*$, respectively. Assume that the parameters in (1.1)-(1.2) satisfy the following assumptions:

$(A_g)$ *The nonlinear function $g$ satisfies the following local Lipschitz condition: Let $B_r(0)$* denote the ball of radius $r$ centered at 0 in $H$ and for some constant $L_{B_r}$ we have

$$||g(\xi) - g(\sigma)|| \leq L_{B_r} ||\xi - \sigma|| \text{ for all } \xi, \sigma \in B_r(0). \tag{2.3}$$

$(A_f)$ *The forcing term $f$ satisfies*

$$f \in L^2(0,T;V^*). \tag{2.4}$$

We define the space of solutions to be

$$\mathcal{U}(0,T) = \{u | u \in L^2(0,T;V), u_t \in L^2(0,T;V), u_{tt} \in L^2(0,T;V^*)\}$$

with norm

$$||u||_{\mathcal{U}(0,T)} = (||u||^2_{L^2(0,T;V)} + ||u_t||^2_{L^2(0,T;V)} + ||u_{tt}||^2_{L^2(0,T;V^*)})^{1/2}.$$

We now define the concept of a weak solution to the problem (1.1)-(1.2).

**Definition 2.1** *We say that a function $w \in \mathcal{U}(0,T)$ is a weak solution of (1.1)-(1.2) if it satisfies*

$$\begin{aligned} &\langle w_{tt}(t), \phi \rangle_{V^*,V} + \kappa_1 \langle w_{xx}(t), \phi_{xx} \rangle + \kappa_2 \langle w_{txx}(t), \phi_{xx} \rangle \\ &= \langle g(w_{xx}(t)), \phi_{xx} \rangle + \langle f(t), \phi \rangle_{V^*,V}, \qquad \forall \phi \in V \end{aligned} \tag{2.5}$$

*and*

$$w(0) = w_0 \in V, \qquad w_t(0) = w_1 \in H. \tag{2.6}$$

Let $P$ be the Hilbert space radial retraction onto the ball (in $H$) of radius 1 centered at $w_{0xx}$. Define $\hat{g}(\xi) = g(P\xi)$. Note that from $(A_g)$ it can be seen that $\hat{g}$ satisfies the following global Lipschitz condition:

$$\begin{aligned} ||\hat{g}\,(\xi) - \hat{g}(\sigma)|| &\leq L_{B_{(1+||w_{0xx}||)}} ||P\xi - P\sigma|| \\ &\leq 2L_{B_{(1+||w_{0xx}||)}} ||\xi - \sigma|| \\ &= L||\xi - \sigma||, \qquad \text{for all } \xi, \sigma \in H. \end{aligned} \tag{2.7}$$

Furthermore, from (2.7) it follows that for any $\xi \in H$

$$||\hat{g}(\xi)|| \leq L||\xi|| + C \tag{2.8}$$

for some $C \geq 0$ depending only on $w_{0xx}$ and $g$. Now, consider the following problem:

$$w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{txxxx} = [\hat{g}(w_{xx})]_{xx} + f, \tag{2.9}$$

with boundary and initial conditions given by

$$\begin{aligned} &w_x(t,0) = w(t,0) = 0, \qquad w_x(t,1) = w(t,1) = 0, \\ &w(0,\cdot) = w_0 \in H_0^2(0,1), \quad w_t(0,\cdot) = w_1 \in L^2(0,1). \end{aligned} \tag{2.10}$$

Concerning (2.9)-(2.10) we develop a Galerkin approximation similar to those used in [4, 5]. Let $\{\psi_i\}_{i=1}^{\infty}$ be any linearly independent total subset of $V$. For each $m$, let

$$V^m = \text{span}\{\psi_1, \ldots, \psi_m\}$$

and let $w_0^m, w_1^m \in V^m$ be chosen so that $w_0^m \to w_0$ in $V$, $w_1^m \to w_1$ in $H$ as $m \to \infty$. For each $m$ we define an approximate solution to the problem (2.9)-(2.10) by $w^m\,(t) = \sum_{i=1}^m C_i^m(t)\psi_i$, where $w^m$ is the unique solution to the $m$- dimensional system

$$\begin{aligned} \langle w_{tt}^m(t), \psi_j\rangle_H + \kappa_1\langle w_{xx}^m(t), \psi_{jxx}\rangle + \kappa_2\langle w_{txx}^m(t), \psi_{jxx}\rangle \\ = \langle \hat{g}(w_{xx}^m(t)), \psi_{jxx}\rangle + \langle f(t), \psi_j\rangle_{V^*,V}, \quad j = 1, 2, \ldots, m, \end{aligned} \tag{2.11}$$

with initial conditions

$$w^m(0) = w_0^m, \qquad w_t^m(0) = w_1^m. \tag{2.12}$$

Using arguments in the spirit of those used in [4, 5], we now establish an *a priori* bound for this approximation. Multiply (2.11) by $\frac{d}{dt}C_j^m(t)$ and summing up over $j$ we obtain

$$\begin{aligned} \langle w_{tt}^m(t), w_t^m(t)\rangle_H + \kappa_1\langle w_{xx}^m(t), w_{txx}^m(t)\rangle + \kappa_2\langle w_{txx}^m(t), w_{txx}^m(t)\rangle \\ = \langle \hat{g}(w_{xx}^m(t)), w_{txx}^m(t)\rangle + \langle f(t), w_t^m(t)\rangle_{V^*,V}. \end{aligned}$$

Hence,

$$\frac{d}{dt}\left[\frac{1}{2}\left\|w_t^m(t)\right\|^2+\frac{\kappa_1}{2}\left\|w_{xx}^m(t)\right\|^2\right]+\kappa_2\left\|w_{txx}^m(t)\right\|^2$$
$$=\langle \hat{g}(w_{xx}^m(t)), w_{txx}^m(t)\rangle+\langle f(t), w_t^m(t)\rangle_{V^*,V}.$$

Upon integrating this equality we obtain

$$\begin{aligned}&\left\|w_t^m(t)\right\|^2+\kappa_1\left\|w_{xx}^m(t)\right\|^2+2\,\kappa_2\int_0^t\left\|w_{\tau xx}^m(\tau)\right\|^2 d\tau=\left\|w_1^m\right\|^2+\kappa_1\left\|w_{0xx}^m\right\|^2\\&+2\int_0^t\langle\hat{g}(w_{xx}^m(\tau)), w_{xx\tau}^m(\tau)\rangle d\tau+2\int_0^t\langle f(\tau), w_\tau^m(\tau)\rangle_{V^*,V} d\tau.\end{aligned} \tag{2.13}$$

Now, using the assumption $(A_f)$, the fourth term on the right side of (2.13) can be bounded as follows:

$$2\int_0^t\langle f(\tau), w_\tau^m(\tau)\rangle_{V^*,V}d\tau\leq\delta\int_0^t\left\|w_{\tau xx}^m(\tau)\right\|^2 d\tau+\frac{1}{\delta}\int_0^t\left\|f(\tau)\right\|_{V^*}^2 d\tau,$$

for any $\delta>0$. Similarly, the third term on the right side of (2.13) satisfies the following estimate:

$$\begin{aligned}2\int_0^t\langle\hat{g}(w_{xx}^m(\tau)), w_{\tau xx}^m(\tau)\rangle d\tau &\leq 2\int_0^t||\hat{g}(w_{xx}^m(\tau))||\,\left\|w_{\tau xx}^m(\tau)\right\| d\tau\\&\leq 2\int_0^t\left(L\,||w_{xx}^m(\tau)||+C\right)\left\|w_{\tau xx}^m(\tau)\right\| d\tau\\&\leq \frac{L^2}{\delta}\int_0^t\left\|w_{xx}^m(\tau)\right\|^2 d\tau+\delta\int_0^t\left\|w_{\tau xx}^m(\tau)\right\|^2 d\tau\\&+\frac{1}{\delta}\int_0^t(C)^2 d\tau+\delta\int_0^t\left\|w_{\tau xx}^m(\tau)\right\|^2 d\tau.\end{aligned}$$

Now choose $\delta=\frac{1}{3}\kappa_2$. Then

$$\left\|w_t^m(t)\right\|^2+\kappa_1\left\|w_{xx}^m(t)\right\|^2+\kappa_2\int_0^t\left\|w_{\tau xx}^m(\tau)\right\|^2 d\tau\leq\frac{3L^2}{\kappa_2}\int_0^t\left\|w_{xx}^m(\tau)\right\|^2 d\tau+\left\|w_1^m\right\|^2$$
$$+\kappa_1\left\|w_{0xx}^m\right\|^2+\frac{3}{\kappa_2}C^2T+\frac{3}{\kappa_2}\left\|f\right\|_{L^2(0,T;V^*)}^2.$$

Recalling that $w_0^m\to w_0$ in $V$, $w_1^m\to w_1$ in $H$ as $m\to\infty$, applying Gronwall inequality we obtain that $\left\|w_{xx}^m(t)\right\|^2$ is bounded. Using this fact we conclude that there exists a positive constant $\tilde{C}$ independent of $m$ such that

$$\left\|w_t^m(t)\right\|^2+\kappa_1\left\|w_{xx}^m(t)\right\|^2+\kappa_2\int_0^t\left\|w_{\tau xx}^m(\tau)\right\|^2 d\tau\leq\tilde{C}. \tag{2.14}$$

We now establish the following theorem. We remark that arguments used to prove Theorem 2.1. below are similar to those used in the linear system case in [6] and the globally Lipschitz nonlinearity case in [7]. However, for completeness we give the necessary details.

**Theorem 2.1** *The problem (2.9)-(2.10) has a unique weak solution.*

**Proof.** From (2.14) it easily follows (see [2, 3]) that there exits a subsequence $\{w^{m_k}\}$ of $\{w^m\}$ and limit functions $w \in W^{1,2}(0,T;V)$ and $\widetilde{g} \in L^2(0,T;H)$ such that

$$w^{m_k} \to w \text{ weakly in } W^{1,2}(0,T;V) \tag{2.15}$$

$$g(w_{xx}^{m_k}(t)) \to \widetilde{g}(t) \text{ weakly in } L^2(0,T;H). \tag{2.16}$$

Arguing as in [2, 4] we can show that for any $\phi \in V$, the limit function satisfies

$$\langle w_{tt}(t), \phi\rangle + \kappa_1\langle w_{xx}(t), \phi_{xx}\rangle + \kappa_2\langle w_{txx}(t), \phi_{xx}\rangle = \langle \widetilde{g}(t), \phi_{xx}\rangle + \langle f(t), \phi\rangle_{V^*,V}, \tag{2.17}$$

with $w(0) = w_0$ and $w_t(0) = w_1$. Hence, to show that the limit function is indeed a weak solution of (2.9)-(2.10), it is left to show that $\widetilde{g}(t) = \hat{g}(w_{xx}(t))$ for a.e. $t \in [0,T]$. Recall that we already proved that $\hat{g}(w_{xx}^m) \to \widetilde{g}$ weakly in $L^2(0,T;H)$ (along a subsequence). Our next goal is to show that this weak convergence is actually a strong one.

Following ideas in [6, page 569], we let $z^m(t) = w^m(t) - w(t)$, where $w^m$ is the unique solution to the finite dimensional system (2.11)-(2.12) and $w$ is the limit function which solves the linear problem (2.17) with $w(0) = w_0$ and $w_t(0) = w_1$. Now, using $w_t^m$ and $w_t$ as test functions in (2.11) and (2.17), respectively and integrating as before, we arrive at the equation:

$$\begin{aligned}
&\|z_t^m(t)\|^2 + \kappa_1\|z_{xx}^m(t)\|^2 + 2\kappa_2\int_0^t \|z_{\tau xx}^m(\tau)\|^2 d\tau = \|w_1^m - w_1\|^2 + \kappa_1\|w_{0xx}^m - w_{0xx}\|^2 \\
&+2\int_0^t \langle(\hat{g}(w_{xx}^m(\tau)) - \widetilde{g}(\tau)), z_{\tau xx}^m(\tau)\rangle d\tau + 2\int_0^t \langle f(\tau), z_\tau^m(\tau)\rangle_{V^*,V} d\tau + X_m(t),
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
X_m(t) &= 2\Big[-\langle w_t(t), w_t^m(t)\rangle - \kappa_1\langle w_{xx}(t), w_{xx}^m(t)\rangle - 2\kappa_2\int_0^t \langle w_{\tau xx}(\tau), w_{\tau xx}^m(\tau)\rangle d\tau \\
&+ \langle w_1, w_1^m\rangle + \kappa_1\langle w_{0xx}, w_{0xx}^m\rangle + \int_0^t \langle \hat{g}(w_{xx}^m(\tau)), w_{\tau xx}(\tau)\rangle d\tau \\
&+ \int_0^t \langle \widetilde{g}(\tau), w_{\tau xx}^m(\tau)\rangle d\tau + 2\int_0^t \langle f(\tau), w_\tau(\tau)\rangle_{V^*,V} d\tau\Big].
\end{aligned}$$

The third term on the right side of (2.18) satisfies the following estimates:

$$\begin{aligned}
2\int_0^t \langle(\hat{g}(w_{xx}^m(\tau)) - \widetilde{g}(\tau)), z_{\tau xx}^m(\tau)\rangle d\tau &= 2\int_0^t \langle(\hat{g}(w_{xx}^m(\tau)) - \hat{g}(w_{xx}(\tau)), z_{\tau xx}^m(\tau)\rangle d\tau \\
&+ 2\int_0^t \langle \hat{g}(w_{xx}(\tau)) - \widetilde{g}(\tau), z_{\tau xx}^m(\tau)\rangle d\tau
\end{aligned}$$

and

$$
\begin{aligned}
2\int_0^t \langle(\hat{g}(w_{xx}^m(\tau)) - \hat{g}(w_{xx}(\tau))), z_{\tau xx}^m(\tau)\rangle d\tau &\le 2\int_0^t \|\hat{g}(w_{xx}^m(\tau)) - \hat{g}(w_{xx}(\tau))\| \, \|z_{\tau xx}^m(\tau)\| \, d\tau \\
&\le 2\int_0^t L\,\|w_{xx}^m(\tau) - w_{xx}(\tau)\| \, \|z_{\tau xx}^m(\tau)\| \, d\tau \\
&\le \frac{L^2}{\delta}\int_0^t \|z_{xx}^m(\tau)\|^2 \, d\tau + \delta\int_0^t \|z_{\tau xx}^m(\tau)\|^2 \, d\tau .
\end{aligned}
$$

From this estimate and (2.18) we obtain the following inequality:

$$
\begin{aligned}
&\|z_t^m(t)\|^2 + \kappa_1 \|z_{xx}^m(t)\|^2 + (2\kappa_2 - \delta)\int_0^t \|z_{\tau xx}^m(\tau)\|^2 \, d\tau \le \frac{L^2}{\delta}\int_0^t \|z_{xx}^m(\tau)\|^2 \, d\tau + \|w_1^m - w_1\|^2 \\
&\quad + \kappa_1 \|w_{0xx}^m - w_{0xx}\|^2 + 2\left|\int_0^t \langle \hat{g}(w_{xx}(\tau)) - \widetilde{g}(\tau), z_{\tau xx}^m(\tau)\rangle d\tau\right| \\
&\quad + 2\left|\int_0^t \langle f(\tau), z_\tau^m(\tau)\rangle_{V^*,V} d\tau\right| + |X_m(t)| \\
&\quad = \frac{L^2}{\delta}\int_0^t \|z_{xx}^m(\tau)\|^2 \, d\tau + \|w_1^m - w_1\|^2 + \kappa_1 \|w_{0xx}^m - w_{0xx}\|^2 + |Y_m(t)| + |X_m(t)| ,
\end{aligned}
$$

where

$$
Y_m(t) = 2\left|\int_0^t \langle \hat{g}(w_{xx}(\tau)) - \widetilde{g}(\tau), z_{\tau xx}^m(\tau)\rangle d\tau\right| + 2\left|\int_0^t \langle f(\tau), z_\tau^m(\tau)\rangle_{V^*,V} d\tau\right| .
$$

Letting $m = m_k$, we clearly have $\|w_1^{m_k} - w_1\|^2 + \kappa_1 \|w_{0xx}^{m_k} - w_{0xx}\|^2 \to 0$. We note that by $z_t^{m_k} = w_t^{m_k} - w_t \to 0$ weakly in $L^2(0,T;V)$ we obtain that $Y_m(t) \to 0$ , and also that $X_m(t) \to 0$ for a.e. $t$ because of the convergences (2.15)-(2.16) and the fact that $w$ satisfies the integrated form of (2.17) with $\phi = w_t$. Ignoring the first and the third term on the left side of the above inequality we see by applying the generalized Gronwall Lemma that for a.e. $t \in [0,T]$, $\|z_{xx}^m(t)\|^2 \to 0$. Hence, from the smoothness properties of $\hat{g}$ inherited from those guaranteed by $(A_g)$ for $g$, we have that $\hat{g}(w_{xx}^{m_k}) \to \hat{g}(w)$ strongly in $L^2(0,T;H)$. Since $\hat{g}(w_{xx}^{m_k}) \to \widetilde{g}$ weakly in $L^2(0,T;H)$ also, we find that $\hat{g}(w_{xx}(t)) = \widetilde{g}(t)$ for a.e. $t \in [0,T]$ and thus the limit function $w$ is a weak solution to (2.9)-(2.10).
We also note that

$$
\begin{aligned}
\phi \to \langle w_{tt}, \phi\rangle_{V^*,V} &= -\int_0^t \langle w_\tau(\tau), \phi_\tau(\tau)\rangle d\tau \\
&= \int_0^t \langle \hat{g}(w_{xx}(\tau)), \phi_{xx}(\tau)\rangle + \langle f(\tau), \phi(\tau)\rangle_{V^*,V} d\tau \\
&- \int_0^t \left(\kappa_1 \langle w_{xx}(\tau), \phi_{xx}(\tau)\rangle + \kappa_2 \langle w_{xx\tau}(\tau), \phi_{xx}(\tau)\rangle\right) d\tau
\end{aligned}
$$

is continuous over $\mathcal{D}(]0,T[;V)$ equipped with the topology of $L^2(0,T;V)$ and thus by density over $L^2(0,T;V)$. (So $w_{tt} \in L^2(0,T;V)^* = L^2(0,T,V^*)$.) Since we have already established that $w \in W^{1,2}(0,T;V)$ we have that $w \in \mathcal{U}(0,T)$. Now by [6, Remark 1. p. 555.] we obtain the additional regularity $w \in C([0,T];V)$ and $w_t \in C([0,T];H)$.

The uniqueness of solutions can be easily derived. Indeed, assume that $w_1$ and $w_2$ are two solutions to our nonlinear problem and define $z = w_1 - w_2$. Then by calculations similar to the ones employed above we deduce that for each $t \in (0,T)$,

$$\|z_t(t)\|^2 + \kappa_1 \|z_{xx}(t)\|^2 + (2\kappa_2 - \delta) \int_0^t \|z_{\tau xx}(\tau)\|^2 d\tau \le \frac{L^2}{\delta} \int_0^t \|z_{xx}(\tau)\|^2 d\tau.$$

Thus dropping the second and the third terms on the left side of the above inequality and applying the Gronwall Lemma, we easily see that $z = 0$ and this completes the proof of Theorem 2.1.

We now prove the existence and uniqueness of weak solutions to our main problem (1.1)-(1.2):

**Theorem 2.2** *There exists a $T^*$ such that the problem (1.1)-(1.2) has a unique weak solution on the interval $[0,T^*]$.*

**Proof.** Since the solution $w$ to the problem (2.9)-(2.10) satisfies $w_{xx}(t)$ is continuous in $t$ we see that there exists a $T^*$ with $0 < T^* \le T$ such that $||w_{xx}(t) - w_{0xx}|| \le 1$ for all $t \in [0,T^*]$. This implies that $Pw_{xx}(t) = w_{xx}(t)$ for all $t \in [0,T^*]$, and therefore $\hat{g}(w_{xx}(t)) = g(w_{xx}(t))$ for all $t \in [0,T^*]$. Hence, $w$ solves (1.1)-(1.2) on $[0,T^*]$. Uniqueness of solutions follows in the same way as in the proof of Theorem 2.1.

# 3 Global Existence

We begin this section by pointing out that in the case that $g$ is globally Lipschitz then the global existence of a unique weak solution is guaranteed by Theorem 2.1. The goal of this section is to show that global existence holds for a class of locally Lipschitz functions as well. To this end, we have the following theorem:

**Theorem 3.1** *Suppose that in addition to $(A_g)$ the nonlinear function also satisfies a boundedness condition:*

$$(A_b) \qquad \|g(\xi)\| \le C_1\|\xi\| + C_2, \quad \xi \in H,$$

*for some constants $C_1, C_2 > 0$. Then the weak solution to the problem (1.1)-(1.2) is global.*

**Proof.** Following the same steps as above, we can define Galerkin approximations $w^m(t) = \sum_{i=1}^m C_i^m(t)\psi_i$ that solve (2.11)-(2.12) with the nonlinear function $g$ instead of $\hat{g}$. By using assumption $(A_b)$ we can develop a similar *a priori* bound:

$$\|w_t^m(t)\|^2 + \kappa_1 \|w_{xx}^m(t)\|^2 + \kappa_2 \int_0^t \|w_{\tau xx}^m(\tau)\|^2 d\tau \le \hat{C}, \tag{3.19}$$

where $\hat{C} = \hat{C}(w_0, w_1, f, T, C_1, C_2)$. Thus convergences (2.15)-(2.16) can be established as above. Additionally, we can argue as in [3, Lemma 5.1 b)] that

$$w_{xx}^m(t) \to w_{xx}(t) \ \text{ weakly in } \ H.$$

We note that the arguments for this are independent of the monotonicity assumptions in [3] and depend only on the *a priori* bound of (2.14) and the general Arzela-Ascoli theorem. Thus by the weak lower semicontinuity of the norm we obtain that

$$\|w_{xx}(t)\|^2 \leq \frac{1}{\kappa_1}\hat{C}.$$

Now the proof can be continued exactly as before, using the locally Lipschitz property of $g$ in the ball $B_{\sqrt{\frac{1}{\kappa_1}\hat{C}}}(0)$ in $H$. Thus we can establish that under the conditions $(A_b), (A_g)$ and $(A_f)$ the problem (1.1)-(1.2) has a unique global weak solution.

For an example of a function which is only locally Lipschitz and satisfies $(A_b)$ consider $g(\xi) = \xi \sin(e^{\xi}) + C$.

# 4 Concluding Remarks

We observe that the arguments and results of the preceding sections readily extend to the general second order system

$$\begin{aligned} & w_{tt} + \mathcal{A}_1 w + \mathcal{A}_2 w_t + \mathcal{N}^* g(\mathcal{N} w) = f \\ & w(0) = w_0, \quad w_t(0) = w_1, \end{aligned}$$

where $\mathcal{A}_1, \mathcal{A}_2, \mathcal{N}$ and $f$ satisfy assumptions A1)-A7) of [3]. That is, for the systems of [3] with $\mathcal{V}_2 = \mathcal{V}$, (i.e., strong damping) the conditions on $g$ of [3] can be replaced by those in Section 2 and 3 of this paper. Our current research efforts are focused on using similar techniques to extend these results to a general system with hysteresis (see [1]). These efforts will be reported in another paper.

Finally, we comment that the results of this note could be obtained using Picard iterate arguments in place of the *a priori* estimates. Indeed, in [1] we combine such arguments with the *a priori* estimates of this note in developing the results for systems with hysteresis (or internal dynamics) investigated in that paper.

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