A Well-posedness Result for a Shear Wave Propagation Model

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Abstract

We consider a nonlinear model for propagation of shear waves in viscoelastic tissue. Existence and uniqueness results for solutions are established.

1 Introduction

In this note we examine the well-posedness of a one-dimensional shear wave propagation model that arises in inverse problems related to the detection and characterization of cardiac artery stenoses. In a previous paper [6], we defined a basic model to emulate shear waves propagating from a coronary stenosis through a homogeneous, soft-tissue like medium. The medium is considered viscoelastic, and the model uses internal strain variables (see, e.g., [1], [3], [4], or [5]) to capture the nonlinear stress-strain relationship. An idealized geometry (based on experimental protocols to test piezoceramic based surface sensors) is depicted in Figure 1.

As outlined in [1], the evolution equation for one-dimensional shear wave propagation through a homogeneous, viscoelastic medium is

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial}{\partial x}\sigma(t,x) = F(x,t), \qquad R_1 < x < R_2, \tag{1}$$

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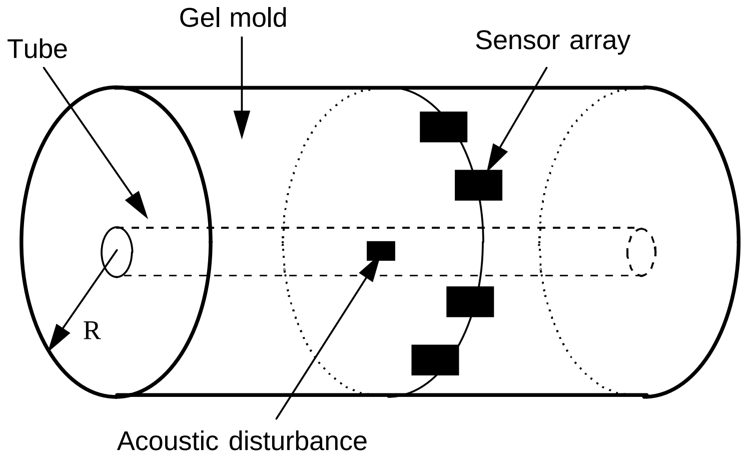

Figure 1: The 1D homogeneous viscoelastic model.

where u represents the shear displacement, σ represents the shear stress, and F represents a body forcing term. For boundary conditions a pure shearing force on the left boundary and a free surface on the right boundary were assumed; hence,

$$\sigma(t, R_1) = f(t), \qquad \sigma(t, R_2) = 0. \tag{2}$$

The initial conditions were $u(0,x) = u_0(x)$, and $u_t(0,x) = u_1(x)$.

The focus of [1] concerns the choice of an effective constitutive equation for this model. In that paper, the authors investigated internal strain variable models as alternatives to the computationally intensive quasi-linear viscoelastic model proposed by Fung [9]. Specifically, they assumed the stress is given as a sum of internal strain variables,

$$\sigma(t) = \sum_{j=1}^{N} \epsilon_j(t). \tag{3}$$

The dynamics of each internal strain variable is modeled dynamically as

$$\frac{d\epsilon_j(t)}{dt} = -\nu_j \epsilon_j + C_j \frac{d\sigma_e}{dt}(u_x(t)), \quad \epsilon_j(0,x) = 0, \qquad j = 1, \ldots, N, \tag{4}$$

where σ_e is the elastic response function defined in ([9],§7), and may be given as

$$\sigma_e(u_x(t,x)) = \gamma + \beta e^{\alpha u_x} \tag{5}$$

where the choices $\gamma = 0$ and $\gamma = -\beta$ are both admissible. We remark that this formulation, with linear internal strain variable models, is equivalent to Fung's formulation with a sum of exponential terms approximating the relaxation function. More generally however, the internal strain variables might be modeled by nonlinear dynamics of the form

$$\frac{d\epsilon_j(t)}{dt} = g_j(\epsilon_j(t)) + C_j \frac{d\sigma_e}{dt}(u_x(t)), \quad \epsilon_j(0,x) = 0, \quad j = 1, \ldots, N. \tag{6}$$

All of these models correspond to a viscoelastic body under either loading or unloading. Each constitutive equation expresses the stress nonlinearly in terms of the infinitesimal strain u_x.

The authors of [1] investigated three particular internal strain variable models as constitutive equations: a one linear internal strain variable model ($\sigma = \epsilon_1$), a two linear internal strain variable model ($\sigma = \epsilon_1 + \epsilon_2$), and one piece-wise linear internal strain variable model. Numerical experiments verified the effectiveness of the internal strain variable models and demonstrated good agreement with simulated data in the case of the two linear internal strain variable model.

In this note, we focus, for simplicity, on theoretical foundations for the one linear internal strain variable formulation. The case of multiple linear internal strain variables is readily treated in the same way. The shear stress, σ, is given by the equation

$$\sigma = \epsilon_1 + C_D u_{tx},$$

where we have added a Kelvin-Voigt damping term with $C_D > 0$ as a first approximation to damping present in viscoelastic materials. The terms ϵ_1 and σ_e are assumed given by

$$\frac{d}{dt}\epsilon_1 + \nu\epsilon_1 = c\frac{d}{dt}\sigma_e(u_x(t,x)), \quad \epsilon_1(0,x) = 0 \tag{7}$$

$$\sigma_e(u_x(t,x)) = \gamma + \beta e^{\alpha u_x}. \tag{8}$$

We then analyze the system

$$u_{tt} - C_D u_{txx} - \epsilon_{1x} = F \qquad \text{in } V^* \tag{9}$$

$$u(0,x) = u_0 \in V \tag{10}$$

$$u_t(0,x) = u_1 \in H, \tag{11}$$

where $H = L^2(\Omega)$, and $V = H^1(\Omega)$, and $\Omega = [R_1, R_2]$. The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. All other norms and inner products will be specifically indicated. Note that V is compactly embedded in H, and H is continuously embedded in V^*, the dual space of V.

The organization of this paper is as follows. We first define weak solutions to system (7)-(11), and list some assumptions in Section 2. In Section 3 we develop the Galerkin approximation, utilizing several lemmas, and in Section 4 we establish the existence and uniqueness of both local and global weak solutions. This work adapts the techniques of [1] and [2] to our system with linear internal strain variables but with nonlinear stress-strain interaction.

2 Preliminaries

We will interpret system (7)-(11) in the V^* sense. In developing a general theory, we will at various times invoke several from among the following assumptions:

- (AF) The forcing term satisfies $F \in L^2(0,T;V^*)$
- (Af) The inner boundary condition satisfies $f \in L^2(0,T)$
- (AL) The elastic response function σ_e satisfies a local Lipschitz condition,

  $$\|\sigma_e(u) - \sigma_e(v)\| \leq L_{B_r}\|u - v\|$$

  for some positive constant L_{B_r} and for all u, v in $B_H(0,r)$, the ball in H centered at 0 of radius r.
- (AG) There exists constants C_1 and C_2 such that

  $$\|\sigma_e(u)\| \leq C_1\|u\| + C_2$$

  for every $u \in H$.

Note that assumption (AL) can be verified for σ_e in [1] (i.e., σ_e given in (5) above) by first computing $\frac{d}{dt}\sigma_e(tu+(1-t)v)$, integrating with respect to t over the interval $[0,1]$, then taking the norm of both sides. Assumption (AG) is a physical bound on the growth of σ_e prior to rupture. It is satisfied by a modified version of σ_e in (5), call it $\tilde{\sigma}_e$, in which $\tilde{\sigma}_e$ accounts for saturation before rupture and agrees with σ_e up to this saturation. However, (AG) is not satisfied by the σ_e of (5) as it is defined there.

We have the following definition of weak solutions for the one linear internal strain variable system (7)-(11).

Definition 2.1 *Let* $\mathcal{L}_T = \{w : [0,T] \to H : w \in C_W([0,T];V) \cap L^2([0,T];V)$ *and* $w_t \in C_W([0,T];H) \cap L^2(0,T;V)\}$. *We define* $u \in \mathcal{L}_T$ *to be a weak solution of system (7)-(11) if it satisfies*

$$\begin{aligned}\int_0^t [-(u_s(s),\eta_s(s)) + C_D(u_{sx}(s),\eta_x(s)) + (\epsilon_1(u_x(s)),\eta_x(s))]\, ds \\ +(u_t(t),\eta(t)) - (u_1,\eta(0)) \\ = \int_0^t [< F(s),\eta(s) >_{V^*,V} - f(s)\eta(s,R_1)]\, ds \end{aligned} \tag{12}$$

for any $t \in [0,T]$ *and* $\eta \in \mathcal{L}_T$, *with the initial conditions* $u_0 \in V$, $u_1 \in H$, *and*

$$\epsilon_1(u_x(t,x)) = c\left\{\sigma_e(u_x(t,x)) - e^{-\nu t}\sigma_e(u_{0x}) - \int_0^t \nu e^{-\nu(t-s)}\sigma_e(u_x(s,x))ds\right\}. \tag{13}$$

Note this notion of weak solution for system (7)-(11) agrees with the usual one in that it yields $u_{tt} \in L^2([0,T];V^*)$ with equation (9) holding in the sense of $L^2([0,T];V^*)$. Here, $C_W([0,T];V)$ refers to the set of weakly continuous functions in V on $[0,T]$.

We first establish existence of local weak solutions under only the assumptions (AF), (Af), and (AL). To deal with the nonlinear elastic response term, we first define the operator P as the radial retraction from the space H onto the ball $B_H(u_{0x},1)$ of radius 1 centered at u_{0x}. Then we define a new elastic response function $\hat{\sigma}_e$ by

$$\hat{\sigma}_e(u) = \sigma_e(Pu), \qquad \forall u \in H.$$

Thus, from assumption (AL), one can easily argue the global conditions:

- (AL2) $\|\hat{\sigma}_e(u) - \hat{\sigma}_e(v)\| \le L_{B_{1+\|u_{0x}\|}}\|u-v\|$ for all $u,v \in H$,
- (AG2) $\|\hat{\sigma}_e(u)\| \le C_1\|u\| + C_2$ for all $u \in H$.

We also define a modified internal strain $\hat{\epsilon}_1$ as follows

$$\hat{\epsilon}_1(u_x(t,x)) = c\left\{\hat{\sigma}_e(u_x(t,x)) - e^{-\nu t}\hat{\sigma}_e(u_{0x}) - \int_0^t \nu e^{-\nu(t-s)}\hat{\sigma}_e(u_x(s,x))ds\right\}. \tag{14}$$

3 Galerkin Approximation

We begin our arguments by developing the standard Galerkin approximation and establishing *a priori* bounds for them. Let $\{\psi_i\}$ be any linearly independent total subset of V. For each m, let $V^m = span\{\psi_1, \psi_2, \ldots, \psi_m\}$. Choose $\{u_0^m\}$ and $\{u_1^m\} \in V^m$ such that $u_0^m \to u_0$ in V and $u_1^m \to u_1$ in H, and let M_0 and M_1 be constants such that

$$\|u_0^m\|_V \leq M_0 \qquad \text{and} \qquad \|u_1^m\| \leq M_1. \tag{15}$$

We define the Galerkin approximation $u^m(t) = \sum_{k=1}^m a_k^m(t)\psi_k$ as the unique solution of the following m-dimensional integro-differential system

$$\begin{aligned} < u_{tt}^m, \psi_j >_{V^*,V} + C_D(u_{tx}^m, \psi_{jx}) + c(\hat{\epsilon}_1(u_x^m), \psi_{jx}) \quad &= \quad < F(t), \psi_j >_{V^*,V} \\ &\quad - f(t)\psi_j(R_1) \end{aligned} \tag{16}$$

for $j = 1, \ldots, m$ on the interval $[0, T]$ for some $T > 0$. We have the following *a priori* estimate for the Galerkin approximation.

Lemma 3.1 *Let $u^m(t)$ be the Galerkin approximation on $[0, T]$. There exists a constant $K > 0$ such that*

$$\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + C_D \int_0^t \|u_{sx}^m(s)\|^2 ds \leq K \tag{17}$$

where K depends on the problem data (i.e., u_0, u_1, f, F, c, α, β, ν, C_D, and T), but is independent of m.

Proof. To obtain the *a priori* estimate we multiply (16) by $\frac{d}{dt}a_j(t)$ and sum for $j = 1, \ldots, m$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_t^m\|^2 + C_D\|u_{tx}^m\|^2 =& < F(t), u_t^m >_{V^*,V} - f(t)u_t^m(R_1) - c(\hat{\sigma}_e(u_x^m), u_{tx}^m) \\ &+ ce^{-\nu t}(\hat{\sigma}_e(u_{0x}^m), u_{tx}^m) + c(\int_0^t \nu e^{-\nu(t-s)}\hat{\sigma}_e(u_x^m(s))ds, u_{tx}^m). \end{aligned}$$

Adding (u_x^m, u_{tx}^m) to each side of the above equation, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_t^m\|^2 + \frac{1}{2}\frac{d}{dt}\|u_x^m\|^2 + C_D\|u_{tx}^m\|^2 \leq& \, | < F(t), u_t^m >_{V^*,V} | + |f(t)u_t^m(R_1)| \\ &+ |(c\hat{\sigma}_e(u_x^m) + u_x^m, u_{tx}^m)| + c|(\hat{\sigma}_e(u_{0x}^m), u_{tx}^m)| \\ &+ c|(\int_0^t \nu e^{-\nu(t-s)}\hat{\sigma}_e(u_x^m(s))ds, u_{tx}^m)|. \end{aligned}$$

We next bound each term on the right side using assumptions (AL2) and (AG2), estimates (15), the embedding $H^1(\Omega) \hookrightarrow C(\Omega)$, and standard inequalities to obtain

$$\frac{d}{dt}(\|u_t^m\|^2 + \|u_x^m\|^2) + C_D\|u_{tx}^m\|^2 \leq C_D\|u_t^m\|^2 + 10(cC_1+1)^2C_D^{-1}\|u_x^m\|^2$$
$$+5C_D^{-1}\|F(t)\|_{V^*}^2 + 5c_3^2C_D^{-1}|f(t)|^2 + 10c^2C_2^2C_D^{-1}$$
$$+5c^2C_D^{-1}(C_1M_0+C_2)^2 + 5(c\nu)^2C_D^{-1}(C_1(M_0+1)+C_2)^2T^2. \quad (18)$$

Integrating from 0 to t, we obtain

$$\|u_t^m\|^2 + \|u_x^m\|^2 + C_D\int_0^t \|u_{sx}^m(s)\|^2\ ds \leq \|u_1^m\|^2 + \|u_{0x}^m\|^2$$
$$+\tilde{K}\int_0^t (\|u_s^m(s)\|^2 + \|u_x^m(s)\|^2)\ ds + 5C_D^{-1}\int_0^T \|F(s)\|_{V^*}^2\ ds$$
$$+5c_3^2C_D^{-1}\int_0^T |f(s)|^2\ ds + 10c^2C_2^2C_D^{-1}T$$
$$+5c^2C_D^{-1}(C_1M_0+C_2)^2T + 5(c\nu)^2C_D^{-1}(C_1(M_0+1)+C_2)^2T^3.$$

Applying Gronwall's inequality we can conclude that the sequences $\{\|u_t^m)\|^2\}$ and $\{\|u_x^m\|^2\}$ are bounded. Hence there exists a positive constant $K = K(M_0, M_1, c, \nu, T, \|F\|_{L^2(0,T;V^*)}, \|f\|_{L^2(0,T)})$ independent of m such that

$$\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + C_D\int_0^t \|u_{sx}^m(s)\|^2 ds \leq K \quad (19)$$

for each $t \in [0,T]$. This proves the lemma. □

Lemma 3.2 *Let $u^m(t)$ be the Galerkin approximation on $[0,T]$. Then $\{u^m\}$ is bounded uniformly in $C([0,T];H) \subset L^2([0,T];H)$.*

Proof. For $\{u_t^m\} \subset L^2([0,T],H)$, we have

$$(u^m(t_2),\phi) - (u^m(t_1),\phi) = \int_{t_1}^{t_2} (u_s^m(s),\phi)ds$$

for all $\phi \in H$ and for any $t_1, t_2 \in [0,T]$. We take a sequence $\{t_1^k\} \in [0,T]$ such that $t_1^k \to 0$ as $k \to \infty$, then

$$(u^m(t_2),\phi) - (u_0^m,\phi) = \int_0^{t_2} (u_s^m(s),\phi)ds.$$

Hence, for $t \in [0,T]$, we have

$$\begin{aligned} |(u^m(t),\phi)| &\le |(u_0^m,\phi)| + \int_0^t |(u_s^m(s),\phi)|ds \\ &\le \|u_0^m\|\|\phi\| + \int_0^T \|u_s^m(s)\|\|\phi\|ds \\ &\le \left(M_0 + TK^{1/2}\right)\|\phi\|. \end{aligned}$$

Thus $\|u^m(t)\| \le M_0 + TK^{1/2}$, and $\{u^m(t)\}$ is bounded uniformly in H for $t \in [0,T]$ and hence in $C([0,T];H)$.

In the following we repeatedly take subsequences of sequences of $\{u^m\}$. In each case we again denote the subsequence as $\{u^m\}$. □

Lemma 3.3 *There exist functions $u \in L^2([0,T];V)$ and $\hat{u} \in L^2([0,T];V)$, and a subsequence $\{u^m\}$ such that*

$$\begin{aligned} u^m &\to u \qquad \textit{weakly in } L^2([0,T];V) \\ u_t^m &\to \hat{u} \qquad \textit{weakly in } L^2([0,T];V) \quad . \end{aligned}$$

Proof. From Lemma 3.1 and Lemma 3.2 we have that $\{u^m\}$ is bounded uniformly in $C([0,T];V) \subset L^2([0,T];V)$. We also have, from Lemma 3.1, that $\{u_t^m\}$ is bounded uniformly in $L^2([0,T];V)$. We then apply the Banach-Alaoglu Theorem to obtain the desired results in the lemma. □

Lemma 3.4 *The set $\{u^m\}$ is an equicontinuous and bounded subset of $C([0,T];V)$, moreover,*

$$u^m(t) \to u(t) \qquad \textit{weakly in } V$$

uniformly in $t \in [0,T]$, i.e., $u^m \to u$ weakly in $C_W([0,T];V)$.

Proof. The boundedness result follows from Lemma 3.1 and Lemma 3.3. To prove the equicontinuity, we have

$$u^m(t+\Delta t) - u^m(t) = \int_t^{t+\Delta t} u_s^m(s)ds$$

for $t, t+\Delta t \in [0,T]$. Using Lemma 3.1, we obtain

$$\begin{aligned} \|u^m(t+\Delta t) - u^m(t)\|_V &\le \int_t^{t+\Delta t} \|u_s^m(s)\|_V ds \\ &\le \Delta t^{1/2}(TK + C_D^{-1}K)^{1/2}. \end{aligned}$$

Thus, for any $\epsilon > 0$ and $t \in [0,T]$, $\exists \quad \delta(\epsilon,t) = (\epsilon/K(T + C_D^{-1}))^2$ such that $|t' - t| < \delta$ implies $\|u^m(t') - u^m(t)\| < \epsilon$.

For the convergence, we use a version of the Arzela-Ascoli theorem (see [2] or [10], Thm. 3.17.24). Let $Y = \overline{B_V(0,K)}^V$, the closure in V of the ball centered at zero with radius K taken with the weak topology. Y is a complete metric space. Let $\mathcal{F} = \{u^m\} \subset C([0,T];Y)$. By the above estimate, equicontinuity in the V sense implies equicontinuity in the Y sense. Also, for each $t \in [0,T]$, the set $\{u^m(t), u^m \in \mathcal{F}\}$ is relatively compact in Y (Banach-Alaoglu Theorem). Then $\mathcal{F}$ is relatively compact in $C([0,T];Y)$, i.e., there exists a subsequence, denoted $\{u^m\}$ again, such that $u^m(t) \to u(t)$ in Y uniformly in $t \in [0,T]$, i.e.,

$$u^m(t) \to u(t) \quad \text{weakly in } V \quad \text{uniformly in } t \in [0,T].$$

□

Lemma 3.5 *The derivative u_t exists in the V sense and $u_t = \hat{u}$ a.e. in $[0,T]$.*

Proof. Now $u_t \in L^2([0,T];V)$ if there exists a $v \in L^2([0,T];V)$ such that

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} v(s)ds, \qquad t_1, t_2 \in [0,T]$$

where $\int_{t_1}^{t_2} v(s)ds$ or $\int_{t_1}^{t_2} u_s(s)ds$ is defined using duality, i.e.,

$$< u(t_2) - u(t_1), \phi >_{V^*,V} = \int_{t_1}^{t_2} < u_s(s), \phi >_{V^*,V} ds$$

for all $\phi \in V$ and for $t_1, t_2 \in [0,T]$. Now

$$< u^m(t_2) - u^m(t_1), \phi >_{V^*,V} = \int_{t_1}^{t_2} < u_s^m(s), \phi >_{V^*,V} ds$$

for all $\phi \in V$ and for $t_1, t_2 \in [0,T]$. By Lemma 3.3 $u_t^m \to \hat{u}$ weakly in $L^2([0,T];V)$, hence

$$\int_{t_1}^{t_2} < u_s^m(s), \phi >_V ds \to \int_{t_1}^{t_2} < \hat{u}(s), \phi >_V ds \qquad \forall \phi \in V.$$

By Lemma 3.4 $u^m(t) \to u(t)$ weakly in V uniformly in $t \in [0,T]$, hence

$$< u(t_2) - u(t_1), \phi >_V = \int_{t_1}^{t_2} < \hat{u}(s), \phi >_V ds$$

for all $\phi \in V$ and for $t_1, t_2 \in [0,T]$. Therefore $u_t(t) = \hat{u}(t)$, a.e. $t \in [0,T]$ in V. □

Lemma 3.6 *The sequence $\{u_t^m\}$ converges to u_t strongly in $L^2([0,T];H)$.*

Proof. Use Aubin's Theorem ([7], Lemma 8.4). We have $\{u_t^m\}$ bounded uniformly in $L^2([0,T];V)$. If we can show that $\{u_{tt}^m\}$ is bounded uniformly in $L^2([0,T];V^*)$, then, since $V \hookrightarrow H$ compactly and $H \hookrightarrow V^*$ continuously, there exists a subsequence, denoted $\{u_t^m\}$ that converges in $L^2([0,T];H)$.

Now $L^2([0,T];V^*) = L^2([0,T];V)^*$. Fix M and let $\Phi_M = \sum_{k=1}^M c_k(t)\psi_k$ with $c_k(t) \in C^1([0,T])$. Then for $m \geq M$,

$$| < u_{tt}^m, \Phi_M >_{L^2([0,T];V)^*, L^2([0,T];V)} | = \left| \int_0^T < u_{ss}^m(s), \Phi_M(s) >_{V^*,V} ds \right|.$$

Using equation (16) to substitute in for $< u_{ss}^m(s), \Phi_M(s) >_{V^*,V}$, we have

$$\begin{aligned} | < u_{tt}^m, \Phi_M > | &\leq \textstyle\int_0^T |< F(s), \Phi_M(s) >_{V^*,V}| \, ds + \int_0^T |f(s)\Phi_M(s, R_1)| \, ds \\ &\quad + \textstyle\int_0^T |(C_D u_{sx}^m(s), \Phi_{Mx}(s))| \, ds + \int_0^T |(\hat{\epsilon}_1(u_x^m(s)), \Phi_{Mx}(s))| \, ds \\ &\leq \left(C_D \|u_{tx}^m\|_{L^2([0,T];H)} + \|\hat{\epsilon}_1\|_{L^2([0,T];H)} + \|F\|_{L^2([0,T];V^*)} \right) \|\Phi_M\|_{L^2([0,T];V)} \\ &\quad + \textstyle\int_0^T |f(s)| \|\Phi_M(s, R_1)\|_{L^\infty} ds, \end{aligned}$$

where we used Cauchy-Schwartz and Young's inequality in the last step. Now $\|\hat{\epsilon}_1\|^2_{L^2([0,T];H)} = \int_0^T \|\hat{\epsilon}_1(u_x^m(s))\|^2 ds$ and, using the definition of $\hat{\epsilon}$ in equation (14), we have

$$\begin{aligned} \|\epsilon_1(u_x^m(t))\| &\leq c \left(\|\hat{\sigma}_e(u_x^m(t))\| + \|\hat{\sigma}_e(u_{0\,x}^m)\| + \int_0^t \nu \|\hat{\sigma}_e(u_x^m(s))\| ds \right) \\ &\leq c \left(C_1(\|u_x^m(t)\| + \|u_{0\,x}^m\|) + 2C_2 + \nu \int_0^T C_1 \|u_x^m(s)\| + C_2 ds \right) \\ &\leq c \left(2C_2 + C_1(M_0 + K + \nu T(C_1 K^{1/2} + C_2) \right) \end{aligned}$$

where we have used the bound for u_0^m and Lemma 3.1 in the last step. Thus, we have $\int_0^T \|\hat{\epsilon}_1(u_x^m(s))\|^2 ds \leq C(c, C_1, C_2, M_0, K, T)$ bounded independent of m. Also, using the fact that, in one dimension, $H^1(\Omega) \hookrightarrow C^0(\Omega)$, we have

$$\int_0^T |f(s)| \|\Phi_M(s, R_1)\|_{L^\infty} ds \leq \|f\|_{L^2([0,T])} \|\Phi_M\|_{L^2([0,T];V)}. \tag{20}$$

Hence,

$$\begin{aligned} &| < u_{tt}^m, \Phi_M >_{L^2([0,T];V)^*, L^2([0,T];V)} | \\ &\qquad \leq \left(K + C + \|F\|_{L^2([0,T];V^*)} + \|f\|_{L^2([0,T])} \right) \|\Phi_M\|_{L^2([0,T];V)}. \end{aligned}$$

Since $\{\Phi_M\}_{M=1}^{\infty}$ form a dense subset of $L^2([0,T];V)$, we have

$$\|u_{tt}^m\|_{L^2([0,T];V^*)} \leq K + C + \|F\|_{L^2([0,T];V^*)} + \|f\|_{L^2([0,T])}$$

and thus $\{u_{tt}^m\}$ is uniformly bounded in $L^2([0,T];V^*)$. □

Lemma 3.7 *The functions $\{u_t^m\}$ are bounded in $C([0,T];H)$ and are equicontinuous in $C_W([0,T];H)$; moreover, for each t,*

$$u_t^m(t) \to u_t(t) \quad \textit{weakly in } H.$$

Proof. The boundedness statement follows from Lemma 3.1. The convergence statement will follow from an application of Arzela-Ascoli in $C_W([0,T];H)$ once we establish the equicontinuity. To do this we first note that, for $v \in V$,

$$\begin{aligned} | < u_t^m(t+\Delta t) - u_t^m(t), v >_{V^*,V} | &\leq \int_t^{t+\Delta t} | < u_{ss}^m(s), v >_{V^*,V} | ds \\ &\leq \|u_{tt}^m\|_{L^2([0,T];V^*)} \|v\|_V \Delta t^{1/2} \\ &\leq \tilde{C} \|v\|_V \Delta t^{1/2} \end{aligned}$$

where we have used the result from the Lemma 3.6 to bound the $\|u_{tt}^m\|$ term in terms of $\tilde{C} = K + C + \|F\|_{L^2([0,T]_V^*)} + \|f\|_{L^2([0,T])}$, independent of m.

Assume now that $\phi \in H$ and fix $\epsilon > 0$. For $v \in V$ and $t, t+\Delta t \in [0,T]$, we have

$$\begin{aligned} |(u_t^m(t+\Delta t) - u_t^m(t), \phi)| &\leq | < u_t^m(t+\Delta t) - u_t^m(t), v >_{V^*,V} | \\ &\quad + |(u_t^m(t+\Delta t) - u_t^m(t), \phi - v > | \\ &\leq \tilde{C} \Delta t^{1/2} \|v\|_V + 2K \|\phi - v\|. \end{aligned}$$

Choose v such that $K\|\phi - v\| \leq \epsilon/4$ and choose $\delta = (\epsilon / 2\tilde{C}\|v\|_V)^2$. Then

$$|(u_t^m(t+\Delta t) - u_t^m(t), \phi)| \leq \epsilon.$$

This proves equicontinuity in $C_W([0,T];H)$. Let $Y = \overline{B_H(0,K)}^H$ with the weak topology, and let $\mathcal{F} = \{u_t^m\} \subset C([0,T];H)$. It follows that $\mathcal{F}$ is equicontinuous in $C_W([0,T];Y)$ and $\{u_t^m(t) : u_t^m \in \mathcal{F}\}$ is relatively compact in Y for each $t \in [0,T]$. We then use an application of Arzela-Ascoli in $C_W([0,T];H)$ to obtain $\mathcal{F}$ relatively compact in $C_W([0,T];Y)$. Thus, there exists a subsequence, denoted $\{u_t^m\}$, such that

$$u_t^m(t) \to u_t(t) \qquad \text{weakly in } H$$

uniformly in $t \in [0,T]$. □

Lemma 3.8 *There exists an $h \in L^2([0,T];H)$ such that*

$$\hat{\epsilon}_1(u_x^m) \to h \quad \textit{weakly in } L^2([0,T];H).$$

Proof. We need to show that $\hat{\epsilon}_1(u_x^m)$ is bounded uniformly in $L^2([0,T];H)$. In Lemma 3.6 we computed $\int_0^T \|\hat{\epsilon}_1(u_x^m(s))\|^2 ds \leq C(c, C_1, C_2, M_0, K, T)$. Thus $\{\hat{\epsilon}_1(u_x^m)\}$ is bounded in $L^2([0,T];H)$ independent of m, and there exists a subsequence and an $h \in L^2([0,T];H)$ such that

$$\hat{\epsilon}_1(u_x^m) \to h \quad \text{weakly in } L^2([0,T];H).$$

□

4 Existence and Uniqueness Theorems

We now state and prove the theorem regarding existence and uniqueness of weak solutions, first establishing a local result.

Theorem 4.1 *Under the assumptions (AF), (Af), and (AL), system (7)-(11) has a unique local weak solution on $[0, t^*]$ for some $t^* > 0$.*

Proof. Denote by $\mathcal{P}_M$ $(M = 1, 2, \ldots)$ the class of functions $\eta \in \mathcal{L}_T$ which can be represented in the form

$$\eta(t) = \sum_{k=1}^{M} c_k(t)\psi_k$$

where $c_k \in C^1([0,T])$. Let $\mathcal{P} = \cup_{M=1}^{\infty} \mathcal{P}_M$. Note $\mathcal{P}$ is dense in $\mathcal{L}_T$.

We start with equation (16), multiply by $c_k(t)$, sum from 1 to M, and integrate over $(0,t)$ to obtain

$$\int_0^t (< u_{ss}^m(s), \eta(s) >_{V^*,V} + C_D(u_{sx}^m(s), \eta_x(s)) + (\hat{\epsilon}_1(u_x^m(s)), \eta_x(s)))\, ds$$
$$= \int_0^t (< F(s), \eta(s) >_{V^*,V} - f(s)\eta(s, R_1))\, ds$$

or, integrating the first term by parts, we find

$$\int_0^t [-(u_s^m(s), \eta_s(s)) + C_D(u_{sx}^m(s), \eta_x(s)) + (\hat{\epsilon}_1(u_x^m(s)), \eta_x(s))]\, ds$$
$$+(u_t^m(t), \eta(t)) - (u_1^m, \eta(0)) = \int_0^t [< F(s), \eta(s) >_{V^*,V} - f(s)\eta(s, R_1)]\, ds$$

which is satisfied for all $\eta \in \mathcal{P}_M$ and for $M \leq m$.

Now fix $\eta \in \mathcal{P}_M$ with $M \leq m$ and pass to the limit as $m \to \infty$, using Lemma 3.3, Lemma 3.6, Lemma 3.7, Lemma 3.8, and the convergence $u_1^m \to u_1$ in H. Hence, on any interval $[0, t]$, with $t \leq T$, we obtain

$$\int_0^t [-(u_s(s), \eta_s(s)) + C_D(u_{sx}(s), \eta_x(s)) + (h(s)), \eta_x(s))] \, ds + (u_t(t), \eta(t)) \quad (21)$$
$$-(u_1, \eta(0)) = \int_0^t [< F(s), \eta(s) >_{V^*,V} - f(s)\eta(s, R_1)] \, ds.$$

We now need to show that

$$\int_0^t (h(s), \eta_x(s)) ds = \int_0^t (\hat{\epsilon}_1(u_x(s)), \eta_x(s)) ds, \qquad \forall \eta \in \mathcal{L}_T.$$

This is accomplished by establishing the strong convergence of $u_x^m(t) \to u_x(t)$ in H as $m \to \infty$. To do this we take u_t^m and u_t as test functions in equations (16) and $\frac{d}{dt}(21)$ respectively, and add a (u_x^m, u_{tx}^m) or (u_x, u_{tx}) term to both sides of their respective equations. We have

$$< u_{tt}^m(t), u_t^m(t) > + C_D(u_{tx}^m(t), u_{tx}^m(t)) + (\hat{\epsilon}_1(u_x^m(t)), u_{tx}^m(t)) + (u_x^m(t), u_{tx}^m(t))$$
$$= (u_x^m(t), u_{tx}^m(t)) + < F(t), u_t^m(t) >_{V^*,V} - f(t) u_t^m(t, R_1)$$

and

$$< u_{tt}(t), u_t(t) > + C_D(u_{tx}(t), u_{tx}(t)) + (h(t), u_{tx}(t)) + (u_x(t), u_{tx}(t))$$
$$= (u_x(t), u_{tx}(t) + < F(t), u_t(t) >_{V^*,V} - f(t) u_t(t, R_1).$$

Let $z^m(t) = u^m(t) - u(t)$ and subtract the two equations to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|z_t^m\|^2 + \|z_x^m\|^2\right) + C_D\|z_{tx}^m\|^2 = -\frac{d}{dt} < z_t^m, u_t > -2C_D < z_{tx}^m, u_{tx} >$$
$$+ < F(t), z_t^m >_{V^*,V} - < \hat{\epsilon}_1(u_x), z_{tx}^m > - < \hat{\epsilon}_1(u_x^m) - h, u_{tx} >$$
$$- f(t) z_t^m(R_1) + < z_x^m, z_{tx}^m > - < \hat{\epsilon}_1(u_x^m) - \hat{\epsilon}_1(u_x), z_{tx}^m > .$$

We then use the Cauchy-Schwartz and Young's inequality on the last two terms on the right and integrate on $(0, t)$ to obtain

$$\|z_t^m\|^2 + \|z_x^m\|^2 \leq \|u_1^m - u_1\|^2 + \|u_{0\,x}^m - u_{0x}\|^2 + 2C_D^{-1} \int_0^t \|z_x^m(s)\|^2 ds$$
$$+ 2C_D^{-1} \int_0^t \|\hat{\epsilon}_1(u_x^m(s)) - \hat{\epsilon}_1(u_x(s))\|^2 ds + X_m(t) + Y_m(t)$$

where

$$X_m(t) = 2 < u_1^m - u_1, u_t(t) > -2 < z_t^m(t), u_t(t) >$$
$$-4C_D \int_0^t < z_{sx}^m(s), u_{sx}(s) > ds + 2\int_0^t < F(s), z_s^m(s) >_{V^*,V} ds$$
$$-2\int_0^t < \hat{\epsilon}_1(u_x(s)), z_{sx}^m(s) > ds - 2\int_0^t < \hat{\epsilon}_1(u_x^m(s)) - h(s), u_{sx} > ds$$

and

$$Y_m(t) = 2\int_0^t |f(s) z_s^m(s, R_1)| ds.$$

Note that $X_m(t) \to 0$ as $m \to \infty$ by Lemma 3.3, Lemma 3.7, Lemma 3.8, and the convergence $u_1^m \to u_1$ in H. To see that $Y_m(t) \to 0$ as $m \to \infty$, we use the embedding $V \hookrightarrow C^0$ and Agmon's inequality (see [11]) to obtain

$$\begin{aligned} Y_m(t) &\leq 2\int_0^t |f(s)| \|z_s^m(s)\|_{L^\infty} ds \\ &\leq 2c\|f\|_{L^2(0,T)} \left(\int_0^t \|z_s^m(s)\| \|z_s^m(s)\|_{H^1} ds \right)^{1/2} \\ &\leq 2c\|f\|_{L^2(0,T)} \|z_t^m\|_{L^2([0,T];H)}^{1/2} \|z_t^m\|_{L^2([0,T];V)}^{1/2}. \end{aligned}$$

Since $z_t^m \to 0$ strongly in $L^2([0,T];H)$ by Lemma 3.6, and since $\{z_t^m\}$ is bounded uniformly in $L^2([0,T];V)$, we have $Y_m(t) \to 0$ as $m \to \infty$. For the integral term on the right side, we use assumptions (AL) and (AL2) to show that

$$\begin{aligned} \int_0^t \|\hat{\epsilon}_1(u_x^m(s)) - \hat{\epsilon}_1(u_x(s))\|^2 ds &\leq 4L^2 T \|u_{0\,x}^m - u_{0x}\|^2 \\ &\quad + 2L^2(2 + \nu T) \int_0^t \|u_x^m(s) - u_x(s)\|^2 ds. \end{aligned}$$

See the Appendix for the proof.

We then have

$$\begin{aligned} \|z_t^m\|^2 + \|z_x^m\|^2 &\leq 2C_D^{-1}(1 + 4L^2 + 2L^2\nu T) \int_0^t \|z_x^m(s)\|^2 ds + \|u_1^m - u_1\|^2 \\ &\quad + (1 + 8C_D^{-1}L^2 T)\|u_{0\,x}^m - u_{0x}\|^2 + X_m(t) + Y_m(t) \end{aligned}$$

We may apply the generalized Gronwall inequality to the above equation to obtain $\|z_x^m(t)\| \to 0$ a.e. $t \in [0,T]$. Thus, $\hat{\epsilon}_1(u_x^m) \to \hat{\epsilon}_1(u_x)$ strongly in $L^2([0,T];H)$, and by the uniqueness of the limit we have $h = \hat{\epsilon}_1(u_x)$ in H.

The limit function u satisfies.

$$\int_0^t [-(u_s(s),\eta_s(s)) + C_D(u_{sx}(s),\eta_x(s)) + (\hat{\epsilon}_1(u_x(s)),\eta_x(s))]\,ds + (u_t(t),\eta(t))$$
$$-(u_1,\eta(0)) = \int_0^t [< F(s),\eta(s) >_{V^*,V} + f(s)\eta(s,R_1)]\,ds$$

which holds for any interval $[0,t]$, for $t \le T$, and for all $\eta \in \mathcal{L}_T$. Thus system (7)-(11) with ϵ_1 replaced by $\hat{\epsilon}_1$ has a solution on any arbitrary interval $[0,T]$. Uniqueness of this weak solution can be shown in the standard way (see, e.g., [1], [4], [8]).

To show that system (7)-(11) with ϵ_1 has a local unique weak solution, we note that the weak solution u has the property that u_x is continuous in t. Thus, there exists a t^* with $0 \le t^* \le T$ such that

$$\|u_x(t) - u_{0x}\| \le 1 \quad \text{for all } t \in [0,t^*],$$

and therefore, from the definition of $\hat{\epsilon}_1$, we have

$$\hat{\epsilon}_1(u_x(t,x)) = \epsilon_1(u_x(t,x)) \quad \text{for all } t \in [0,t^*].$$

Hence u is a weak solution of system (7)-(11) on $[0,t^*]$. Uniqueness is again shown in the standard way. This completes the proof of local existence under assumptions (AF), (Af), and (AL). □

If we add the growth condition (AG) on σ_e (recall that (8) does not satisfy this condition unless it is modified for large u_x), then we can use arguments similar to those above to establish global existence.

Theorem 4.2 *Under the assumptions (AF), (Af), (AL), and (AG), system (7), (9)-(11) has a unique global weak solution on any finite interval* $[0,T]$.

Proof. Under the additional assumption (AG), we can argue as in Theorem 2 of [1] to establish that this local unique solution actually exists on any arbitrary interval [0,T]. Essentially, one uses the condition (AG) to establish *a priori* bounds similar to (17) for approximations involving σ_e, not $\hat{\sigma}_e$, and then argues a pointwise bound on $u_x(t)$. The local Lipschitz condition (AL) can then be used on σ_e and arguments similar to those above carried out. Thus we arrive at the results of Theorem 4.2. □

Appendix

We argue the inequality

$$\int_0^t \|\hat\epsilon_1(u_x^m(s)) - \hat\epsilon_1(u_x(s))\|^2 ds \;\leq\; 4L^2T\|u_{0\;x}^m - u_{0x}\|^2 + 2L^2(2+\nu T)\int_0^t \|u_x^m(s) - u_x(s)\|^2 ds.$$

Using equation (14) to define $\hat\epsilon$, we have

$$\begin{aligned}
\int_0^t \|\hat\epsilon_1(u_x^m(r)) - \hat\epsilon_1(u_x(r))\|^2 dr \leq \int_0^t \{ &\|\hat\sigma_e(u_x^m(r)) - \hat\sigma_e(u_x(r))\| \\
&+ \|e^{-\nu r}(\hat\sigma_e(u_{0\;x}^m) - \hat\sigma_e(u_{0x}))\| \\
&+ \|\int_0^r \nu e^{-\nu(r-s)}(\hat\sigma_e(u_x^m(s)) - \hat\sigma_e(u_x(s))ds\|\}^2 dr \\
\leq 4\int_0^t \|\hat\sigma_e(u_x^m(r)) - \hat\sigma_e(u_x(r))\|^2 dr &+ 4\int_0^t \|(\hat\sigma_e(u_{0\;x}^m) - \hat\sigma_e(u_{0x}))\|^2 dr \\
&+ 4\int_0^t \left(\int_0^r \nu^2 e^{-2\nu(r-s)}\|(\hat\sigma_e(u_x^m(s)) - \hat\sigma_e(u_x(s))\|ds\right)^2 dr \\
\leq 4L^2\int_0^t \|u_x^m(r) - u_x(r)\|^2 dr &+ 4L^2\int_0^t \|u_{0\;x}^m - u_{0x}\|^2 dr \\
&+ 4\int_0^t \left(\int_0^r \nu e^{-\nu(r-s)} L\|u_x^m(s) - u_x(s)\|ds\right)^2 dr,
\end{aligned}$$

where $L = L_{B_{1+\|u_{0x}\|}}$ and we have used assumption (AL2) in the last step. Now

$$\int_0^t \|u_{0\;x}^m - u_{0x}\|^2 dr \leq \|u_{0\;x}^m - u_{0x}\|^2 T$$

for $t \in [0,T]$. Also, using Cauchy-Schwartz, we have

$$\begin{aligned}
\int_0^t &\left(\int_0^r \nu e^{-\nu(r-s)} L\|u_x^m(s) - u_x(s)\|^2 ds\right) dr \\
&\leq \int_0^t \nu^2 L^2 e^{-2\nu r}\left(\int_0^r e^{2\nu s} ds\right)\left(\int_0^r \|u_x^m(s) - u_x(s)\|^2 ds\right) dr \\
&\leq \frac{\nu L^2}{2}\int_0^t\int_0^r \|u_x^m(s) - u_x(s)\|^2 ds dr \\
&\leq \frac{\nu L^2 T}{2}\int_0^t \|u_x^m(s) - u_x(s)\|^2 ds
\end{aligned}$$

for $t \in [0,T]$. Hence

$$\int_0^t \|\hat{\epsilon}_1(u_x^m(s)) - \hat{\epsilon}_1(u_x(s))\|^2 ds \leq 4L^2T\|u_0^m{}_x - u_{0x}\|^2 + 2L^2(2+\nu T)\int_0^t \|u_x^m(s) - u_x(s)\|^2 ds.$$

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