A NOTE ON PRECONDITIONING NON-SYMMETRIC MATRICES

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In [2] preconditioners for real indefinite matrices of KKT form

\[ A \equiv \begin{pmatrix} A & B^* \\ C & 0 \end{pmatrix} \]

are presented\(^1\). The preconditioners \( P \) are of the following form

\[ \begin{pmatrix} A & B^* \\ 0 & \pm CA^{-1}B^* \end{pmatrix}, \begin{pmatrix} A & B^* \\ C & 2CA^{-1}B^* \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & CA^{-1}B^* \end{pmatrix}. \]

The preconditioned matrices \( P^{-1}A \) have minimal polynomials of degree at most 4. Hence a Krylov subspace method like GMRES applied to a preconditioned linear system with coefficient matrix \( P^{-1}A \) converges in 4 iterations or less, in exact arithmetic.

We extend the preconditioners \( P \) in [2] to general matrices \( A \), by deriving them from LU decompositions of \( A \). As before, the preconditioned matrices \( P^{-1}A \) and \( AP^{-1} \) have minimal polynomials of degree at most four.

Let

\[ A \equiv \begin{pmatrix} A & B^* \\ C & D \end{pmatrix} \]

be a complex, non-singular matrix where the leading principal submatrix \( A \) is non-singular. Let \( S \equiv D - CA^{-1}B^* \) be the Schur complement with respect to \( A \). Since \( A \) is non-singular, so is \( S \). The idea is to factor \( A = LDU \) such that the preconditioned matrix \( L^{-1}A^{-1}U \) has a minimal polynomial of small degree.

Proposition 1 (Extension of Remark 2 in [2]). If

\[ P \equiv \begin{pmatrix} A & B^* \\ 0 & S \end{pmatrix} \]

then\(^2\)

\[ AP^{-1} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}, \]

and \( P^{-1}A \) and \( AP^{-1} \) have the minimal polynomial \( (\lambda - 1)^2 \).

\(^1\) The superscript \( * \) denotes the conjugate transpose.

\(^2\) \( I \) denotes the identity matrix.

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PROPOSITION 2 (Extension of (5) in [2]). If
\[ \mathcal{P} \equiv \begin{pmatrix} A & B^* \\ 0 & -S \end{pmatrix} \]
then
\[ \mathcal{A} \mathcal{P}^{-1} = \begin{pmatrix} I & 0 \\ C \mathcal{A}^{-1} & -I \end{pmatrix}, \]
and \( \mathcal{P}^{-1} \mathcal{A} \mathcal{P}^{-1} \) have the minimal polynomial \( (\lambda - 1)(\lambda + 1) \).

The preconditioned matrix below is the same, up to permutations, as the one in [1, §2.1].

PROPOSITION 3. If
\[ \mathcal{P}_1 \equiv \begin{pmatrix} I & 0 \\ C \mathcal{A}^{-1} & -I \end{pmatrix}, \quad \mathcal{P}_2 \equiv \begin{pmatrix} A & B^* \\ 0 & S \end{pmatrix} \]
then
\[ \mathcal{P}_1^{-1} \mathcal{A} \mathcal{P}_2^{-1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \]

The preconditioned matrix is also similar to \( \mathcal{P}^{-1} \mathcal{A} \mathcal{P}^{-1} \), where
\[ \mathcal{P} \equiv \begin{pmatrix} A & B^* \\ C & D - 2S \end{pmatrix}, \]
which is an extension of the preconditioner in [2, p. 7].

REMARK 1. Extending the preconditioner in [2, Proposition 1] to general matrices gives
\[ \mathcal{P} \equiv \begin{pmatrix} A & -S \\ -S \end{pmatrix}. \]
It can be derived from the scaled LU decomposition \( \mathcal{A} = \mathcal{LUP} \), where
\[ \mathcal{L} \equiv \begin{pmatrix} I & 0 \\ C \mathcal{A}^{-1} & I \end{pmatrix}, \quad \mathcal{U} \equiv \begin{pmatrix} I & -B^* S^{-1} \\ 0 & -I \end{pmatrix}, \quad \mathcal{P} \equiv \begin{pmatrix} A & -S \\ -S \end{pmatrix}. \]
The preconditioned matrix is
\[ \mathcal{T} \equiv \mathcal{A} \mathcal{P}^{-1} = \mathcal{L} \mathcal{U} = \begin{pmatrix} I & -B^* S^{-1} \\ C \mathcal{A}^{-1} & -DS^{-1} \end{pmatrix}. \]
If \( \mathcal{A} \) is of KKT form with \( D = 0 \) then
\[ \mathcal{T}^2 - \mathcal{T} = \begin{pmatrix} -B^* S^{-1} C \mathcal{A}^{-1} & 0 \\ 0 & I \end{pmatrix}. \]
Since \( (\mathcal{T}^2 - \mathcal{T})^2 = \mathcal{T}^2 - \mathcal{T} \), the preconditioned matrix \( \mathcal{T} \) has a minimal polynomial of degree 4.

REFERENCES