

PSEUDO-TRANSIENT CONTINUATION AND DIFFERENTIAL-ALGEBRAIC EQUATIONS *

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Abstract. Pseudo-transient continuation is a practical technique for globalizing the computation of steady-state solutions of nonlinear differential equations. The technique employs adaptive time-stepping to integrate an initial value problem derived from an underlying ODE or PDE boundary value problem until sufficient accuracy in the desired steady-state root is achieved to switch over to Newton’s method and gain a rapid asymptotic convergence. The existing theory for pseudo-transient continuation includes a global convergence result for differential equations written in semidiscretized method-of-lines form. However, many problems are better formulated or can only sensibly be formulated as differential-algebraic equations (DAEs). These include systems in which some of the equations represent algebraic constraints, perhaps arising from the spatial discretization of a PDE constraint.

Multirate systems, in particular, are often formulated as differential-algebraic systems to suppress fast time scales (acoustics, gravity waves, Alfvén waves, near equilibrium chemical oscillations, etc.) that are irrelevant on the dynamical time scales of interest. In this paper we present a global convergence result for pseudo-transient continuation applied to DAEs of index 1, and we illustrate it with numerical experiments on model incompressible flow and reacting flow problems, in which a constraint is employed to step over acoustic waves.

Key words. pseudo-transient continuation, nonlinear equations, steady-state solutions, global convergence, differential-algebraic equations, multi-rate systems

AMS subject classifications. 65H10, 65J15, 65L80, 65N12, 65N22

1. Introduction. Pseudo-transient continuation (Ψtc) is a globalization technique for the nonlinearly implicit computation of steady-state solutions of partial differential equations. Mathematically, it is a homotopy that embeds the steady-state problem in a space-time setting. The technique is popular in the aerodynamics [31], magnetohydrodynamics [20], radiation transport [28], and reacting flow [30] modeling communities, among others, because conventional methods for solving nonlinear equations when the initial iterate is far from a solution, such as line-search and trust region methods [8, 15, 25], can converge to nonphysical solutions or local minima of the norm of the steady-state residual [6, 18]. This is particularly the case when the solution has complex features, such as shocks or discontinuities, that are not present in the initial iterate [24]. Ψtc is natural to apply in systems that originally derive from transient dynamics. While following a physical transient accurately may be inefficient when only the steady state is required, the existence of a physical space-time trajectory provides a basis for robust nearby pseudo-transient approaches.

In general terms, pseudo-transient continuation is a predictor-corrector method for temporal integration in which the time step is increased as the iteration progresses with the objective of

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fast convergence near a solution. Temporal accuracy is not the objective, but in most cases the method begins by integrating accurately in time, closely following the transient behavior of the solution in the early stages of the iteration until an approximate steady state is reached. The time step is thereafter increased, sacrificing temporal accuracy in the terminal phase in favor of rapid convergence to steady state.

There are many other forms of continuation, some of which can be employed directly on the steady-state boundary value problem of interest. Parameter continuation [27], mesh sequencing [29], and model sequencing [18] can be effective in globalizing Newton’s method applied to large-scale nonlinear problems. Another complementary approach towards globalizing Newton’s method is nonlinear preconditioning of additive Schwarz type, recently introduced in [5]. In practice, we advocate mesh sequencing as a primary strategy in any highly-resolved nonlinear problem — solving first on a relatively coarse mesh, interpolating to the next finer mesh to produce an initial iterate for Newton’s method that lies within or close to the ball of convergence on the finer mesh (where iteration is more expensive), and so forth recursively [29]. Any of these globalization techniques can, in principle, be combined with each other and with Ψ tc .

A particularly effective algorithmic combination is Ψ tc on each level of a mesh-sequenced approach. Depending upon the physical understanding that can be brought to bear on a specific application, any of the techniques that work directly on the steady-state formulation might be more effective than Ψ tc alone, but Ψ tc may have a more complete general mathematical theory than some of these other methods, especially with the recent additions of this paper and [9, 14]. Theoretical exploration of the interaction of Ψ tc with these other techniques is likely to be fruitful, but is beyond the scope of the present contribution.

1.1. ODE Dynamics. For problems with ODE dynamics, sufficient conditions for convergence of Ψ tc are given in [16], using rules for growth of the time step that are common in practice. We give a brief review of those results here, both to introduce notation and to put the new results in perspective.

If we express the initial value problem as $u' = F(u)$, $u(0) = u_0$, we seek a root, if it exists, of $F(u)$ that is the limit of $u(t)$ as $t \rightarrow \infty$. Conventional nonlinear equation algorithms can (and do) find other solutions; hence the need for a continuation method that respects the physical solution.

Assuming that a stable steady-state solution exists, global convergence and local superlinear convergence to that solution are proved in [16] for a class of methods that integrate

$$u' = -V^{-1}F(u), \quad u(0) = u_0 \tag{1.1}$$

by a variable time step method that attempts to increase the time step as the integration progresses and steady state is approached. In (1.1), V is a scaling or preconditioning matrix. One method considered in [16] is

$$u_{n+1} = u_n - (\delta_n^{-1}V + F'(u_n))^{-1} F(u_n), \tag{1.2}$$

where F' is the Jacobian and δ_n is the time step. The Newton step may be computed inexactly in the sense of [7, 15] and the convergence results change in an expected way.

A common choice for δ_n , covered by the results in [16], is the “switched evolution relaxation” (SER) method [23],

$$\delta_n = \delta_{n-1} \|F(u_{n-1})\| / \|F(u_n)\| = \delta_0 \|F(u_0)\| / \|F(u_n)\|. \tag{1.3}$$

SER is commonly used in computational aerodynamics [17,24,31]. In practice, the growth of $\{\delta_n\}$ may be limited from above, relative to (1.3), which can otherwise be rather aggressive. The results in the present paper, as with those in [16], assume an update of the form

$$\delta_n = \phi \left(\delta_{n-1} \frac{\|F(u_{n-1})\|}{\|F(u_n)\|} \right). \quad (1.4)$$

In (1.4)

$$\phi(\xi) = \begin{cases} \xi & \xi \leq \xi_t \\ \delta_{\max} & \xi > \xi_t \end{cases}. \quad (1.5)$$

where either $\xi_t = \delta_{\max}$ or $\xi_t < \infty$ and $\delta_{\max} = \infty$.

As a method for integrating in time, Ψ_{tc} is a Rosenbrock method [11] when δ is fixed. One may also think of this as a predictor-corrector method, where the simple predictor (result from the previous time) and a Newton corrector are used.

The motivating example in [16] is Euler flow over an airfoil. Ψ_{tc} is a natural for this problem since the time-dependent Euler equations have physical meaning. The main result of [16] is that if the desired steady-state solution is a stable steady state for the time-dependent problem and the initial step is sufficiently small, then either the Ψ_{tc} iteration stagnates with $\inf_n \delta_n = 0$ or the iteration converges to the desired steady-state solution and the convergence in the terminal phase is as fast as an inexact-Newton method with the same algorithmic parameters. Of course, stagnation is an undesirable outcome, and ultimately requires the numerical analyst to exercise ingenuity in finding a better initial iterate. However, this “graceful” termination is preferable to allowing an erratic Newton process to enter an infeasible regime, where perhaps certain operations would generate floating point exceptions, perhaps leading to the unexplained hanging of a large-scale parallel execution.

There have been other recent theoretical studies of Ψ_{tc} . Trust-region methods are used in an analysis of gradient flows in [14] to control the growth of the time step and prove convergence. Affine invariance ideas are applied to problems with linear dynamical invariants in [9]. The time step is controlled with a method that seeks to increase the step but, at the same time, demands a decrease in the norm of the nonlinear residual. We did not ask for such a decrease in [16], nor do we here. In the transient phase, early in the iteration, the norm of the steady-state residual may well increase, as indeed may happen in a time-accurate integration.

1.2. DAE Dynamics. One of the limitations of the theory developed in [9, 14, 16] is the ODE dynamics. Many problems are best formulated, or can only be formulated, as differential-algebraic equations (DAEs). In § 3 we give two examples from fluid mechanics, an incompressible flow and a variable-density reacting flow. We seek a steady-state balance of the discretized conservation laws of mass, momentum, energy, and (as applicable) chemical species. For all equations but the mass (“continuity”) equation, we evolve the solution through pseudo-time stepping. In effect, we parabolize these nonlinear elliptic PDEs by prepending a time derivative term in the appropriate dependent variable. This is not only a physically natural form of nonlinear relaxation; it also leads to superior linear conditioning of the discrete systems resulting at each implicit time step (see, e.g., [10]).

However, we argue on the basis of physical intuition that it could be harmful to allow the dependent variables of the mass conservation equation to relax by prepending a similar tendency

term. Instead, we force the progression from initial iterate to steady-state solution to occur on a manifold of continuity-satisfying states. We require this because linear acoustic waves in a time-dependent hydrodynamical system arise from a combination of the tendency and convective terms in the continuity equation and the tendency and convective terms in the momentum equations [22]. We can suppress acoustic-like oscillations in the solution trajectory by not equipping our pseudo-transient system with this mechanism. Indeed, the formulations we choose, velocity-vorticity and streamfunction-vorticity, are historically motivated, in part, by their suppression of the acoustic mechanism. We do not want to readmit this mechanism by relaxing continuity. In computing the steady-states of other multi-rate physical phenomena possessing fast waves or stiff restoring forces, we may similarly choose to omit a tendency-term piece of the fast mechanism. A combination analogous to the continuity equation and acoustic waves is the hydrostatic potential equation and gravity waves in shallow water wave theory.

We therefore need to be allowed to formulate our governing system as a partial differential algebraic equation (PDAE) of index 1 [4]. Let the vector of unknowns be $x = [u, v]^T$ and the conformally partitioned vector of steady conservation laws be $F = [f(u, v), g(u, v)]^T$, where the first equation set takes the pseudo-transient tendency terms. The steady-state equation is then $F(x) = 0$ and the time-dependent equation is (allowing for different scalings of the transient terms)

$$Dx' = -F(x), \quad (1.6)$$

where

$$D = \begin{pmatrix} V & \\ & 0 \end{pmatrix}$$

and V is a nonsingular diagonal matrix.

The purpose of this paper is to extend the results from [16] to the semi-explicit index-one DAE case and thereby understand the historical success of Ψ tc as a solver for such systems. In the remainder of this paper we state and prove new results for the DAE case in § 2, and illustrate with a pair of examples in § 3.

2. Ψ tc for Differential Algebraic Equations. In this section we extend the current theory for Ψ tc to include a global convergence result for semi-explicit index-1 DAEs. We consider DAEs of the form

$$\begin{aligned} Vu' &= -f(u, v) \\ 0 &= g(u, v) \end{aligned} \quad (2.1)$$

with initial value $(u(0), v(0)) = (u_0, v_0)$, $u \in R^{N_1}$, $v \in R^{N_2}$, where V is an $N_1 \times N_1$, nonsingular scaling matrix and $\partial g/\partial v$ is nonsingular. Our result follows from the work in [16], and we prove a theorem corresponding to the final stage of the Ψ tc convergence.

We write the DAE (2.1) in the form of (1.6), where $x = [u, v]^T \in R^N$, $N = N_1 + N_2$, $F = [f, g]^T$, and

$$Dx' = -F(x), \quad x(0) = x_0. \quad (2.2)$$

The pseudo-transient continuation procedure will be defined by the iteration

$$x_{n+1} = x_n - (\delta_n^{-1}D + F'(x_n))^{-1}F(x_n), \quad (2.3)$$

where $\{\delta_n\}$ is given by (1.5).

We assume that the DAE has an index of one, has a global solution in time, and that the solution converges to a steady state.

ASSUMPTION 2.1.

1. The initial values (u_0, v_0) are consistent, i.e. $g(u_0, v_0) = 0$.
2. F is twice Lipschitz continuously differentiable.
3. The solution $x(t) = [u(t), v(t)]^T$ of (2.2) exists for all $t > 0$ and $\lim_{t \rightarrow \infty} x(t) = x^* \in R^N$.
4. $g_v(u(t), v(t)) = g_v(x(t))$ is a nonsingular $N_2 \times N_2$ matrix for all $t \geq 0$ and $\|g_v(x(t))^{-1}\|$ is uniformly bounded on $(0, \infty)$.

The analysis hinges on the fact that the Ψ tc iteration remains in a neighborhood of the solution of (2.2),

$$S(\epsilon) = \{z \mid \inf_{t \geq 0} \|z - x(t)\| \leq \epsilon\}. \quad (2.4)$$

Given $\epsilon > 0$, we will show that if δ_0 is sufficiently small, then the Ψ tc iteration remains within $S(\epsilon)$ for all n and accurately approximates $\{x(t_n)\}$ until the iteration is within the ball of convergence for Newton's method. This corresponds to the first phase of the iteration as identified in [16]. The complete convergence analysis requires more. Once the iteration is near x^* , one must show that the δ_n increases so as to obtain fast convergence and that the Ψ tc iteration does not diverge while this increase is in progress. Once this intermediate phase is complete, the final phase is a rapidly convergent Newton or inexact Newton iteration [16]. We describe the latter two phases first. These can be studied with a local convergence analysis that depends on the DAE dynamics only in a trivial way.

In order for the Ψ tc iteration to be well defined, we must assume that the linear systems to be solved are nonsingular and that the steady-state solution is stable.

ASSUMPTION 2.2. There are $\epsilon_S, \epsilon_G > 0$ such that

1. For all $z_0 \in S(\epsilon_S)$, the solution of $Dz' = -F(z)$, $z(0) = z_0$ exists, $z(t) \in S(\epsilon_G)$ for all t , and $\lim_{t \rightarrow \infty} z(t) = x^*$, $t \rightarrow \infty$.
2. Moreover, there are $M_D, M_I > 0$ such that for all $\delta > 0$,
 - (a) $(\delta^{-1}D + F'(x))$ is nonsingular for all $x \in S(\epsilon_G)$,
 - (b) $\|(\delta^{-1}D + F'(x))\| \leq M_D$ for all $x \in S(\epsilon_G)$, and
 - (c) $\|(\delta^{-1}D + F'(x))^{-1}\| \leq M_I$ for all $x \in S(\epsilon_G)$.

The foregoing assumptions are mathematically reasonable and conducive to a compact theory. However, they may be too severe in practice, even for relatively simple nonlinear problems containing material interfaces, discretized with nondifferentiable flux limiters, etc. Pseudo-transient continuation has been effectively used well beyond these restrictions on the model for $F(u)$, including in mild violations in examples of this paper.

2.1. Local Convergence. The assumptions for local convergence differ from the standard ones for nonlinear equations [15] in that we must allow for the possibility that δ can be small even if x is near x^* .

Let

$$B(\epsilon) = \{x \mid \|x - x^*\| < \epsilon\}.$$

Our assumptions on continuity and differentiability of F and nonsingularity of F' near x^* are

ASSUMPTION 2.3. There are $\beta, \epsilon_L > 0$ such that for all $x \in B(\epsilon_L)$ and all $\delta > 0$

$$\|(D + \delta F'(x))^{-1}D\| \leq 1/(1 + \beta\delta).$$

Assumption 2.3 is the DAE analog of Assumption 2.1.3 in [16], and is critical for stability in the middle phase of the iteration.

Theorem 2.1 describes the middle and terminal phases of the iteration in the way that Theorems 2.1 and 2.3 of [16] do in the ODE case. The proof differs only in the local convergence estimate.

Once the approximate solution is near x^* , then the iteration will converge and δ will increase.

THEOREM 2.1. *Let Assumptions 2.1, 2.2 and 2.3 hold. Let $\{\delta_n\}$ be given by (1.4). Then there are $C_T, \epsilon_T > 0$ so that if $x_0 \in B(\epsilon_T)$ then either $\inf_n \delta_n = 0$ or $\delta_n \rightarrow \delta_{max}$, the Ψ tc iteration converges, and for n sufficiently large*

$$\|x_{n+1} - x^*\| \leq C_T \|x_n - x^*\| (\delta_{max}^{-1} + \|x_n - x^*\|). \quad (2.5)$$

Proof. As is standard [15], we describe the progress of the iteration in terms of the transition from a current iterate x_c to a new one x_+ . We let $e = x - x^*$. Let $\epsilon_T < \epsilon_L$ be small enough so that the local convergence theory for Newton's method holds for $F(x) = 0$. Then, if $x_c \in B(\epsilon_T)$

$$\begin{aligned} e_+ &= e_c - (\delta_c^{-1}D + F'(x_c))^{-1}F(x_c) \\ &= e_c - (\delta_c^{-1}D + F'(x_c))^{-1}(\delta_c^{-1}D + F'(x_c))e_c \\ &\quad + (\delta_c^{-1}D + F'(x_c))^{-1}(F'(x_c)e_c - F(x_c)) \\ &\quad + (\delta_c^{-1}D + F'(x_c))^{-1}\delta_c^{-1}De_c. \end{aligned} \quad (2.6)$$

The standard local convergence theory for Newton's method and Assumption 2.3 imply that there is C_N such that

$$\|(\delta_c^{-1}D + F'(x_c))^{-1}(F'(x_c)e_c - F(x_c))\| \leq C_N \|e_c\|^2$$

and

$$\|e_c - (\delta_c^{-1}D + F'(x_c))^{-1}(\delta_c^{-1}D + F'(x_c))e_c\| \leq C_N \|e_c\|^2.$$

Assumption 2.3 implies that

$$\|(\delta_c^{-1}D + F'(x_c))^{-1}\delta_c^{-1}De_c\| \leq \|e_c\|/(1 + \beta\delta_c).$$

Now assume that $\{\delta_n\}$ is bounded from below by δ^* . Since $\delta_c \geq \delta^*$,

$$\|e_+\| \leq (2C_N \|e_c\| + (1 + \beta\delta^*)^{-1}) \|e_c\|.$$

Reducing ϵ_T if needed so that

$$(2C_N \epsilon_T + (1 + \beta\delta^*)^{-1}) < (1 + \beta\delta^*/2)^{-1}$$

implies that the local convergence is q-linear provided $\{\delta_n\}$ is bounded from below.

The q-linear convergence will eventually drive δ_n to δ_{max} if $\delta_{max} < \infty$. Hence, the standard local convergence theory for Newton's method [15] will imply (2.5). \square

2.2. Global Convergence. The objective of this section is to show that if δ_0 is sufficiently small, then $x_n \in S(\epsilon)$ for some $\epsilon < \min(\epsilon_S, \epsilon_T)$ until $x_n \in B(\epsilon_T)$, where the local convergence theory holds.

THEOREM 2.2. *Let Assumptions 2.1, 2.2, and 2.3 hold. If δ_0 is sufficiently small and $\{\delta_n\}$ is bounded from below, then $x_n \rightarrow x^*$ and (2.5) holds for n sufficiently large.*

Proof. Assume that $\delta_n \geq \delta^*$. Let

$$t_n = \sum_{j=0}^{n-1} \delta_j.$$

The assumptions imply that there is T such that $x(t) \in B(\epsilon_T/2)$ for all $t \geq T$, where $B(\epsilon_T)$ is the ball of local convergence from Theorem 2.1. Hence if $N \geq T/(2\delta^*)$, $x(t_N) \in B(\epsilon_T/2)$. The proof will show that for δ_0 sufficiently small, $x_n \in S(\epsilon_T/2)$, that is

$$\|x_n - x(t_n)\| < \epsilon_T/2$$

for all $n \leq N$. This will imply that $x_N \in B(\epsilon_T)$ and complete the proof.

We will show that the Ψ tc iteration is a consistent first-order explicit method for the ODE

$$\begin{bmatrix} u \\ v \end{bmatrix}' = x' = G(x) = \begin{bmatrix} -V^{-1}f \\ g_v^{-1}g_u V^{-1}f \end{bmatrix}, \quad (2.7)$$

which is equivalent to (2.1). Our assumptions imply that G is Lipschitz differentiable and hence the explicit Euler method is first-order accurate.

We will show that the local error of Euler's method for (2.7) is within $O(\delta^2)$ of that for the Ψ tc method. This will complete the proof.

Assume that $x_n = x(t_n)$, then the Euler step for (2.7) can be written as $x_{n+1} = x_n + \delta_n s$, where

$$s = \begin{bmatrix} s_u \\ s_v \end{bmatrix} = \begin{bmatrix} -V^{-1}f \\ g_v^{-1}g_u V^{-1}f \end{bmatrix}.$$

Hence, for the Euler discretization

$$g_u s_u + g_v s_v = 0. \quad (2.8)$$

The Ψ tc discretization is $x_{n+1} = x_n + \delta_n \sigma$, where

$$\sigma = \begin{bmatrix} \sigma_u \\ \sigma_v \end{bmatrix} = -(D + \delta_n F'(x_n))^{-1} F(x_n).$$

The equations for σ_u and σ_v are

$$V\sigma_u + \delta_n(f_u\sigma_u + f_v\sigma_v) = -f \quad (2.9)$$

and

$$\delta_n(g_u\sigma_u + g_v\sigma_v) = -g. \quad (2.10)$$

Since $x_n = x(t_n)$, $g = 0$ and

$$g_u\sigma_u + g_v\sigma_v = 0,$$

which is similar to (2.8) and σ_v can be computed from σ_u in exactly the same way s_v can be computed from s_u . Therefore the proof will be complete if we can show that

$$\sigma_u = s_u + O(\delta_n). \quad (2.11)$$

Since $Vs_u = -f$, (2.11) follows from (2.9) and the nonsingularity of V . \square

3. Computational Examples. In this section we illustrate the DAE form of Ψ_{tc} with two examples of simple elliptically-constrained systems: incompressible Boussinesq flow in a lid- and buoyancy-driven cavity and reacting flow in a laminar diffusion flame with single-step infinitely fast kinetics — a classic flamesheet model. The first example is freely downloadable as part of the release of the PETSc toolkit [1]. For the first example, we compare a fully parabolized Ψ_{tc} formulation and a PDAE version and show that the latter has an advantage in computational efficiency.

3.1. Incompressible Flow. Our incompressible flow example is a combination of two classic problems, lid-driven flow and buoyancy-driven flow in a two-dimensional rectangular cavity. The lid, moving with a steady and spatially uniform velocity, sets up a principal vortex and subsidiary corner vortices by viscous forces and the differentially heated lateral walls of the cavity set up a buoyant vortex flow, which opposes the principal lid-driven vortex. Our parameterization and discretization are inspired by [2]; however, in this example we do not exploit local adaptive refinement. The PETSc implementation contains uniform-refinement mesh-sequencing capabilities, applied both in a nonlinear continuation sense in the outer iteration, as described in the introduction, and as an inner iteration as part of a multigrid solver.

The governing system consists of the following system of four elliptic partial differential equations in two dimensions:

$$-\Delta u - \frac{\partial \omega}{\partial y} = 0, \quad (3.1)$$

$$-\Delta v + \frac{\partial \omega}{\partial x} = 0, \quad (3.2)$$

$$-\Delta \omega + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} - \text{Gr} \frac{\partial T}{\partial x} = 0, \quad (3.3)$$

$$-\Delta T + \text{Pr} \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = 0, \quad (3.4)$$

where $(u(x, y), v(x, y))$ are the velocity fields in the (x, y) directions, $\omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ is the component of the vorticity normal to the xy -plane (representing the in-plane rotation of an infinitesimal fluid element), and $T(x, y)$ is the temperature. Pr is a Prandtl number and Gr is a Grashof number. (For those who would rather see a Peclet number than a Prandtl number in the energy equation, Reynolds number is unity, so Pe and Pr are identical.)

These equations are subject to boundary conditions:

1. Along the bottom ($0 < x < 1, y = 0$): $u = v = 0, \frac{\partial T}{\partial y} = 0$.
2. Along the top ($0 < x < 1, y = 1$): $u = V_{lid}, v = 0, \frac{\partial T}{\partial y} = 0$.
3. Along the left ($x = 0, 0 < y < 1$): $u = v = 0, T = 0$.
4. Along the right ($x = 1, 0 < y < 1$): $u = v = 0, T = 1$ if $\text{Gr} > 0$ or 0 otherwise.

On each boundary, ω is given by its definition, $\omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, based on the normal gradients of either u and v (the tangential gradients being zero, according to the velocity boundary conditions). We make the standard assumption of Boussinesq perturbation theory, namely that the effect of density variation is significant only in the body force term, when multiplied by the acceleration of gravity (in Gr), and not in the mass or momentum flux terms.

Nonlinear iterative methods for (3.1)-(3.4) require an initial iterate, while Ψtc methods require an initial condition. For either, we start with a motionless, vorticity-free initial flow field. This initial condition is incompatible with the forcing boundary conditions on a set of measure zero for the continuous problem. Parameter continuation could be used to eliminate the resulting “impulsive start” but we do not pursue this possibility here.

In the interests of space, we leave the derivation of this velocity-vorticity form of the Navier-Stokes and energy governing equations (3.1)-(3.4) to the literature (e.g., [2] and citations therein). However, to motivate our main point, we note that the two equations with velocity components under the Laplacian operators come from differentiating the continuity equation and substituting from the definition of vorticity. Thus, for instance, we differentiate the continuity equation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, by x and the definition of vorticity, $\omega(x, y) = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, by y and eliminate the term in v to get (3.1).

We discretize the system (3.1)-(3.4) with a standard five-point stencil for each component, on a uniform Cartesian grid, using standard first-order upwinding for the convective terms and central differencing for the other first-order gradients. All four solution components are vertex centered. This discretization, the source code for which can be examined in the PETSc release, is famously diffusive at large Reynolds numbers and would not be of acceptable order for deriving accurate engineering results on efficient meshes. However, it serves our purpose of leading straightforwardly to a model problem with interesting nonlinearities. These nonlinearities may be tuned to be strong or weak, depending upon the magnitude of V_{lid} and/or the Grashof number.

PETSc [1] allows an enormous variety of solvers to be assembled via command-line options to solve (3.1)-(3.4). The reader is encouraged to download and build PETSc and experiment. We prefer various forms of preconditioned Jacobian-free Newton-Krylov solvers [21]. Such a solver has as its heart an inexact Newton method [7]. The Newton-correction equation is solved approximately with a Krylov method, such as GMRES [26], which requires the action of the Jacobian of $F(u)$ only in the form of Jacobian-vector products. Such Jacobian-vector products are available either in approximate finite-difference (Fréchet derivative) form, via construction of an explicit Jacobian matrix from analytical or numerical means, or via automatic differentiation (AD). The PETSc distribution example is of the AD variety, exploiting the automatic differentiation package ADIC [3]. In practice, the Jacobian-based linear systems for the Newton corrections are ill-conditioned, and require preconditioning. PETSc is equipped with many forms of preconditioning for this Cartesian grid discretization. We use the default, which is a block ILU relaxation sweep within a multilevel context. The block ILU kernel treats all four components common to each vertex implicitly. This is a well known practice in multicomponent problems with cross-coupling in transport and source terms, as in (3.1)-(3.4). (It also has excellent cache locality properties, since it leads to small BLAS3-like kernels that can be unrolled into registers.) Such solver issues have been extensively studied elsewhere, e.g., [13]. In the context of this paper, the aspect of performance about which we are directly concerned is the outer nonlinear convergence.

We readily observe from experimentation that Newton’s method struggles when this simulation is started at high values of V_{lid} and Gr , on any but the coarsest grids. For illustrative purposes, we use dimensionless values of $V_{lid} = 100$ and $Gr = 10^5$ below. Without Ψtc , the standard way to solve this system in PETSc is to use mesh sequencing, working up from a grid containing just 4×4 cells at its coarsest. Starting from a “cold” initial iterate even for a grid only as fine as 32×32 with these high parameter values for V_{lid} and Gr , Newton stagnates.

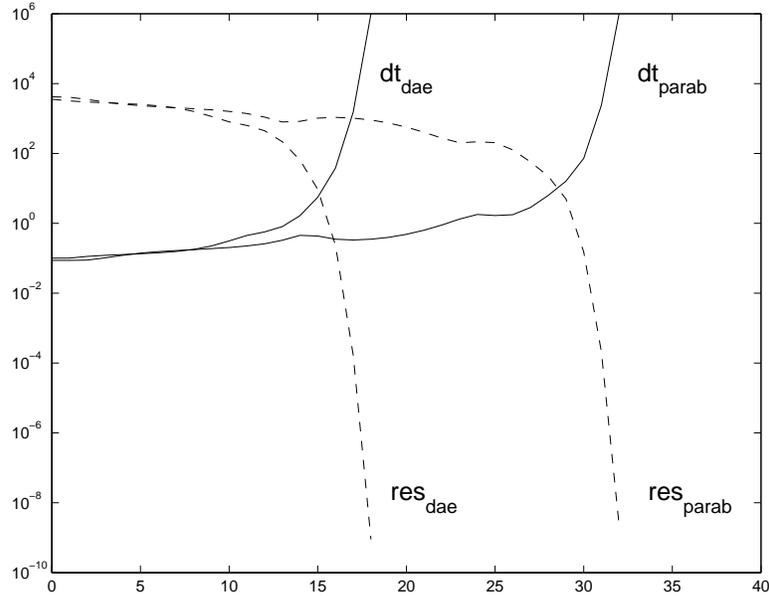


FIG. 3.1. Convergence of residual norm (dashed curve) and growth of pseudo time step (solid curve) for two versions of Ψtc on the driven cavity example. The DAE form of Ψtc converges faster than the fully parabolized ODE form.

We can prepend a tendency term to each equation in (3.1)-(3.4) and attempt to overcome this problem by Ψtc . The result is plotted in Figure 3.1. This figure contains two convergence plots of the Euclidean norm of the steady state residual, $\|F(x_n)\|$, as a function of iteration n , and corresponding plots of δ_n . The longer-running history pertains to the case in which (3.1)-(3.4) are fully parabolized. If, instead, we constrain the two velocity equations to be satisfied “exactly” (to within the limits of the overall inexact Newton method linear convergence), by not adding any time-stepping to those equations, we obtain the shorter-running history in Figure 3.1. Both techniques (ODE-based and DAE-based Ψtc) lead to convergence where Newton alone fails, apart from mesh-sequencing. However, the DAE form of the technique is superior. It converges after half as many pseudo-time steps and does not suffer any nonmonotonicity in $\|F(x_n)\|$ or δ_n .

3.2. Variable-density Reacting Flow. As a second example, we consider the axisymmetric flamesheet model from [19], expressed as a system of three steady-state PDEs. Instead of the velocity-vorticity treatment of the driven cavity above, we use a streamfunction-vorticity formulation, following the the development of [12]. With r and z for the radial and axial directions in cylindrical coordinates and primitive variables v_r and v_z as the radial and axial velocities, respectively, we define the vorticity component normal to the axisymmetric plane, representing in-plane rotation, $\omega = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$, and the variable-density Stokes streamfunction, ψ , in terms of which

$$\rho r v_r = -\frac{\partial \psi}{\partial z} \quad \text{and} \quad \rho r v_z = \frac{\partial \psi}{\partial r}.$$

The three equations are mass conservation:

$$-\frac{\partial}{\partial z} \left(\frac{1}{r\rho} \frac{\partial \psi}{\partial z} \right) - \frac{\partial}{\partial r} \left(\frac{1}{r\rho} \frac{\partial \psi}{\partial r} \right) - \omega = 0, \quad (3.5)$$

momentum conservation:

$$\begin{aligned}
& -\frac{\partial}{\partial r} \left(r^3 \frac{\partial}{\partial r} \left(\frac{\mu\omega}{r} \right) \right) - \frac{\partial}{\partial z} \left(r^3 \frac{\partial}{\partial z} \left(\frac{\mu\omega}{r} \right) \right) + r^2 \left[\frac{\partial}{\partial z} \left(\frac{\omega}{r} \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial r} \left(\frac{\omega}{r} \frac{\partial \psi}{\partial z} \right) \right] \\
& + r^2 g \frac{\partial \rho}{\partial r} + r^2 \nabla \cdot \left(\frac{v_r^2 + v_z^2}{2} \right) \cdot \text{iso } \rho = 0,
\end{aligned} \tag{3.6}$$

and species conservation:

$$-\frac{\partial}{\partial r} \left(r \rho D \frac{\partial S}{\partial r} \right) - \frac{\partial}{\partial z} \left(r \rho D \frac{\partial S}{\partial z} \right) + \frac{\partial}{\partial z} \left(S \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial r} \left(S \frac{\partial \psi}{\partial z} \right) = 0. \tag{3.7}$$

Here, μ is mixture viscosity, D is molecular diffusivity, and the notation “iso” is defined by $\text{iso } \rho = \left(-\frac{\partial \rho}{\partial z}, -\frac{\partial \rho}{\partial r} \right)^T$. The system is closed by constitutive laws and an equation of state that express μ , D , and ρ as nonlinear functions of S .

A single conserved scalar, S , is capable of describing the complete composition and thermodynamic state of the fluid under the assumption of infinitely fast single-step kinetics and the additional assumptions that the binary diffusion coefficients of all pairs of species are equal and that the Lewis number (ratio of thermal to species diffusivity) is unity. S is a so-called Shvab-Zeldovich or mixture-fraction variable, as defined, for instance, in [32]. The flame front, Γ , is defined as a level set of S : $\Gamma \equiv \{(r, z) | S(r, z) = S_f\}$, on one side of which fuel cannot exist and on the other side of which oxidizer cannot exist, because each is consumed at the interface. The value of S_f comes from the stoichiometry of the reaction. For a detailed derivation of this formulation in the combustion literature in the case of a methane-air flame, see [18].

The geometry is that of a co-flowing gaseous burner, with an inner tube out of which flows a methane-air mixture and an outer annulus out of which flows a standard atmosphere. The boundary conditions are those of symmetry for $r = 0$, a stress-free free-stream for $r = r_{max}$, inhomogeneous Dirichlet values for velocities and chemical species at the inflow where $z = 0$, and extrapolative exit conditions at $z = z_{max}$. The inflow velocities are piecewise constant in r , different on either side of the radius that divides the inner tube from the outer co-flowing annulus. The inflow condition on S is constructed to represent a Gaussian peak of combustion products (carbon dioxide and water vapor) centered at the dividing radius between the co-flowing fuel and oxidizer streams and a corresponding temperature profile achieving the stoichiometrically appropriate peak value at the dividing radius. These conditions are swept downstream to form an initial condition throughout the domain. Thus, the problem of ignition is avoided. The flame is “underventilated” in the language of combustion theory, which implies that the initial flame tube must find a final bulbous pinched steady-state shape characteristic of the flame attached to a bunsen burner. There are numerous technical issues in flameholding whose discussion we suppress here, with the justification that this model is well published, its limitations well understood, and various remedies for those limitations available. (See, e.g., [30] and [10].) Though highly simplified, the problem is practically motivated in that it serves as a useful asymptotic precursor for finite-rate detailed kinetics flame structure models.

Because of the polynomial nonlinearities that are evident in (3.5)-(3.7) and the more insidious nonlinearity of the different state functions of S as a function of the *a priori* unknown interface where $S(r, z) = S_f$, this system is virtually impossible to solve by Newton’s method from any readily specified initial iterate, including the relatively sophisticated, already ignited state described

above. A “natural” pseudo-transient approach prepends a term $r\rho\frac{\partial S}{\partial t}$ to (3.7) and a term $r^3\rho\frac{\partial \omega}{\partial t}$ to (3.6).

We argue on the basis of physical intuition and the experience of the prior example that it might be harmful to allow ψ to relax by prepending a similar tendency term in the streamfunction to (3.5). Like (3.1)-(3.2), Eq. (3.5) derives from the underlying primitive variable expression of the conservation of mass for the reacting flow. Due to chemical reaction and heat release, the density ρ varies (by up to an order of magnitude in common terrestrial hydrocarbon-air or hydrogen-air diffusion flame applications). This variation cannot be neglected outside of the buoyancy term, as it was in the earlier example. However, we do not want to admit density variations arising from transient momentum imbalances, which could lead to acoustic-like waves. While there are many applications, such as turbulent combustion, in which thermal fluctuations and acoustic waves interact in dynamically relevant ways, these applications require far more complex models than (3.5)-(3.7) to describe. On the other hand, there are many applications for which (3.5)-(3.7) is a useful model. For these situations, we exploit our theory for Ψ_{tc} that retains (3.5) as it is, an elliptic constraint enforced at every time step to implicitly equilibrate pressure and suppress acoustic stiffness.

A convergence history for DAE-based Ψ_{tc} for the flamesheet problem is shown in Figure 3.2. There is an initial rapid plunge of the steady-state residual on the first pseudo-time step. This corresponds to creating a consistent initial condition for the DAE (Assumption 2.1, part 1). There is no assurance in the limited theory of this paper that this first Newton step should be successful, which is why we start with a relatively small initial time step, $\delta_0 = 10^{-5}$. Following this point, Ψ_{tc} convergence is fairly uneventful. However, it should be noted that the success of the method is relatively sensitive to the initial time step. If we attempt $\delta_0 \geq 10^{-4}$, for instance, the Newton iteration dies on the first step with an infeasible iterate (components of S are driven out of bounds). In particular, unassisted Newton iteration on the steady-state system is hopeless. If we try $\delta_0 < 10^{-5}$, Newton’s method assisted by SER-style Ψ_{tc} will converge, but will take more pseudo-time steps to build δ_n up to a value at which a switchover to a full Newton iteration can safely occur.

In this implementation, the maximum increase of δ_n from one time step to the next is limited to a factor of 2, except at the final switchover to a full Newton step, which is triggered by the pseudo-time step reaching unity. This succession of doublings is observed in Fig. 3.2 between iterations 2 and 13, during which time the residual norm is struggling nonmonotonically, but at a level that is substantially below $\|F(x_0)\|$, so the criterion (1.3) permits a growth in δ_n that is limited only by the ratio bound. Obviously, many refinements of the strategy to pick δ_n are possible but the simple one we have adopted is sufficient, if possibly inefficient.

We remark that Ψ_{tc} is more crucial for this flamesheet example than it was for the driven cavity example. For the driven cavity, mesh sequencing leads to a steady-state solution with or without Ψ_{tc} . For the flamesheet, the problem has too rich a geometric structure to be represented on a grid much coarser than 16×16 in the axisymmetric plane, and the choice of initial mesh must be constructed rather carefully (with refinement in r around the mixing radius and gradual expansion in the z -direction downstream) for the flame to be sufficiently accurately discretized with second-order finite differences on such a coarse mesh. Even on a 16×16 grid, Ψ_{tc} is crucial. After interpolation of the 16×16 solution to a refined 32×32 grid, Ψ_{tc} is again crucial to the success of Newton’s method, as it is following interpolation of the 32×32 method to a 64×64 grid.

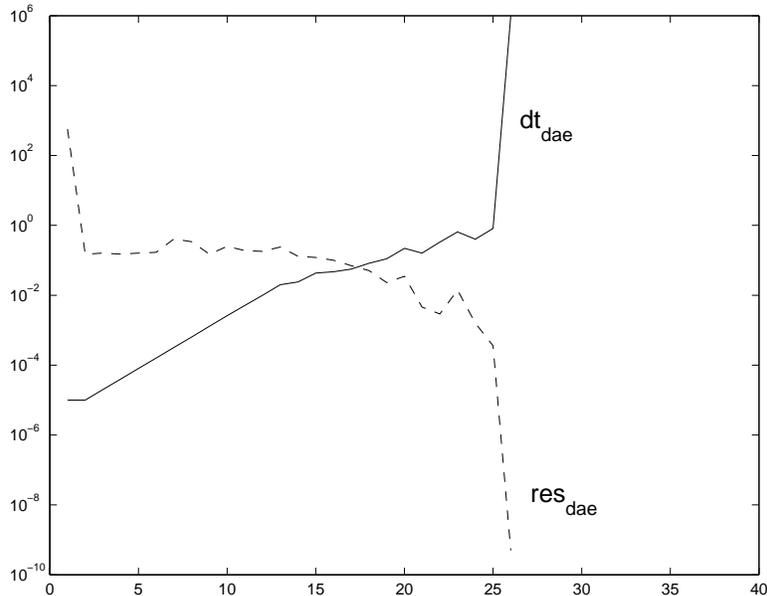


FIG. 3.2. Convergence of residual norm (dashed curve) and growth of pseudo time step (solid curve) for the DAE form of Ψtc on the flamesheet example.

It takes many doublings of grid density before the bilinearly interpolated solution on the refined grid lies initially in the ball of convergence of Newton's method applied directly to the steady-state system.

4. Conclusion. We have shown, under natural assumptions, that Ψtc is a globally convergent method for semi-explicit index-1 differential-algebraic equations. We have also argued that the index-1 DAE formulation is an important complement to the standard method-of-lines ODE formulation of Ψtc . Both propositions are illustrated by numerical experiments on model nonlinear fluid mechanical applications, which show, first, that Ψtc is effective and, second, that the DAE formulation can be more effective than the ODE formulation. Our model problems are of relatively low discrete dimension but are already sufficient to illustrate the challenges of applying Newton's method without Ψtc .

The objective of this paper is to analyze and illustrate Ψtc by itself with a fixed spatial discretization, approached with standard inexact Newton-Krylov solvers, and without much attention to the preconditioning of the resulting inner linear problems. In large-scale practice, linear preconditioning is important and is made easier by the addition of the implicit time step. Finally, to recapitulate the introduction, we believe that Ψtc is best exploited in combination with other techniques, especially of the multilevel variety, both for nonlinear robustness and for solution efficiency.

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