A NOTE ON PRECONDITIONING NON-SYMMETRIC MATRICES

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Abstract. The preconditioners for indefinite matrices of KKT form in [M.F. Murphy, G.H. Golub, and A.J. Wathen: A Note on Preconditioning for Indefinite Systems, *SIAM J. Sci. Comput.*, vol. 21, no. 6, pp 1969-1972 (2000)] are extended to general non-symmetric matrices.

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In [2] preconditioners for real indefinite matrices of KKT form

$$\mathcal{A} \equiv \begin{pmatrix} A & B^* \\ C & 0 \end{pmatrix}$$

are presented[1]. The preconditioners $\mathcal{P}$ are of the following form

$$\begin{pmatrix} A & B^* \\ 0 & \pm CA^{-1}B^* \end{pmatrix}, \quad \begin{pmatrix} A & B^* \\ C & 2CA^{-1}B^* \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & CA^{-1}B^* \end{pmatrix}.$$

The preconditioned matrices $\mathcal{P}^{-1}\mathcal{A}$ have minimal polynomials of degree at most 4. Hence a Krylov subspace method like GMRES applied to a preconditioned linear system with coefficient matrix $\mathcal{P}^{-1}\mathcal{A}$ converges in 4 iterations or less, in exact arithmetic.

We extend the preconditioners $\mathcal{P}$ in [2] to general matrices $\mathcal{A}$, by deriving them from LU decompositions of $\mathcal{A}$. As before, the preconditioned matrices $\mathcal{P}^{-1}\mathcal{A}$ and $\mathcal{A}\mathcal{P}^{-1}$ have minimal polynomials of degree at most four.

Let

$$\mathcal{A} \equiv \begin{pmatrix} A & B^* \\ C & D \end{pmatrix}$$

be a complex, non-singular matrix where the leading principal submatrix A is non-singular. Let $S \equiv D - CA^{-1}B^*$ be the Schur complement with respect to A. Since $\mathcal{A}$ is non-singular, so is S. The idea is to factor $\mathcal{A} = \mathcal{L}\mathcal{D}\mathcal{U}$ such that the preconditioned matrix $\mathcal{L}^{-1}\mathcal{A}\mathcal{U}^{-1} = \mathcal{D}$ has a minimal polynomial of small degree.

Proposition 1 (Extension of Remark 2 in [2]). *If*

$$\mathcal{P} \equiv \begin{pmatrix} A & B^* \\ 0 & S \end{pmatrix}$$

then[2]

$$\mathcal{A}\mathcal{P}^{-1} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix},$$

and $\mathcal{P}^{-1}\mathcal{A}$ *and* $\mathcal{A}\mathcal{P}^{-1}$ *have the minimal polynomial* $(\lambda - 1)^2$.

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[1] The superscript * denotes the conjugate transpose.

[2] I denotes the identity matrix.

PROPOSITION 2 (EXTENSION OF (5) IN [2]). *If*

$$\mathcal{P} \equiv \begin{pmatrix} A & B^* \\ 0 & -S \end{pmatrix}$$

then

$$\mathcal{A}\mathcal{P}^{-1} = \begin{pmatrix} I & 0 \\ CA^{-1} & -I \end{pmatrix},$$

and $\mathcal{P}^{-1}\mathcal{A}$ *and* $\mathcal{A}\mathcal{P}^{-1}$ *have the minimal polynomial* $(\lambda - 1)(\lambda + 1)$.

The preconditioned matrix below is the same, up to permutations, as the one in [1, §2.1].

PROPOSITION 3. *If*

$$\mathcal{P}_1 \equiv \begin{pmatrix} I & 0 \\ CA^{-1} & -I \end{pmatrix}, \qquad \mathcal{P}_2 \equiv \begin{pmatrix} A & B^* \\ 0 & S \end{pmatrix}$$

then

$$\mathcal{P}_1^{-1}\mathcal{A}\mathcal{P}_2^{-1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The preconditioned matrix is also similar to $\mathcal{P}^{-1}\mathcal{A}$ and $\mathcal{A}\mathcal{P}^{-1}$, where

$$\mathcal{P} \equiv \begin{pmatrix} A & B^* \\ C & D - 2S \end{pmatrix},$$

which is an extension of the preconditioner in [2, p 7].

REMARK 1. *Extending the preconditioner in [2, Proposition 1] to general matrices gives*

$$\mathcal{P} \equiv \begin{pmatrix} A & \\ & -S \end{pmatrix}.$$

It can be derived from the scaled LU decomposition $\mathcal{A} = \mathcal{L}\mathcal{U}\mathcal{D}$, *where*

$$\mathcal{L} \equiv \begin{pmatrix} I & \\ CA^{-1} & I \end{pmatrix}, \qquad \mathcal{U} \equiv \begin{pmatrix} I & -B^*S^{-1} \\ & -I \end{pmatrix}, \qquad \mathcal{D} \equiv \begin{pmatrix} A & \\ & -S \end{pmatrix}.$$

The preconditioned matrix is

$$\mathcal{T} \equiv \mathcal{A}\mathcal{P}^{-1} = \mathcal{L}\mathcal{U} = \begin{pmatrix} I & -B^*S^{-1} \\ CA^{-1} & -DS^{-1} \end{pmatrix}.$$

If $\mathcal{A}$ *is of KKT form with* $D = 0$ *then*

$$\mathcal{T}^2 - \mathcal{T} = \begin{pmatrix} -B^*S^{-1}CA^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

Since $(\mathcal{T}^2 - \mathcal{T})^2 = \mathcal{T}^2 - \mathcal{T}$, *the preconditioned matrix* $\mathcal{T}$ *has a minimal polynomial of degree 4.*

REFERENCES

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[2] M. MURPHY, G. GOLUB, AND A. WATHEN, *A note on preconditioning for indefinite systems*, SIAM J. Sci. Comput., 21 (2000), pp. 1969–72.