Computational Methods for Nonsmooth Acoustic Systems

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Abstract

We consider numerical techniques for a nonsmooth acoustic pressure system arising in electromagnetic interrogation of dielectric materials. We first describe several formulations of coupled electromagnetic/ acoustic systems that arise naturally when traveling acoustic interfaces are used as reflectors for pulsed microwave input signals for dielectric property and geometry identification. We then develop fully Galerkin schemes in a somewhat nonstandard variational formulation of only the acoustic system with distributional inputs. Sample computational findings using the resulting large algebraic system are given.

1 Introduction to electromagnetic/ acoustic problems

An increasingly important class of electromagnetic inverse problems entails using reflected microwave input impulse responses to characterize material dielectric properties as well as geometry (e.g., see [5]). These reflections satisfy a dissipative form of Maxwell's equations and one goal involves identification of the polarization mechanism represented by a hysteresis term in Maxwell's equations [8], [6], [11], [3]. For a number of polarization mechanisms (including models such as the standard Debye and Lorentz), these systems have the form (again see [5], pages 20-21)

$$\begin{aligned} &\tilde{\epsilon}_r \ddot{E}(t,z) + b\dot{E}(t,z) + hE(t,z) \\ &+ \int_0^t k(t-s,z)E(s,z)\ ds - c^2 E''(t,z) = \mathcal{J}(t,z) \end{aligned} \tag{1}$$

where the hysteresis kernel k is the second time derivative of a polarization susceptibility kernel. For the problems of interest, this kernel is to be identified using the information contained in reflections of the input pulse. As explained in [5], the formulation (1) is a useful conceptual and theoretical formulation for identifying the internal polarization dynamics and thereby characterizing the dielectric material.

In certain classes of electromagnetic interrogation techniques, one may employ a perfectly conductive backing, such as metal, as a reflector for an oncoming electromagnetic wave. However, there are many applications in which this is either undesirable or even impossible, for example in searching for an underground nonmetal object or in detecting a brain tumor. In such cases, it may be possible for a traveling acoustic wave, perhaps even one occurring naturally, to serve as a virtual interface. In [5], the authors describe models and applications for techniques which employ perfectly conductive metal backings and *standing* acoustic waves as reflectors for the electromagnetic waves. In addition, they suggest the possibility of a technique in which a *traveling* acoustic wave might be used as a virtual interface to reflect an oncoming electromagnetic wave.

An essential feature of the aforementioned models is the interaction between the electromagnetic and acoustic waves. This interaction can be modeled in various ways. Here we briefly outline several modeling approaches (not all equivalent) found in the literature. We use interchangeably the notation used by the original authors (e.g., $\ddot{E}$ and $\frac{\partial^2}{\partial t^2}E$ are the same) to facilitate cross-referencing. We begin by noting that in [5] the authors assume that the dielectric material obeys the generalized pressure dependent polarization rule

$$\frac{1}{\epsilon_0}\frac{\partial^2 P}{\partial t^2} = f_0(p)E + f_1(p)\frac{\partial E}{\partial t} + f_2(p)\frac{\partial^2 E}{\partial t^2}$$

and make the simplification

$$f_0(p) = 0, \quad f_1(p) = 0, \quad f_2(p) = \chi_0 + \kappa p(t,z).$$

This reduces the model to

$$\frac{1}{\epsilon_0}\frac{\partial^2 P}{\partial t^2} = (\chi_0 + \kappa p(t,z))\,\frac{\partial^2 E}{\partial t^2}, \tag{2}$$

which is used with standing acoustic waves in both [5] and [1].

As an alternative to (1), Maxwell's equations may be written in the form

$$\tilde{\epsilon}_r\ddot{E}(t,z) + b\dot{E}(t,z) + e\ddot{P}(t,z) - c^2E''(t,z) = \mathcal{J}(t,z). \tag{3}$$

We note that if

$$P(t,z) = \int_0^t g(t-s,z)E(s,z)\ ds$$

for some (sufficiently differentiable) polarization susceptibility kernel g, equations (1) and (3) are equivalent, up to the form of the coefficient functions. We see that the polarization model (2) can be used to replace $\ddot{P}$ in (3) to create a pressure-dependent electromagnetic system (e.g., a coupled electromagnetic/ acoustic system, given the dynamics for p).

In contrast, the authors of [10] begin with the electromagnetic wave equation

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E = 0.$$

We note that in one dimension this is equivalent to (3) with $\tilde{\epsilon}_r = 1$, $b = 0$, and no polarization or source term. They then suggest that a change in pressure will produce a change in the index of refraction; they describe this perturbation in the refraction index in terms of a variation $\delta\epsilon/\epsilon_0$ in the dielectric constant. This leads then to the following equation

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E = \frac{\delta\epsilon/\epsilon_0}{c^2}\frac{\partial^2}{\partial t^2}E. \tag{4}$$

Reduced to one dimension, equation (4) can be written in the form of (3), with $\tilde{\epsilon}_r = 1$, $b = 0$, $\mathfrak{J} = 0$ and

$$e\ddot{P} = \frac{\delta\epsilon}{\epsilon_0}\ddot{E}.$$

The dielectric constant ϵ can be thought of as a function of the pressure and entropy of the system, $\tilde{P}$ and S respectively. Thus

$$\delta\epsilon = \left(\frac{\partial\epsilon}{\partial\tilde{P}}\right)_S \delta\tilde{P} + \left(\frac{\partial\epsilon}{\partial S}\right)_{\tilde{P}} \delta S.$$

If the system is assumed to be at constant entropy, this reduces to

$$\delta\epsilon = \left(\frac{\partial\epsilon}{\partial\tilde{P}}\right)_S \delta\tilde{P}$$

which can then be used in (4) to obtain

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E = \frac{1}{c^2}\frac{1}{\epsilon_0}\left(\frac{\partial\epsilon}{\partial\tilde{P}}\right)_S p\frac{\partial^2}{\partial t^2}E, \tag{5}$$

where $p = \delta\tilde{P}$ is the pressure variation. We note that if $\left(\frac{\partial\epsilon}{\partial\tilde{P}}\right)_S$ is constant, the polarization model in (5) is of the same form as (2) with $\chi_0 = 0$.

An approach similar to that in [10] is found in [4]. The author considers the case where light is scattered due to fluctuations in the dielectric constant and assumes that these fluctuations are the result of fluctuations in thermodynamic variables, such as pressure, within the system. We follow his arguments to present a macroscopic view of the problem. This begins with the assumption that the scattered field $\vec{E}$ is described by the equation (after conversion from gaussian to MKS units)

$$\nabla^2\vec{E} - \frac{n^2}{c^2}\ddot{\vec{E}} = \frac{\epsilon_0}{c^2}\ddot{\vec{P}}, \tag{6}$$

where n is the index of refraction. We note that in one dimension this can be written as

$$\frac{n^2}{c^2}\ddot{E} - E'' = -\frac{\epsilon_0}{c^2}\ddot{P}. \tag{7}$$

We then let $\Delta\epsilon$ be a fluctuation in the dielectric constant and $\Delta\chi$ be a fluctuation in electric susceptibility. Since

$$\epsilon = \epsilon_0(1+\chi),$$

it follows that

$$\Delta\chi = \frac{1}{\epsilon_0}\Delta\epsilon.$$

We next suppose that the polarization due to the fluctuation is given by

$$\vec{P} = \Delta\chi\vec{E}_0 = \frac{1}{\epsilon_0}\Delta\epsilon\vec{E}_0 \tag{8}$$

where $\vec{E}_0$ is the incident optical field.

We further assume that density and temperature, ρ and T, are the independent thermodynamic variables in order to represent the dielectric constant fluctuation as

$$\Delta\epsilon = \left(\frac{\partial\epsilon}{\partial\rho}\right)\Delta\rho + \left(\frac{\partial\epsilon}{\partial T}\right)\Delta T.$$

Under assumption that the dielectric constant has a stronger dependence on density than on temperature [4], we can approximate this relationship by

$$\Delta\epsilon = \left(\frac{\partial\epsilon}{\partial\rho}\right)\Delta\rho. \tag{9}$$

If we then treat the density as dependent on pressure and entropy, p and s (which are now the independent thermodynamic variables), we find that the fluctuation in density can be written

$$\Delta\rho = \left(\frac{\partial\rho}{\partial p}\right)\Delta p + \left(\frac{\partial\rho}{\partial s}\right)\Delta s.$$

Finally since our main interest is the scattering due to variations in acoustic pressure, as opposed to entropy, we neglect the second term and arrive at the relationship

$$\Delta\rho = \left(\frac{\partial\rho}{\partial p}\right)\Delta p. \tag{10}$$

Using relations (9) and (10) in equation (8), we obtain

$$\vec{P} = \frac{1}{\epsilon_0}\frac{\partial \epsilon}{\partial \rho}\frac{\partial \rho}{\partial p}\Delta p \vec{E}_0,$$

which can then be used in equations (6) or (7). We note that this results in an equation very similar to (5).

In our alternate approach, we consider the ideas developed in [2]. Here, the authors propose that the coefficients in the polarization model (Debye, Lorentz, etc.) can be represented as a linear function of pressure. For the Debye polarization model, this leads to the differential equation

$$\dot{P} = -\frac{1}{(\tau_0 + \kappa_\tau p)}P + \frac{(\alpha_0 + \kappa_\alpha p)}{(\tau_0 + \kappa_\tau p)}E.$$

Similarly, the Lorentz polarization model can be expressed as the following second order differential equation

$$\ddot{P} + \frac{1}{(\tau_0 + \kappa_\tau p)}\dot{P} + (\gamma_0 + \kappa_\gamma p)P = (\beta_0 + \kappa_\beta p)E.$$

Either of these two models can be coupled with equation (3) through the $\ddot{P}$ term to describe the electromagnetic/acoustic interaction.

Before any of these interaction models can be employed, however, we must begin assessment of this proposed interrogation technique by investigating impulse generated pressure waves in a heterogeneous medium. In particular, we consider here an acoustic pressure wave initiated by a windowed sine wave impulse traveling through a layered medium and formulate the equations and boundary conditions describing the system. We explore several approaches to solving the problem with the finite element method and settle on a (somewhat nonstandard) fully Galerkin scheme. We then discuss numerical findings obtained with this method.

2 Wave system formulation

We first present the decoupled acoustic system of interest in the problems formulated above. We initially write the equations that describe the behavior of the traveling acoustic wave in a strong form. Then we develop a variational formulation for the system and discuss difficulties that arise while doing so.

We consider the wave equation for acoustic pressure in a material consisting of three homogeneous layers. We assume that in the left and right layers of the material the wave propagates with the same wave speed, but that the wave travels at a different speed in the middle layer. The boundary conditions are given by the input of windowed sine wave at $z = 1$ and a no reflection, or total absorbing, condition at $z = 0$. Since discontinuities (at z_1 and z_2) are present in the propagating medium, we also must introduce interface conditions. We do this by requiring continuity of $p(t, \cdot)$ and $c^2 p'(t, \cdot)$ at $z = z_1$ and $z = z_2$. The continuity on p will be an essential condition while the continuity of c^2p' will be a natural condition in our weak formulation below. A schematic of the geometry is given in Figure 1. We suppose that the system is initially at rest. Then the equations that govern this system are given by

$$\ddot{\tilde{p}} - c^2(z)\tilde{p}'' = 0 \tag{11}$$

$$\begin{array}{ll} \tilde{p}(0,z) = 0 & \tilde{p}(t,1) = f(t) \\ \dot{\tilde{p}}(0,z) = 0 & \dot{\tilde{p}}(t,0) - c(0)\tilde{p}'(t,0) = 0 \end{array}$$

where

$$c(z) = \begin{cases} c_1 & 0 \le z < z_1 \\ c_2 & z_1 \le z \le z_2 \\ c_1 & z_2 < z \le 1, \end{cases} \qquad f(t) = \begin{cases} 0 & 0 \le t \le \tau, t \ge 2\tau \\ \sin(\frac{2\pi}{\tau}(t-\tau)) & \tau < t < 2\tau. \end{cases}$$

for $0 < z_1 \le z_2 \le 1$ and $0 < \tau$.

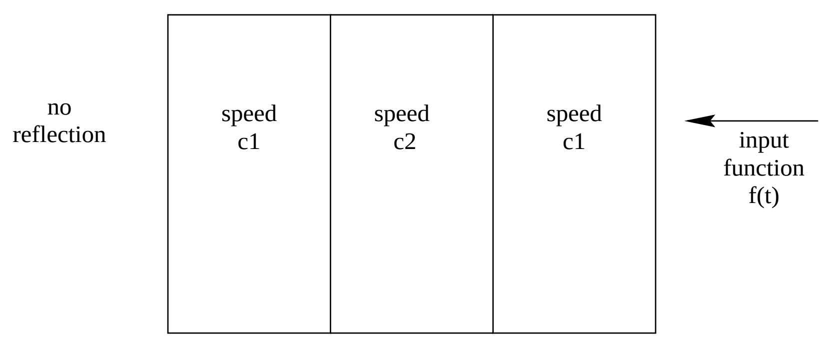

Figure 1 Schematic diagram of geometry

Since finding a solution to the wave equation is normally an easy exercise in solving partial differential equations, computing a numerical solution to this system would appear to be a simple task. However, unique characteristics of this system make solving it a somewhat more challenging task.

To treat the nonhomogeneous time dependent Dirichlet boundary condition at $z = 1$, we make a change of variables which facilitates finite element solutions. To obtain a new equation with a homogeneous boundary condition at $z = 1$, we introduce a new state variable p defined by

$$p(t,z) = \tilde{p}(t,z) - zf(t). \tag{12}$$

In this new state our system is

$$\ddot{p} - c^2(z)p'' + z\ddot{f}(t) = 0 \tag{13}$$

$$\begin{aligned} &p(0,z) = 0 && p(t,1) = 0 \\ &\dot{p}(0,z) = 0 && \dot{p}(t,0) - c_1 p'(t,0) - c_1 f(t) = 0 \\ &p(t,z_1-) = p(t,z_1+) && c^2(z_1-)p'(t,z_1-) = c^2(z_1+)p'(t,z_1+) \\ &p(t,z_2-) = p(t,z_2+) && c^2(z_2-)p'(t,z_2-) = c^2(z_2+)p'(t,z_2+) \end{aligned} \tag{14}$$

where $c(z)$ and $f(t)$ are as defined above. We observe that this change of variable does provide the desired boundary condition at $z = 1$.

Since $c(z)$ is only piecewise continuous in z, we do not expect solutions to the above equation in strong form in space (i.e., C^2 or even only H^2 in z). Therefore, for both theoretical and computational purposes, it is useful to write the system in weak or variational form in the spatial variable. This approach is standard. However, we note that in our change of variables, we have introduced the term $\ddot{f}(t)$ into the wave equation. If we recall that the function f is a windowed sine wave, we realize that its second derivative $\ddot{f}(t)$ includes a delta impulse in time (see Figure 2). One thus observes that we also may not be able to expect solutions in strong form in time. Thus, we may expect distributional derivatives in both time and space.

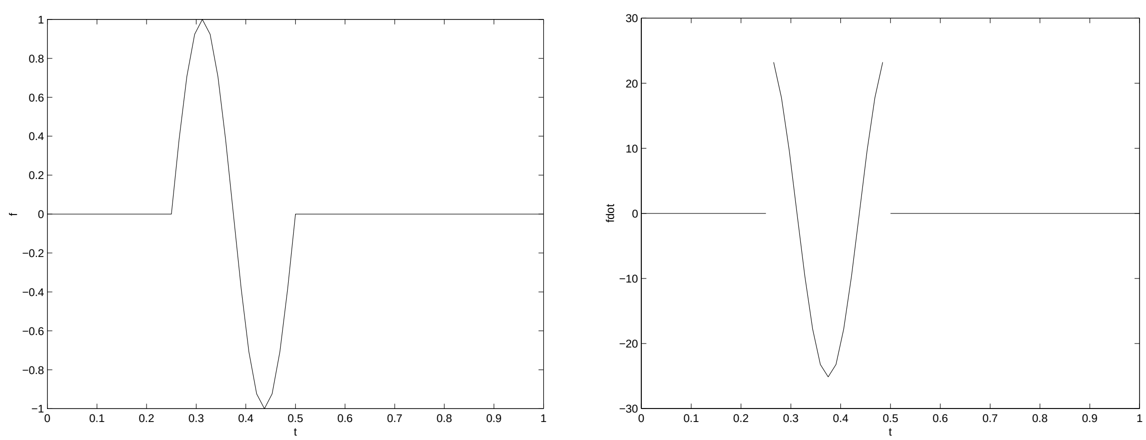

Figure 2(a) f(t) **Figure 2(b)** First derivative of f(t)

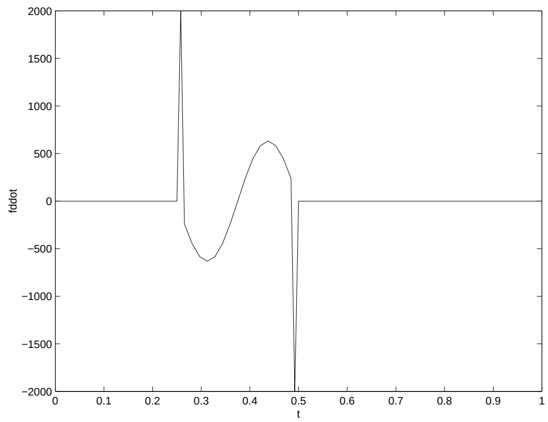

Figure 2(c) Second derivative of f(t)

We tried two different approaches to deal with potential difficulties due to lack of smoothness of solutions. First, we ignored the lack of smoothness of $\ddot{f}$ and proceeded with a standard semi-Galerkin finite element method. Since we know that our solution should be a traveling sine wave (at least if we assume $c_1 = c_2$), it was clear from the resulting simulations that this solution technique was not adequate. For our second approach, we used mollifiers to smooth the "windowing" of the function f and again continued in the traditional way using a standard semi-Galerkin finite element method. However, this approach led to solutions that failed to converge to the known solution. We concluded that an appropriate way to solve the problem might be to use a fully Galerkin finite element scheme. For further discussion of fully Galerkin methods, see, for example, [7], [9].

We let $< \cdot, \cdot >$ denote the usual L^2 inner product on $(0,1)$, i.e., $< f, g >= \int_0^1 f(z)g(z)\ dz$, and we let $< \cdot, \cdot >_{(a,b)}$ denote the L^2 inner product on the specified interval (a,b). We define the spaces $H^1_R(a,b) = \{\phi \in H^1(a,b) | \phi(b) = 0\}$ and $H^1_L(a,b) = \{\phi \in H^1(a,b) | \phi(a) = 0\}$.

We suppose that p satisfies (13), (14) and the following hold:

$$\begin{aligned} &p \in H^1_L(0,T; H^1_R(0,1)) \\ &p(\cdot, z) \in H^2(0,T) \text{ almost everywhere in } (0,1) \\ &p(t,\cdot) \in \tilde{H} \equiv \{\phi \in C(0,1) : \phi \in H^2(\tilde{\Omega})\} \text{ almost everywhere in } (0,T), \\ &\quad \text{where } \tilde{\Omega} \equiv (0,z_1) \cup (z_1,z_2) \cup (z_2,1). \end{aligned} \tag{15}$$

Then

$$\int_0^T < \ddot{p}, \phi > \psi\ dt\ - \int_0^T < c^2(z)p'', \phi > \psi\ dt\ + \int_0^T \ddot{f}\psi\ dt < z, \phi >= 0$$

holds for all $\phi \in H^1_R(0,1)$ and for all $\psi \in H^1_R(0,T)$.

Integrating by parts, we have

$$-\int_0^T < \dot{p},\phi > \dot{\psi}\ dt\ +\int_0^T \left(c_1^2 < p',\phi' >_{(0,z_1)}\ +c_2^2 < p',\phi' >_{(z_1,z_2)}\ +c_1^2 < p',\phi' >_{(z_2,1)}\right)\psi\ dt$$

$$-\int_0^T \dot{f}\dot{\psi}\ dt < z,\phi > + < \dot{p},\phi > \psi|_0^T\ -\ \int_0^T \left(c_1^2 p'\phi|_0^{z_1-} + c_2^2 p'\phi|_{z_1+}^{z_2-} + c_1^2 p'\phi|_{z_2+}^{1}\right)\psi\ dt$$

$$+ < z,\phi > \dot{f}\psi|_0^T = 0.$$

We may then substitute our boundary, interface, and initial conditions (14), as well as the conditions on ϕ and ψ, into the above equation to obtain

$$-\int_0^T < \dot{p},\phi > \dot{\psi}\ dt\ +\int_0^T \left(c_1^2 < p',\phi' >_{(0,z_1)}\ +c_2^2 < p',\phi' >_{(z_1,z_2)}\ +c_1^2 < p',\phi' >_{(z_2,1)}\right)\psi\ dt$$

$$-\int_0^T \dot{f}\dot{\psi}\ dt < z,\phi > +c_1\int_0^T \dot{p}(\cdot,0)\phi(0)\psi\ dt - c_1^2\int_0^T f(\cdot)\phi(0)\psi\ dt = 0.$$

This implies

$$-\int_0^T < \dot{p},\phi > \dot{\psi}\ dt\ +\int_0^T < c^2(z)p',\phi' > \psi\ dt\ -\int_0^T \dot{f}\dot{\psi}\ dt < z,\phi >$$

$$+c_1\int_0^T \dot{p}(\cdot,0)\phi(0)\psi\ dt - c_1^2\int_0^T f(\cdot)\phi(0)\psi\ dt = 0.$$

This suggests that our weak solution with $p(0,z) = 0$ and $p(t,1) = 0$ should satisfy

$$-\int_0^T < \dot{p},\phi > \dot{\psi}\ dt\ +\int_0^T < c^2(z)p',\phi' > \psi\ dt\ -\int_0^T \dot{f}\dot{\psi}\ dt < z,\phi >$$

$$+c_1\int_0^T \dot{p}(\cdot,0)\phi(0)\psi\ dt - c_1^2\int_0^T f(\cdot)\phi(0)\psi\ dt = 0$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$.

Thus, we seek solutions $p \in H_L^1(0,T;V)$, where $V \equiv H_R^1(0,1)$, that satisfy

$$\begin{aligned} &-\int_0^T < \dot{p},\phi > \dot{\psi}\ dt\ +\int_0^T < c^2(z)p',\phi' > \psi\ dt\ -\int_0^T \dot{f}\dot{\psi}\ dt < z,\phi > \\ &+c_1\int_0^T \dot{p}(\cdot,0)\phi(0)\psi\ dt - c_1^2\int_0^T f(\cdot)\phi(0)\psi\ dt = 0 \end{aligned} \tag{16}$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$.

If we assume that our solutions have enough smoothness, i.e., $p(t,\cdot) \in \tilde{H}$ and $p(\cdot,z) \in H^2(0,T)$, we can verify that this is, in fact, a desired weak form of our equation. Assuming this smoothness and integrating the weak form by parts, we find

$$\int_0^T < \ddot{p}, \phi > \psi \; dt - \int_0^T \left(c_1^2 < p'', \phi >_{(0,z_1)} + c_2^2 < p'', \phi >_{(z_1,z_2)} + c_1^2 < p'', \phi >_{(z_2,1)} \right) \psi \; dt$$

$$+ < z, \phi > \int_0^T \ddot{f} \psi \; dt + c_1 \int_0^T \dot{p}(\cdot, 0)\phi(0)\psi \; dt - c_1^2 \int_0^T f(\cdot)\phi(0)\psi \; dt$$

$$- < \dot{p}, \phi > \psi|_0^T - < z, \phi > \dot{f}\psi|_0^T$$

$$+ \int_0^T \left(c_1^2 p'\phi|_0^{z_1-} + c_2^2 p'\phi|_{z_1+}^{z_2-} + c_1^2 p'\phi|_{z_2+}^{1} \right) \psi \; dt = 0$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$ with $p(0,z) = 0$ and $p(t,1) = 0$.

Then

$$\int_0^T < \ddot{p}, \phi > \psi \; dt - \int_0^T < c^2(z)p'', \phi > \psi \; dt + < z, \phi > \int_0^T \ddot{f}\psi \; dt$$

$$+ c_1 \int_0^T \dot{p}(\cdot, 0)\phi(0)\psi \; dt - c_1^2 \int_0^T f(\cdot)\phi(0)\psi \; dt + < \dot{p}(0, \cdot), \phi > \psi(0)$$

$$- \int_0^T c_1^2 p'(\cdot, 0)\phi(0)\psi \; dt \tag{17}$$

$$+ \int_0^T \left[\phi(z_1) \left(c_1^2 p'(\cdot, z_1-) - c_2^2 p'(\cdot, z_1+) \right) - \phi(z_2) \left(c_1^2 p'(\cdot, z_2+) - c_2^2 p'(\cdot, z_2-) \right) \right] \psi \; dt$$

$$= 0$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$ with $p(0,z) = 0$ and $p(t,1) = 0$.

If we choose $\phi \in H_I^1(0,1) = \{\phi \in H^1(0,1) | \phi(0) = \phi(1) = 0, \; \phi(z_1) = \phi(z_2) = 0\} \subset H_R^1(0,1)$ and $\psi \in H_0^1(0,T) = \{\psi \in H^1(0,T) | \psi(0) = \psi(T) = 0\} \subset H_R^1(0,T)$, then we have

$$\int_0^T < \ddot{p}, \phi > \psi \; dt - \int_0^T < c^2(z)p'', \phi > \psi \; dt - < z, \phi > \int_0^T \ddot{f}\psi \; dt = 0$$

for all $\phi \in H_I^1(0,1)$ and for all $\psi \in H_0^1(0,T)$.

Since $\psi \in H_0^1(0,T)$ is arbitrary, this implies that

$$< \ddot{p}, \phi > - < c^2(z)p'', \phi > - \ddot{f} < z, \phi > = 0$$

for all $\phi \in H_I^1(0,1)$.

Hence, since $H_I^1(0,1)$ is dense in $L^2(0,1)$,

$$\ddot{p} - c^2(z)p'' - \ddot{f}z = 0$$

in the L^2 sense and hence almost everywhere.

We thus have that (17) reduces to

$$\begin{aligned} c_1 \int_0^T \dot{p}(\cdot,0)\phi(0)\psi \ dt - c_1^2 \int_0^T f(\cdot)\phi(0)\psi \ dt + < \dot{p}(0,\cdot), \phi > \psi(0) \\ - \int_0^T c_1^2 p'(\cdot,0)\phi(0)\psi \ dt \\ + \int_0^T \left[\phi(z_1)\left(c_1^2 p'(\cdot,z_1-) - c_2^2 p'(\cdot,z_1+)\right) - \phi(z_2)\left(c_1^2 p'(\cdot,z_2+) - c_2^2 p'(\cdot,z_2-)\right)\right]\psi \ dt \\ = 0 \end{aligned} \tag{18}$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$.

If we again choose $\phi \in H_I^1(0,1)$, we have

$$< \dot{p}(0,\cdot), \phi >= 0$$

for all $\phi \in H_I^1(0,1)$, since $\psi(0)$ is arbitrary.

Thus, since $H_I^1(0,1)$ is dense in $L^2(0,1)$,

$$\dot{p}(0,z) = 0$$

almost everywhere in $z \in [0,1]$.

Returning to (18), we have

$$\begin{aligned} c_1 \int_0^T \dot{p}(\cdot,0)\phi(0)\psi \ dt - c_1^2 \int_0^T f(\cdot)\phi(0)\psi \ dt - \int_0^T c_1^2 p'(\cdot,0)\phi(0)\psi \ dt \\ + \int_0^T \left[\phi(z_1)\left(c_1^2 p'(\cdot,z_1-) - c_2^2 p'(\cdot,z_1+)\right) - \phi(z_2)\left(c_1^2 p'(\cdot,z_2+) - c_2^2 p'(\cdot,z_2-)\right)\right]\psi \ dt \\ = 0 \end{aligned} \tag{19}$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$.

We choose $\phi \in \{\phi \in H_R^1(0,1) : \phi(z_1) = \phi(z_2) = 0\}$ and note that since $\phi(0)$ is arbitrary in this set

$$\int_0^T \left(-c_1^2 p'(\cdot,0) + c_1\dot{p}(\cdot,0) - c_1^2 f(\cdot)\right)\psi \ dt = 0$$

for all $\psi \in H_R^1(0,T)$.

Since $H_R^1(0,T)$ is dense in $L^2(0,T)$,

$$\dot{p}(t,0) - c_1 p'(t,0) - c_1 f(t) = 0$$

almost everywhere in $t \in [0, T]$.

Finally, we now have that (19) becomes

$$\int_0^T \left[\phi(z_1)\left(c_1^2 p'(\cdot, z_1-) - c_2^2 p'(\cdot, z_1+)\right) - \phi(z_2)\left(c_1^2 p'(\cdot, z_2+) - c_2^2 p'(\cdot, z_2-)\right)\right] \psi \; dt = 0$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$.

If we choose $\phi \in H_R^1(0,1)$ such that $\phi(z_2) = 0$, then we have

$$\int_0^T \phi(z_1)\left(c_1^2 p'(\cdot, z_1-) - c_2^2 p'(\cdot, z_1+)\right) \psi \; dt = 0$$

for all $\psi \in H_R^1(0,T)$.

Since $\phi(z_1)$ is arbitrary and $H_R^1(0,T)$ is dense in $L^2(0,T)$, we have

$$c_1^2 p'(t, z_1-) = c_2^2 p'(t, z_1+)$$

almost everywhere in $[0,T]$, and by similar arguments

$$c_1^2 p'(t, z_2+) = c_2^2 p'(t, z_2-)$$

almost everywhere in $[0,T]$.

Hence, we have shown that if $p \in H^1(0,T;V)$ (which implies $p(t,\cdot)$ is continuous at z_1 and z_2) satisfies the weak form (16) and p possesses the additional smoothness $p(t,\cdot) \in \tilde{H}$ and $p(\cdot, z) \in H^2(0,T)$, then

$$\ddot{p} - c^2(z)p'' + z\ddot{f}(t) = 0 \text{ almost everywhere}$$

$$\begin{array}{ll}
p(0,z) = 0 & p(t,1) = 0 \\
\dot{p}(0,z) = 0 \text{ almost everywhere} & \dot{p}(t,0) - c_1 p'(t,0) - c_1 f(t) = 0 \text{ almost everywhere} \\
 & \\
p(t, z_1-) = p(t, z_1+) & c^2(z_1-)p'(t, z_1-) = c^2(z_1+)p'(t, z_1+) \text{ almost everywhere} \\
p(t, z_2-) = p(t, z_2+) & c^2(z_2-)p'(t, z_2-) = c^2(z_2+)p'(t, z_2+) \text{ almost everywhere.}
\end{array}$$

Thus the variational form (16) has been verified with the pointwise interface conditions at $z = z_1$ and $z = z_2$ being weakly satisfied whenever p is a solution of (16).

3 An approximate system with computational examples

In view of the weak formulation of the previous section for the system, we develop a dual finite element approximation to the solution. Since we have written the equations weakly in both time and space, it is natural to use a fully (time and space) Galerkin scheme.

Since we seek solutions $p \in H_L^1(0,T;V)$ with $V \equiv H_R^1(0,1)$ that satisfy the weak form, it is also natural to approximate p by a linear combination of piecewise linear basis elements in both time and space. That is,

$$p(t,z) \approx p^{MN}(t,z) = \sum_{i=1}^{M} \sum_{j=0}^{N-1} a_{ij} \psi_i(t) \phi_j(z), \tag{20}$$

where $\psi_i \in H_L^1(0,T)$ and $\phi_j \in H_R^1(0,1)$ are the standard piecewise linear spline functions.

We may substitute this approximation into our weak form to obtain defining equations for the coefficients a_{ij} given by

$$\sum_{i=1}^{M} \sum_{j=0}^{N-1} a_{ij} \left\{ -\int_0^T \dot{\psi}_i \dot{\psi}\ dt < \phi_j, \phi > + \int_0^T \psi_i \psi\ dt < c^2(z)\phi_j', \phi' > + c_1 \int_0^T \dot{\psi}_i \psi\ dt\ \phi_j(0)\phi(0) \right\}$$

$$- \left\{ < z, \phi > \int_0^T \dot{f} \dot{\psi}\ dt + c_1^2 \int_0^T f \psi\ dt\ \phi(0) \right\} = 0$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$. However this results in too many equations, but as usual, one restricts the families of ϕ and ψ for which we require the system to hold. In particular, we require it for $\phi = \phi_l \in H_R^1(0,1), l = 0,1,\ldots,N-1$ and for $\psi = \psi_m(t) \in H_R^1(0,T), m = 0,\ldots,M-1$, where ϕ_l and ψ_m are piecewise linear spline functions. This yields the reduced system of equations

$$\sum_{i=1}^{M} \sum_{j=0}^{N-1} a_{ij} \left\{ -\int_0^T \dot{\psi}_i \dot{\psi}_m\ dt < \phi_j, \phi_l > + \int_0^T \psi_i \psi_m\ dt < c^2(z)\phi_j', \phi_l' > + c_1 \int_0^T \dot{\psi}_i \psi_m\ dt\ \phi_j(0)\phi_l(0) \right\}$$

$$- \left\{ < z, \phi_l > \int_0^T \dot{f} \dot{\psi}_m\ dt + c_1^2 \int_0^T f \psi_m\ dt\ \phi_l(0) \right\} = 0$$

for each $l = 0,\ldots,N-1$, and for each $m = 0,\ldots,M-1$.

For each $j = 0,\ldots,N-1$, for each $i = 1,\ldots,M$, for each $l = 0,\ldots,N-1$, and for each $m = 0,\ldots,M-1$, we define

$$G_{lm}^{ij} = -\int_0^T \dot{\psi}_i \dot{\psi}_m\ dt < \phi_j, \phi_l > + \int_0^T \psi_i \psi_m\ dt < c^2(z)\phi_j', \phi_l' >$$

$$+ c_1 \int_0^T \dot{\psi}_i \psi_m\ dt\ \phi_j(0)\phi_l(0)$$

and for each $l = 0,\ldots,N-1$, and $m = 0,\ldots,M-1$, we define

$$H_{lm} = < z, \phi_l > \int_0^T \dot{f} \dot{\psi_m} \; dt + c_1^2 \int_0^T f \psi_m \; dt \; \phi_l(0).$$

Thus, we can write our algebraic system of defining equations as

$$\sum_{i=1}^{M} \sum_{j=0}^{N-1} a_{ij} G_{lm}^{ij} = H_{lm}$$

for each $l = 0, \ldots, N-1$, $m = 0, \ldots, M-1$.

Next, for all $i = 1, \ldots, M$, we let

$$\vec{a}_i = [a_{i0} \quad a_{i1} \quad \cdots \quad a_{iN-1}],$$

and for all $i = 1, \ldots, M$; $l = 0, \ldots, N-1$; and $m = 0, \ldots, M-1$, we let

$$\vec{G}_{lm}^{i} = \begin{bmatrix} G_{lm}^{i0} \\ G_{lm}^{i1} \\ \vdots \\ G_{lm}^{iN-1} \end{bmatrix}.$$

Then for all $i = 1, \ldots, M$; $l = 0, \ldots, N-1$; and $m = 0, \ldots, M-1$,

$$\sum_{j=0}^{N-1} a_{ij} G_{lm}^{ij} = \vec{a}_i \vec{G}_{lm}^{i}.$$

So, for all $l = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$,

$$\begin{aligned} \sum_{i=1}^{M} \sum_{j=0}^{N-1} a_{ij} G_{lm}^{ij} &= \sum_{i=1}^{M} \vec{a}_i \vec{G}_{lm}^{i} \\ &= \vec{a}_1 \vec{G}_{lm}^{1} + \vec{a}_2 \vec{G}_{lm}^{2} + \ldots + \vec{a}_M \vec{G}_{lm}^{M} \\ &= H_{lm}. \end{aligned}$$

Furthermore, we define

$$\alpha = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_{M-1}]$$

$$\mathcal{G} = \begin{bmatrix} \vec{G}^1_{00} \cdots \vec{G}^1_{N-10} & \vec{G}^1_{01} \cdots \vec{G}^1_{N-11} & \cdots & \vec{G}^1_{0M-1} \cdots \vec{G}^1_{N-1M-1} \\ \vec{G}^2_{00} \cdots \vec{G}^2_{N-10} & \vec{G}^2_{01} \cdots \vec{G}^2_{N-11} & \cdots & \vec{G}^2_{0M-1} \cdots \vec{G}^2_{N-1M-1} \\ & & \vdots & \\ \vec{G}^M_{00} \cdots \vec{G}^M_{N-10} & \vec{G}^M_{01} \cdots \vec{G}^M_{N-11} & \cdots & \vec{G}^M_{0M-1} \cdots \vec{G}^M_{N-1M-1} \end{bmatrix}$$

and

$$\mathcal{H} = [H_{00} \cdots H_{N-10} \quad H_{01} \cdots H_{N-11} \quad \cdots \quad H_{0M-1} \cdots H_{N-1M-1}].$$

Thus we find that our finite element scheme can be written

$$\alpha \mathcal{G} = \mathcal{H} \tag{21}$$

and hence

$$\alpha = \mathcal{H}\mathcal{G}^{-1}.$$

As usual in finite element approximations, if we choose N, M sufficiently large, we expect that by computing α from (21), we can obtain coefficients a_{ij} such that

$$p^{MN}(t,z) = \sum_{i=1}^{M} \sum_{j=0}^{N-1} a_{ij}\psi_i(t)\phi_j(z)$$

is a good approximation for p on the time interval $[0, T]$. Thus the corresponding $\tilde{p}$ defined via (12) sufficiently approximates the behavior of the solution to (11). However, we find that in practice it is difficult to accurately approximate p over a given interval $[0, T_F]$ in one step due to the conditioning of the system (21) whenever T_F is large. (We discuss the details of the implementation later in this section.) Instead we first approximate p over a shorter interval $[0, t_1]$, where $t_1 < T_F$ and the windowed sine wave is *entirely within the material by the time* $t = t_1$. Note that we can change the time T in the weak form to accommodate any interval over which we wish to solve. Then, since we can find a sufficient approximation for p over this smaller interval, we are able to accurately approximate $\tilde{p}$ and describe the pressure over the interval $[0, t_1]$.

In order to determine the behavior of the pressure on the entire given interval $[0, T_F]$, we need to describe the pressure on the interval $(t_1, T_F]$. This is equivalent to considering the original wave equation for pressure with boundary conditions given by a zero input at $z = 1$ and a no reflection condition at $z = 0$ but now initially there is a windowed sine wave already propagating through the material. The equations that govern this system are given by

$$\ddot{\tilde{y}} - c^2(z)\tilde{y}'' = 0$$

$$\begin{array}{ll} \tilde{y}(0,z) = g(z) & \tilde{y}(t,1) = 0 \\ \dot{\tilde{y}}(0,z) = h(z) & \dot{\tilde{y}}(t,0) - c_1\tilde{y}'(t,0) = 0 \\ & \\ \tilde{y}(t,z_1-) = \tilde{y}(t,z_1+) & c^2(z_1-)\tilde{y}'(t,z_1-) = c^2(z_1+)\tilde{y}'(t,z_1+) \\ \tilde{y}(t,z_2-) = \tilde{y}(t,z_2+) & c^2(z_2-)\tilde{y}'(t,z_2-) = c^2(z_2+)\tilde{y}'(t,z_2+) \end{array}$$

where

$$g(z) = \tilde{p}(t,z)|_{t=t_1}, \qquad h(z) = \dot{\tilde{p}}(t,z)|_{t=t_1}.$$

Observe that we have a nonhomogeneous initial condition at $t = 0$. When using a semi-Galerkin finite element scheme, nonhomogeneous initial conditions are of little consequence, but this is not the case for fully Galerkin schemes. To treat this case, it is desirable to make another change of variables. To this end, we let

$$y(t,z) = \tilde{y}(t,z) + (t-1)g(z),$$

and the resulting equations for y are

$$\ddot{y} - c^2(z)y'' + c^2(z)(t-1)g''(z) = 0$$

$$\begin{array}{ll} y(0,z) = 0 & y(t,1) = 0 \\ \dot{y}(0,z) = h(z) + g(z) & \dot{y}(t,0) - c_1 y'(t,0) = 0 \\ & \\ y(t,z_1-) = y(t,z_1+) & c^2(z_1-)y'(t,z_1-) = c^2(z_1+)y'(t,z_1+) \\ y(t,z_2-) = y(t,z_2+) & c^2(z_2-)y'(t,z_2-) = c^2(z_2+)y'(t,z_2+) \end{array}$$

where $c(z)$, $g(z)$, and $h(z)$ are as defined previously, and we use that fact that $g'(0) = g(1) = g(0) = 0$ due to the location of the pressure impulse entirely within the material at $t = t_1$.

Using similar notation and techniques as before, we find that the weak form of our equation is

$$\begin{gathered} -\int_0^T < \dot{y}, \phi > \dot{\psi}\ dt \ + \int_0^T < c^2(z)y', \phi' > \psi\ dt \\ - \int_0^T (t-1)\psi\ dt < c^2(z)g', \phi' > - < h+g, \phi > \psi(0) \ + \int_0^T c_1 \dot{y}(\cdot,0)\phi(0)\psi\ dt = 0 \end{gathered} \tag{22}$$

for all $\phi \in H_R^1(0,1)$ and for all $\psi \in H_R^1(0,T)$ where $T = T_F - t_1$ with $y(0,z) = 0$ and $y(t,1) = 0$.

Thus, we seek solutions $y \in H_L^1(0,T;V)$, where $V \equiv H_R^1(0,1)$, that satisfy (22).

Given the weak form (22) of our equation, we can again approximate y by a linear combination of basis elements in time and space

$$y(t,z) \approx y^{MN}(t,z) = \sum_{i=1}^{M} \sum_{j=0}^{N-1} \gamma_{ij} \psi_i(t) \phi_j(z),$$

where $\psi_i \in H_L^1(0,T)$ and $\phi_j \in H_R^1(0,1)$ are piecewise linear spline functions. We note that in our computations M, N need not be the same as those for the approximation of p on the interval $[0, t_1]$, although we use the same notation here for ease of exposition. Following the same procedure as before, we substitute the approximation into the weak form of our equation, finding that

$$\sum_{i=1}^{M} \sum_{j=0}^{N-1} \gamma_{ij} \left\{ - < \phi_j, \phi_l > \int_0^T \dot{\psi}_i \dot{\psi_m} \ dt + \ < c^2(z) \phi_j', \phi_l' > \int_0^T \psi_i \psi_m \ dt + c_1 \phi_j(0) \phi_l(0) \int_0^T \dot{\psi}_i \psi_m \ dt \right\}$$

$$= \int_0^T (t-1) \psi_m \ dt < c^2(z) g', \phi_l' > + \ < h + g, \phi_l > \psi_m(0)$$

holds for the piecewise linear spline functions $\psi_m \in H_R^1(0,T), \ m = 0, \ldots, M-1$, and $\phi_l \in H_R^1(0,1), \ l = 0, \ldots, N-1$. Then, as before, we can write these equations as a system

$$\Gamma \mathcal{G} = \mathcal{K}$$

where Γ contains the coefficients γ_{ij}, $\mathcal{G}$ is as defined previously, and $\mathcal{K}$ is analogous to the previously defined $\mathcal{H}$. This equation can be solved for the coefficient vector Γ, and the coefficients can in turn be used to determine approximations for y and $\tilde{y}$.

Summarizing, we use

$$\begin{aligned} p(t,z) &\approx p^{M_p N_p}(t,z) = \sum_{i=1}^{M_p} \sum_{j=0}^{N_p-1} a_{ij} \psi_i(t) \phi_j(z) \quad \text{for } z \in [0,1], \quad t \in [0, t_1] \\ y(t,z) &\approx y^{M_y N_y}(t,z) = \sum_{i=1}^{M_y} \sum_{j=0}^{N_y-1} \gamma_{ij} \psi_i(t) \phi_j(z) \quad \text{for } z \in [0,1], \quad t \in [0, T_F - t_1] \end{aligned}$$

and

$$\begin{aligned} \tilde{p}^{M_p N_p}(t,z) &= p^{M_p N_p}(t,z) + z f(t) \quad \text{for } z \in [0,1], \ t \in [0, t_1] \\ \tilde{y}^{M_y N_y}(t,z) &= y^{M_y N_y}(t,z) - (t-1) g(z) \quad \text{for } z \in [0,1], \ t \in [0, T_F - t_1] \\ \tilde{p}^{M_p N_p}(t,z) &= \tilde{y}^{M_y N_y}(t - t_1, z) \quad \text{for } z \in [0,1], \ t \in [t_1, T_F] \end{aligned}$$

to define an appropriate approximation to the behavior of the wave on the spatial interval $[0,1]$ and the entire given time interval $[0, T_F]$.

Prior to presenting some solutions obtained from this approximation, we discuss briefly the implementation of these approximation techniques. The calculations were performed using code written in MATLAB, version 5.3 (The MathWorks, Inc., Natick, MA), and the computations were carried out on a Sun Sparc Ultra 10 workstation. The coefficient vectors α and Γ were computed with MATLAB's *slash* command, which finds solutions by Gaussian elimination. We recall that the elements in $\mathcal{K}$ are

$$\int_0^T (t-1)\psi_m \, dt < c^2(z)g', \phi_l' > + \; < h+g, \phi_l > \psi_m(0).$$

Here, g represents $\tilde{p}^{M_pN_p}(t,\cdot)$ at a specified time t_1. As a result, we only have access to values of g at the nodal points, z_k. Furthermore, $h(z)$ represents $\dot{\tilde{p}}^{M_pN_p}(t,z)$ at t_1, but these values must be approximated at the appropriate spatial nodes. So, in order to compute the terms in $\mathcal{K}$, we must first numerically approximate $\dot{g}$ (which is the same as h) and g' from the known data points and then calculate the inner products. We use a centered difference method to approximate g' at the nodal points and a backward difference method to approximate $\dot{g}$ at the same points. Then we use linear interpolation via the MATLAB command *interp1* to obtain values for $g(z), g'(z)$, and $\dot{g}(z)$ at intermediate values z between the nodes. With these "enhanced" data sets, we can use the trapezoidal method, via MATLAB's *trapz* command, to compute the inner products.

Finally, we show plots to illustrate the behavior of the wave as it passes through the layered medium. Each of the plots in Figure 3 is a snapshot in time of the pressure in the medium for the parameters given in Table 1 with $M = N = 256$ basis elements. Looking at the snapshots sequentially, we can see the pressure wave move through the layers. The noise in front of and behind the wave is a result of approximation error and should not have significant impact on the electromagnetic interrogation process when used in the problem described in Section 1.

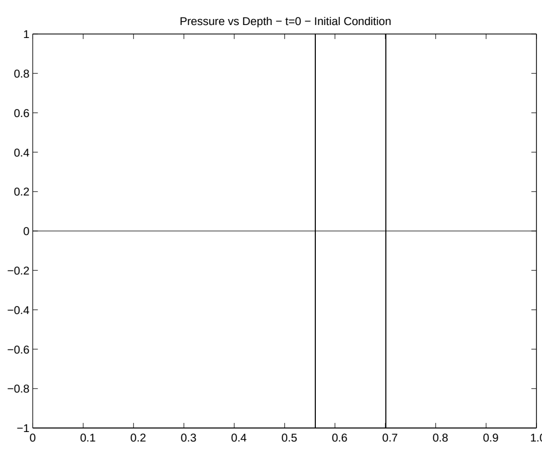

Figure 3(a) Pressure vs depth – t=0

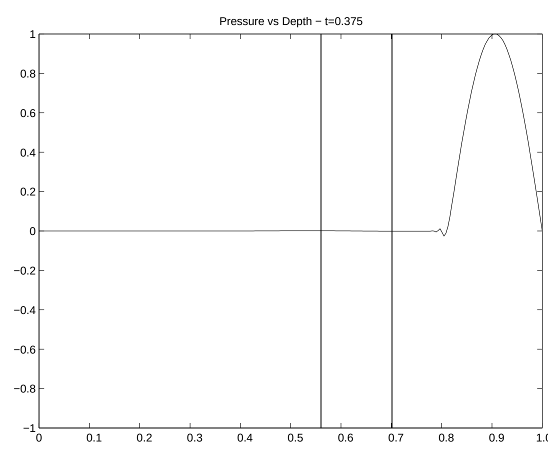

Figure 3(b) Pressure vs depth – t=0.375

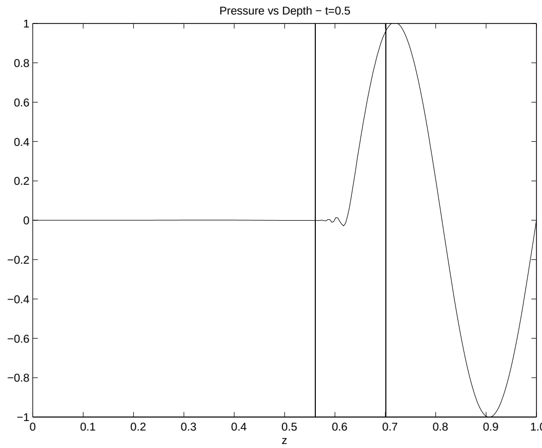

Figure 3(c) Pressure vs depth – t=0.5

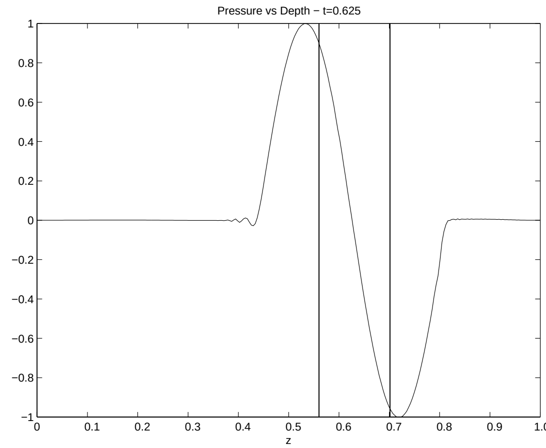

Figure 3(d) Pressure vs depth – t=0.625

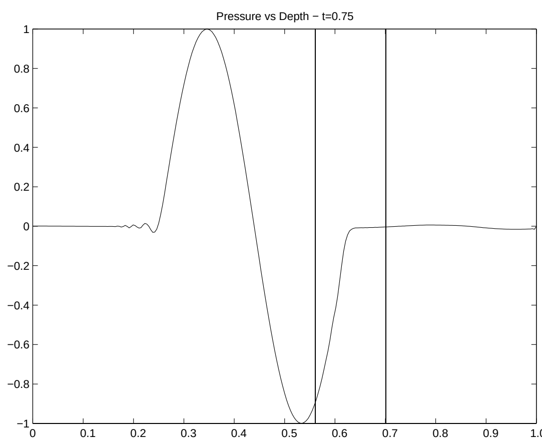

Figure 3(e) Pressure vs Depth – t=0.75

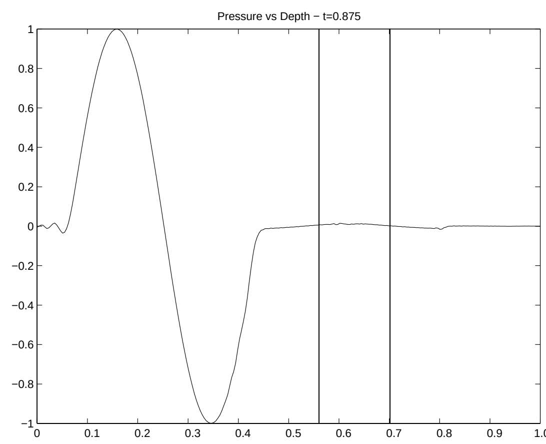

Figure 3(f) Pressure vs depth – t=0.875

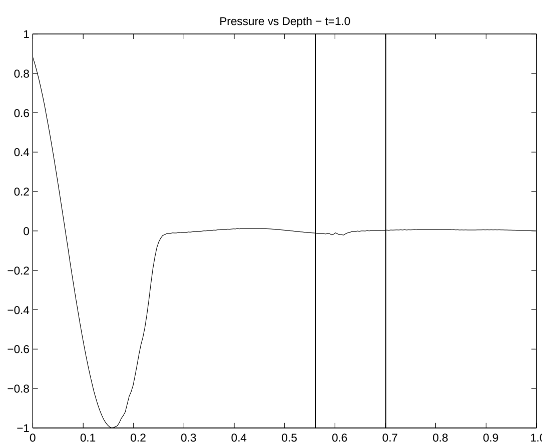

Figure 3(g) Pressure vs depth – t=1.0

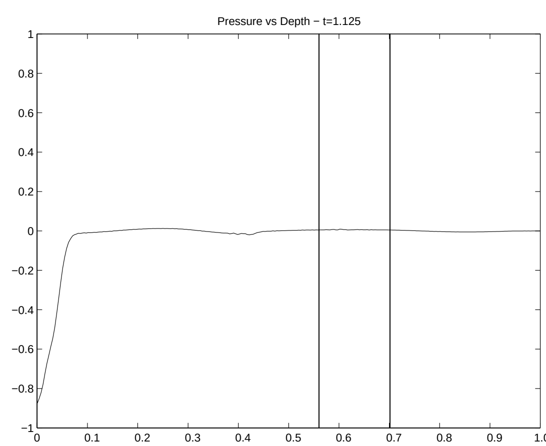

Figure 3(h) Pressure vs depth – t=1.125

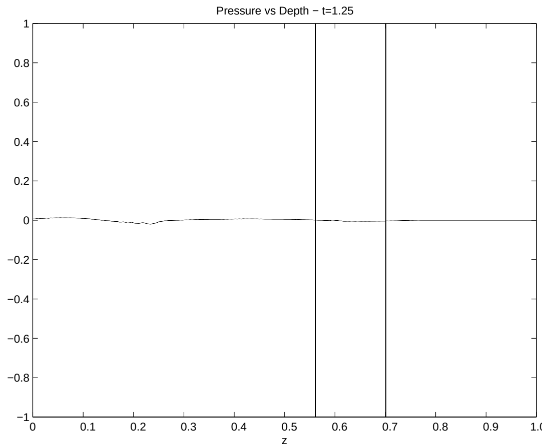

Figure 3(i) Pressure vs depth – t=1.25

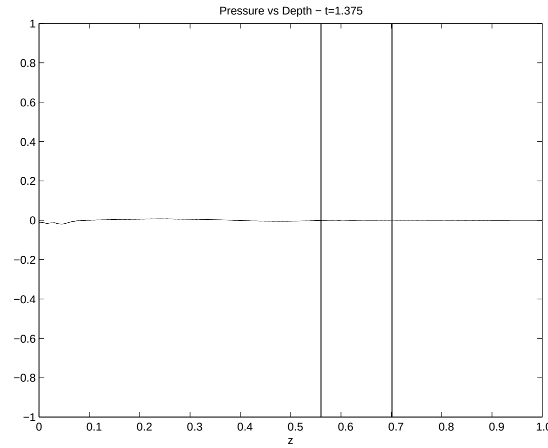

Figure 3(j) Pressure vs depth – t=1.375

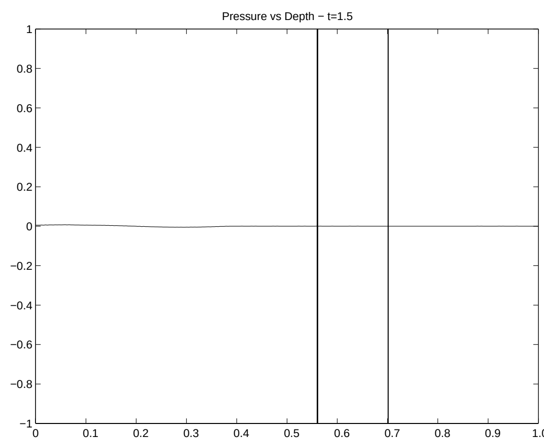

Figure 3(k) Pressure vs depth – t=1.5

Parameter	Value
c_1	1.5
c_2	1.485
z_1	0.5605
z_2	0.7012
τ	0.25
t_1	0.5
T_F	1.5

Table 1 Parameter values for computations in Figure 3

To address approximation error, we compare solutions as the number of basis elements increases. In Figure 4, we see the solutions of pressure versus depth for a fixed time computed with varying number of basis elements. We see that as the number of basis elements increases from $N = M = 64$

(denoted by $\circ$) to $N = M = 128$ (denoted by $\times$) to $N = M = 256$ (denoted by $*$), the solutions appear to converge. Figure 5 gives the corresponding plots for solutions of pressure versus time at a fixed depth. Again, we see apparent convergence as the number of basis elements increases. This suggests that any error in the approximate solution is due to approximation error. The values of parameters used in these computations are given in Table 2. Moreover, Table 3 illustrates the convergence in norm we see as we increase the number of basis elements. The norms used to obtain the results given in the table are defined as follows

$$|f - g|_{l^\infty} = sup_k sup_l |f(t_k, z_l) - g(t_k, z_l)|$$

$$|f - g|_{l^2} = \left(\frac{1}{N_k N_l} \sum_k \sum_l (f(t_k, z_l) - g(t_k, z_l))^2\right)^{\frac{1}{2}}$$

where (t_k, z_l) are the nodal points of the piecewise linear elements $\{\psi_i\}, \{\phi_j\}$ of the approximation (20) and N_k, N_l are the number of nodal points.

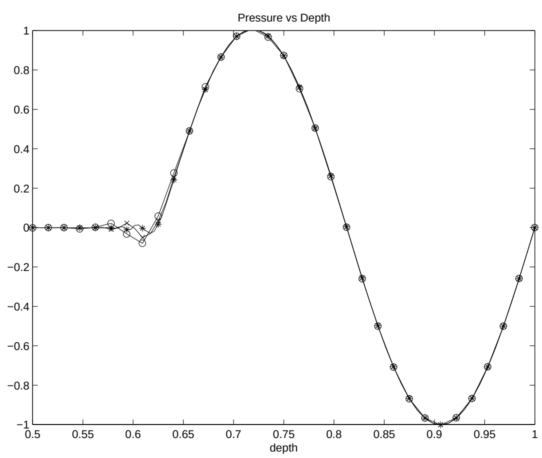

Figure 4(a) Convergence of elements in depth

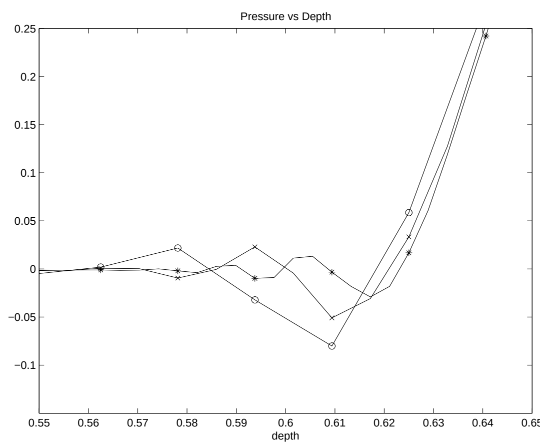

Figure 4(b) A close-up of Figure 4(a)

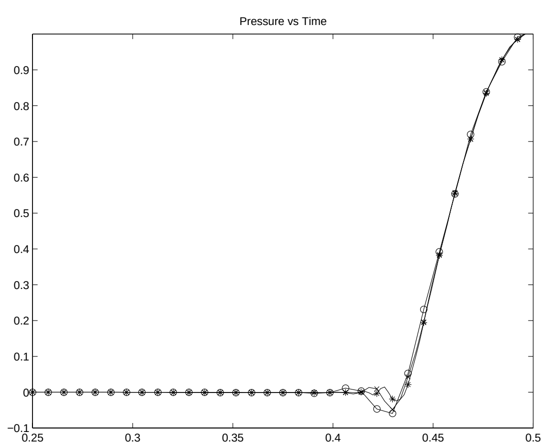

Figure 5(a) Convergence of elements in time

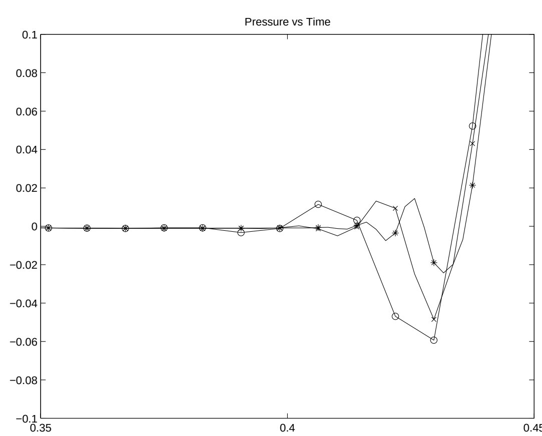

Figure 5(b) A close-up of Figure 5(a)

Parameter	Value
c_1	1.5
c_2	1.485
z_1	0.5605
z_2	0.7480
τ	0.25
t_1	0.5
T_F	0.5

Table 2 Parameter values for computations in Figures 4 and 5

Norm	Value	Norm	Value
$\|\tilde{p}^{32,32} - \tilde{p}^{16,16}\|_{l^\infty}$	0.119126	$\|\tilde{p}^{32,32} - \tilde{p}^{16,16}\|_{l^2}$	0.048853
$\|\tilde{p}^{64,64} - \tilde{p}^{32,32}\|_{l^\infty}$	0.080170	$\|\tilde{p}^{64,64} - \tilde{p}^{32,32}\|_{l^2}$	0.032970
$\|\tilde{p}^{128,128} - \tilde{p}^{64,64}\|_{l^\infty}$	0.069472	$\|\tilde{p}^{128,128} - \tilde{p}^{64,64}\|_{l^2}$	0.025776
$\|\tilde{p}^{256,256} - \tilde{p}^{128,128}\|_{l^\infty}$	0.048119	$\|\tilde{p}^{256,256} - \tilde{p}^{128,128}\|_{l^2}$	0.021199

Table 3 Convergence in norm

4 Conclusions and future work

We have derived and tested numerically an adequate approximation method for the behavior of the pressure wave. The next task is to couple the acoustic system with the appropriate electromagnetic system to develop theory and computation for the interrogation technique described in Section 1. This will involve deriving a model similar to those outlined in Section 1 for the interaction between the two waves.

In this paper, we have developed an approximation scheme for the acoustic wave equations. Questions of the well-posedness of the decoupled pressure system remain to be addressed. Since the variational form of the equation is weak in both time and space, this is a non-trivial issue and will be treated in a forthcoming paper. Moreover, a theoretical and computational treatment of the coupled electromagnetic/ acoustic interrogation problem must be pursued.

5 Acknowledgements

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