DETERMINANT APPROXIMATIONS

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Abstract. A sequence of approximations for the determinant of a complex matrix is derived, along with relative error bounds. The first approximation in this sequence represents an extension of Fischer's and Hadamard's inequalities to indefinite non-Hermitian matrices. The approximations are based on expansions of $\det(X) = \exp(\text{trace}(\log(X)))$.

Key words. determinant, trace, spectral radius, determinantal inequalities, tridiagonal matrix

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1. Introduction. The determinant approximations presented here were motivated by a problem in computational quantum field theory. Usually it is recommended that the determinant be computed via a LU decomposition with partial pivoting [4, §14.6], [10, §3.18]. However, in the context of this physics application, it is desirable to work with expansions of $\det(X) = \exp(\text{trace}(\log(X)))$.

To approximate the determinant $\det(M)$ of a complex square matrix M, decompose $M = M_0 + M_E$ so that M_0 is non-singular. Then $\det(M) = \det(M_0)\det(I + M_0^{-1}M_E)$, where I is the identity matrix. In

$$\det(I + M_0^{-1}M_E) = \exp(\text{trace}(\log(I + M_0^{-1}M_E))),$$

we expand $\log(I + M_0^{-1}M_E)$, obtaining a sequence of increasingly accurate approximations, and relative error bounds for these approximations. The accuracy of the approximations is determined by the spectral radius of $M_0^{-1}M_E$.

If M_0 is the diagonal or a block-diagonal of M then the first approximation in this sequence amounts to an extension of Hadamard's inequality [5, Theorem 7.8.1], [3, Theorem II.3.17] and Fischer's inequality [5, Theorem 7.8.3], [3, §II.5] to indefinite non-Hermitian matrices.

Literature. In [9] determinant approximations for symmetric positive-definite matrices are constructed from sparse approximate inverses. Relative perturbation bounds for determinants that involve the condition number of the matrix are given in [4, Problem 14.15] and for symmetric positive-definite matrices in [9, Lemma 2.1]. For integer matrices a statistical analysis in [1] estimates the tightness of Hadamard's inequality. In [7] lower and upper bounds for the determinant are presented for matrices whose trace has sufficiently large magnitude.

Overview. Our main results, the determinant approximations and their relative error bounds, are presented in §2. We start with approximations from block diagonals in §2.1, and extend them to a sequence of more general, higher order approximations in §2.2. In §2.3 we show how they simplify for block tridiagonal matrices. The idea for the approximations is sketched in §3. Auxiliary determinantal inequalities are derived in §4, and the proofs of the results from §2 are given in §5.

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Notation. The eigenvalues of a complex square matrix A are $\lambda_j(A)$ and its spectral radius is $\rho(A) \equiv \max_j |\lambda_j(A)|$. The identity matrix is I, and A^* is the conjugate transpose of A.

We will often (but not always :-) use the following convention: $\log(X)$ and $\exp(X)$ denote logarithm and exponential function of a matrix X, and $\ln(x)$ and e^x denote the natural logarithm and exponential function of a scalar x.

2. Main Results. In this section we present the determinant approximations and their relative error bounds. The proofs are postponed until §5.

2.1. Diagonal Approximations. We bound the determinant of a complex matrix by the determinant of a block diagonal. This represents an extension of the fact that the determinant of a positive-definite matrix is bounded above by the determinant of its diagonal blocks, as the two well-known inequalities below show.

Fischer's Inequality [5, Theorem 7.8.3], [3, §II.5]. If M is a Hermitian positive-definite matrix, partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}$$

so that M_{11} and M_{22} are square, but not necessarily of the same dimension, then

$$\det(M) \leq \det(M_{11}) \det(M_{22}).$$

Repeated application of Fischer's inequality leads to diagonal blocks of dimension 1 and Hadamard's inequality.

Hadamard's Inequality [5, Theorem 7.8.1], [3, Theorem II.3.17]. If M is Hermitian positive-definite with diagonal elements m_{jj} then

$$\det(M) \leq \prod_j m_{jj}.$$

We extend Hadamard's and Fischer's inequalities to indefinite non-Hermitian matrices.

Let M be a complex square matrix partitioned as a $k \times k$ block matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1k} \\ M_{21} & M_{22} & \dots & M_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ M_{k1} & M_{k2} & \dots & M_{kk} \end{pmatrix},$$

where the diagonal blocks M_{jj} are square but not necessarily of the same dimension.

Decompose $M = M_D + M_{\text{off}}$ into diagonal blocks M_D and off-diagonal blocks M_{off},

$$M_D = \begin{pmatrix} M_{11} & & & \\ & M_{22} & & \\ & & \ddots & \\ & & & M_{kk} \end{pmatrix}, \qquad M_{\text{off}} = \begin{pmatrix} 0 & M_{12} & \dots & M_{1k} \\ M_{21} & 0 & \dots & M_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ M_{k1} & M_{k2} & \dots & 0 \end{pmatrix}.$$

The block diagonal matrix M_D is called a pinching of M [3, §II.5]. In this section we approximate $\det(M)$ by the determinant of a pinching, $\det(M_D)$.

Theorem 2.1. *If* $\det(M)$ *is real,* M_D *is non-singular with* $\det(M_D)$ *real, and all eigenvalues* $\lambda_j(M_D^{-1}M_{\text{off}})$ *are real with* $\lambda_j(M_D^{-1}M_{\text{off}}) > -1$ *then*

$$0 < \det(M) \le \det(M_D) \qquad \text{or} \qquad \det(M_D) \le \det(M) < 0.$$

Corollary 2.2. *Theorem 2.1 implies Hadamard's and Fischer's inequalities.*

Theorem 2.1 implies an obvious relative error bound for the determinant of a pinching,

$$0 < \frac{\det(M_D) - \det(M)}{\det(M_D)} \le 1.$$

The upper bound can be tightened. Denote by n the dimension of M.

Theorem 2.3. *If* $\det(M)$ *is real,* M_D *is non-singular with* $\det(M_D)$ *real, and all eigenvalues* $\lambda_j(M_D^{-1}M_{\text{off}})$ *are real with* $\lambda_j(M_D^{-1}M_{\text{off}}) > -1$*, then*

$$0 < \frac{\det(M_D) - \det(M)}{\det(M_D)} \le 1 - e^{-\frac{n\rho^2}{1+\lambda_{\min}}},$$

where $\rho \equiv \rho(M_D^{-1}M_{\text{off}})$ *and* $\lambda_{\min} \equiv \min_{1\le j\le n} \lambda_j(M_D^{-1}M_{\text{off}})$.

Theorem 2.3 gives a bound on the relative error of the approximation $\det(M_D)$ to $\det(M)$. The upper bound on the error is small if the eigenvalues of $M_D^{-1}M_{\text{off}}$ are small in magnitude and not too close to -1. Note that $\lambda_{\min} < 0$ because $M_D^{-1}M_{\text{off}}$ has a zero diagonal, hence $\text{trace}(M_D^{-1}M_{\text{off}}) = 0$. In the argument of the exponential function

$$\frac{n\rho^2}{1+\lambda_{\min}} > n\rho^2.$$

In particular, we can expect the pinching $\det(M_D)$ to be a bad approximation to $\det(M)$ when $I + M_D^{-1}M_{\text{off}}$ is close to singular.

The bounds in Theorem 2.3 can be tightened when the spectral radius is sufficiently small.

Theorem 2.4. *If, in addition to the assumptions of Theorem 2.3, also* $n\rho^2 < 1$ *then*

$$0 < \frac{\det(M_D) - \det(M)}{\det(M_D)} \le n\rho^2.$$

This means, if M is 'diagonally dominant' in the sense that the spectral radius ρ of $M_D^{-1}M_{\text{off}}$ is sufficiently small then the relative error in $\det(M_D)$ is proportional to ρ^2.

Theorems 2.3 and 2.4 imply relative error bounds for Fischer's and Hadamard's inequalities.

Corollary 2.5 (Error for Fischer's Inequality). *If*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}$$

is Hermitian positive-definite then

$$0 < \frac{\det(M_{11})\det(M_{22}) - \det(M)}{\det(M_{11})\det(M_{22})} \le 1 - e^{-\frac{n\,\rho^2}{1+\lambda_{\min}}},$$

where[1]

$$\rho \equiv \|M_{11}^{-1/2} M_{12} M_{22}^{-1/2}\|_2, \qquad \lambda_{\min} \equiv \min_{1\le j\le n} \lambda_j(M_{11}^{-1/2} M_{12} M_{22}^{-1/2}).$$

If also $n^2\rho < 1$ *then*

$$0 < \frac{\det(M_{11})\det(M_{22}) - \det(M)}{\det(M_{11})\det(M_{22})} \le n\rho^2.$$

Corollary 2.6 (Error for Hadamard's Inequality). *If* M *is Hermitian positive-definite with diagonal elements* m_{jj}, $1 \le j \le n$, *and* ρ *is the spectral radius of the matrix* B *with elements*

$$b_{ij} \equiv \begin{cases} 0 & \text{if } i = j \\ m_{ij}/\sqrt{m_{ii}m_{jj}} & \text{if } i \ne j \end{cases}$$

then

$$0 < \frac{m_{11}\cdots m_{nn} - \det(M)}{m_{11}\cdots m_{nn}} \le 1 - e^{-\frac{n\rho^2}{1+\lambda_{\min}}},$$

where $\lambda_{\min} \equiv \min_{1\le j\le n} \lambda_j(B)$.

If also $n^2\rho < 1$ *then*

$$0 < \frac{m_{11}\cdots m_{nn} - \det(M)}{m_{11}\cdots m_{nn}} \le n\rho^2.$$

The following example shows that $|\det(M)| \le |\det(M_D)|$ may not hold when $M_D^{-1}M_{\text{off}}$ has complex eigenvalues or real eigenvalues that are smaller than -1.

Example 1. *Even if all eigenvalues* $\lambda_j(M_D^{-1}M_{\text{off}})$ *satisfy* $|\lambda_j(M_D^{-1}M_{\text{off}})| < 1$, *it is still possible that* $|\det(M)| > |\det(M_0)|$ *when some* $\lambda_j(M_D^{-1}M_{\text{off}})$ *are complex.*

Consider

$$M = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}, \qquad M_D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_{\text{off}} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} = M_D^{-1}M_{\text{off}}.$$

Then $\lambda_j(M_D^{-1}M_{\text{off}}) = \pm\alpha$, *and* $\det(M) = 1 - \alpha^2$. *Choose* $\alpha = \frac{1}{2}\imath$, *where* $\imath = \sqrt{-1}$. *Then both eigenvalues of* $M_D^{-1}M_{\text{off}}$ *are complex,* $\lambda_j(M_D^{-1}M_{\text{off}}) = \pm\frac{1}{2}\imath$ *and* $|\lambda_j(M_D^{-1}M_{\text{off}})| < 1$. *But* $\det(M) = 1.25 > 1 = \det(M_D)$.

The situation $\det(M_D) > \det(M)$ *can also occur when* $M_D^{-1}M_{\text{off}}$ *has a real eigenvalue that's less than* -1. *If* $\alpha = 3$ *in the matrices above then one eigenvalue of* $M_D^{-1}M_{\text{off}}$ *is* -2, *and* $|\det(M)| = 8 > \det(M_D) = 1$. *In general,* $|\det(M)|/\det(M_D) \to \infty$ *as* $|\alpha| \to \infty$.

[1] $\|\cdot\|_2$ denotes the Euclidean two-norm.

This example illustrates that, unless the eigenvalues of $M_0^{-1}M_E$ are real and greater than -1, $\det(M_D)$ is, in general, not a bound for $\det(M)$. In the case of complex eigenvalues, however, we can still determine how well $\det(M_D)$ *approximates* $\det(M)$.

Below is a relative error bound for $\det(M_D)$ for the case when the spectral radius of $M_D^{-1}M_{\text{off}}$ is sufficiently small. The eigenvalues are allowed to be complex.

THEOREM 2.7 (Complex Eigenvalues). *If M_D is non-singular and $\rho \equiv \rho(M_D^{-1}M_{\text{off}}) < 1$ then*

$$\frac{|\det(M) - \det(M_D)|}{|\det(M_D)|} \leq c\rho\, e^{c\rho}, \qquad \text{where} \quad c \equiv -n\ln(1-\rho).$$

If also $c\rho < 1$ then

$$\frac{|\det(M) - \det(M_D)|}{|\det(M_D)|} \leq 2c\rho.$$

As before this means, if M is 'diagonally dominant' in the sense that the eigenvalues of $M_D^{-1}M_{\text{off}}$ are small in magnitude then we can get a relative error bound for $\det(M_D)$. The bound in Theorem 2.7 is worse than the bound for real eigenvalues in Theorem 2.4 because it is only proportional to ρ rather than ρ^2, and the multiplicative factors are larger.

2.2. A Sequence of General Higher Order Approximations. We extend the diagonal approximations in §2.1 to a sequence of more general approximations that become increasingly more accurate.

Let $M = M_0 + M_E$ be *any* decomposition where M_0 is non-singular and $\rho(M_0^{-1}M_E) < 1$ (here 'E' stands for 'expendable'). Below we give a sequence of approximations Δ_m for $\det(M)$.

THEOREM 2.8. *Let M_0 be non-singular and $\rho \equiv \rho(M_0^{-1}M_E) < 1$. Define*

$$\Delta_m \equiv \det(M_0)\ \exp\left(\sum_{p=1}^{m} \frac{(-1)^{p-1}}{p}\,\text{trace}((M_0^{-1}M_E)^p)\right).$$

Then

$$\frac{|\det(M) - \Delta_m|}{|\Delta_m|} \leq c\rho^m\, e^{c\rho^m}, \qquad \text{where} \quad c \equiv -n\ln(1-\rho).$$

If also $c\rho^m < 1$ then

$$\frac{|\det(M) - \Delta_m|}{|\Delta_m|} \leq 2c\,\rho^m.$$

The accuracy of the approximations is determined by the spectral radius ρ of $M_0^{-1}M_E$. In particular, the relative error bound for the approximation Δ_m is proportional to ρ^m, and the approximations tend to improve with increasing m. The approximations can be determined from successive updates

$$\Delta_0 = \det(M_0), \qquad \Delta_m = \Delta_{m-1} * \exp\left(\frac{(-1)^{m-1}}{m}\,\text{trace}((M_0^{-1}M_E)^m)\right), \quad m \geq 1.$$

Theorem 2.8 represents an extension of Theorem 2.7 because the diagonal approximations in Theorem 2.7 correspond to $\Delta_1 = \det(M_0)\exp\big(\text{trace}(M_0^{-1}M_E)\big)$. The nice thing there is that $\text{trace}(M_0^{-1}M_E) = 0$, hence $\Delta_1 = \det(M_0)$.

We can derive a better error bound for the odd-order approximations when the eigenvalues of $M_0^{-1}M_E$ are real.

THEOREM 2.9 (Real Eigenvalues). *If, in addition to the conditions of Theorem 2.8, the eigenvalues of $M_0^{-1}M_E$ are also real and m is odd then*

$$\frac{|\det(M) - \Delta_m|}{|\Delta_m|} \leq 1 - e^{-\frac{n}{m+1}\rho^{m+1}}.$$

If also $\frac{2n}{m+1}\rho^{m+1} < 1$ then

$$\frac{|\det(M) - \Delta_m|}{|\Delta_m|} \leq \frac{2n}{m+1}\rho^{m+1}.$$

Theorem 2.9 represents an extension of Theorem 2.4 because the diagonal approximations in Theorem 2.4 correspond to $\Delta_1 = \det(M_0)\exp\big(\text{trace}(M_0^{-1}M_E)\big)$, where $\text{trace}(M_0^{-1}M_E) = 0$.

2.3. (Block) Tridiagonal Matrices. For block tridiagonal matrices T the expressions for the approximations Δ_m simplify, because the traces of the odd matrix powers turn out to be zero.

When T is a complex block tridiagonal matrix,

$$T = \begin{pmatrix} A_1 & B_1 & & \\ C_1 & A_2 & \ddots & \\ & \ddots & \ddots & B_{k-1} \\ & & C_{k-1} & A_k \end{pmatrix},$$

decompose $T = T_D + T_{\text{off}}$ with

$$T_D = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}, \quad T_{\text{off}} = \begin{pmatrix} 0 & B_1 & & \\ C_1 & 0 & \ddots & \\ & \ddots & \ddots & B_{k-1} \\ & & C_{k-1} & 0 \end{pmatrix}$$

where the diagonal blocks A_i have the same dimension.

Since the odd powers of $T_D^{-1}T_{\text{off}}$ have zero diagonal blocks, only half of the approximations contribute to an increase in accuracy.

THEOREM 2.10. *If T_D is non-singular and $\rho(T_D^{-1}T_{\text{off}}) < 1$ the approximations in Theorem 2.8 reduce to*

$$\Delta_0 = \det(T_D), \qquad \Delta_m = \begin{cases} \Delta_{m-1} & \text{if } m \text{ is odd} \\ \Delta_{m-2}/\exp\left(\text{trace}\left(\frac{(T_D^{-1}T_{\text{off}})^m}{m}\right)\right) & \text{if } m \text{ is even.} \end{cases}$$

Theorem 2.10 shows that an odd-order approximation is equal to the previous even-order approximation. Hence the odd-order approximations lose one order of accuracy.

Moreover, the approximations Δ_m can be determined from individual blocks. For instance,

$$\text{trace}((T_D^{-1}T_{\text{off}})^2) = 2\sum_{j=1}^{k-1}\text{trace}(A_j^{-1}B_jA_{j+1}^{-1}C_j)$$

while $\text{trace}(T_D^{-1}T_{\text{off}}) = \text{trace}((T_D^{-1}T_{\text{off}})^3) = 0$.

In one of our applications we have complex non-Hermitian block tridiagonal matrices with complex eigenvalues, where, for instance, $n = 512$ and $\rho \approx 10^{-1}$. The bound in Theorem 2.7 predicts the relative error extremely well when $m = 2$. The exact relative error (computed in Matlab) is about 7×10^{-4}, while the error bound in Theorem 2.7 gives $\frac{7}{4}c\,\rho^2 \approx 9 \times 10^{-4}$.

3. Idea.

The idea for the approximations in §2 came about as follows.

If M_0 is non-singular then $M = M_0(I + M_0^{-1}M_E)$. Hence $\det(M) = \det(M_0)\det(I + M_0^{-1}M_E)$. We express the determinant of $I + M_0^{-1}M_E$ as

$$\det(I + M_0^{-1}M_E) = \exp(\text{trace}(\log(I + M_0^{-1}M_E))).$$

For which matrices X is the above expression valid? If X is singular there does not exist a matrix W such that $X = \exp(W)$ [6, Theorem 6.4.15(b)]. Hence the expression can only be valid for nonsingular X.

Lemma 3.1. *If X is non-singular then* $\det(X) = \exp(\text{trace}(\log(X)))$.

Proof. If X is nonsingular then there exists a matrix W such that $X = \exp(W)$ [6, Theorem 6.4.15(a)]. With $\log(X) := W$ we get $X = \exp(\log(X))$. Since for any square matrix W, $\det(\exp(W)) = \exp(\text{trace}(W))$ [6, Problem 6.2.4], we can write $\det(X) = \exp(\text{trace}(\log(X)))$. □

4. Auxiliary Determinant Bounds.

We derive approximations and bounds for $\det(I + A)$. Let A be a complex square matrix of order n, with eigenvalues $\lambda_i(A)$ and spectral radius $\rho(A) \equiv \max_{1\le i\le n} |\lambda_i(A)|$.

Lemma 4.1. *If either A has real eigenvalues with $\lambda_i(A) > -1$, $1 \le i \le n$, or if $\rho(A) < 1$ then*

$$\det(I + A) = \exp\left(\sum_{i=1}^{n} \ln(1 + \lambda_i(A))\right).$$

Proof. If all $\lambda_i(A) > -1$ or if $\rho(A) < 1$ then $I + A$ is non-singular. Lemma 3.1 implies

$$\det(I + A) = \exp(\text{trace}(\log(I + A))).$$

If A has real eigenvalues with $\lambda_i(A) > -1$ then $\lambda_i(A)$ are in the interior of the domain of the real logarithm $\ln(1 + x)$. Thus [8, Theorem 9.4.6] the eigenvalues of $\log(I + A)$ are $\ln(1 + \lambda_i(A))$ and

$$\text{trace}(\log(I + A)) = \sum_{i=1}^{n} \ln(1 + \lambda_i).$$

If $\rho(A) < 1$ then [8, §9.8, p 329]

$$\log(I + A) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} A^p.$$

From the linearity of the trace [8, §1.8] and the fact that $\text{trace}(A^p) = \sum_{i=1}^{n} \lambda_i(A)^p$ follows

$$\text{trace}(\log(I + A)) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p}\, \text{trace}(A^p) = \sum_{i=1}^{n}\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p}\, \lambda_i(A)^p = \sum_{i=1}^{n} \ln(1 + \lambda_i(A)).$$

□

LEMMA 4.2. *If A has real eigenvalues with $\lambda_i(A) > -1$, $1 \leq i \leq n$, then*

$$\exp(\operatorname{trace}(A))\, e^{-\frac{n\rho(A)^2}{1+\lambda_{\min}}} \leq \det(I+A) \leq \exp(\operatorname{trace}(A)),$$

where $\lambda_{\min} \equiv \min_{1\leq j\leq n} \lambda_j(A)$.

If also $\operatorname{trace}(A) = 0$ *then*

$$e^{-\frac{n\rho(A)^2}{1+\lambda_{\min}}} \leq \det(I+A) \leq 1.$$

Proof. Abbreviate $\lambda_i \equiv \lambda_i(A)$. Lemma 4.1 implies

$$\det(I+A) = \exp\left(\sum_{i=1}^{n} \ln(1+\lambda_i)\right).$$

For $x > -1$ we have [2, 4.1.33] $\frac{x}{x+1} \leq \ln(1+x) \leq x$. Hence

$$\operatorname{trace}(A) - \sum_{i=1}^{n} \frac{\lambda_i^2}{1+\lambda_i} \leq \sum_{i=1}^{n} \ln(1+\lambda_i) \leq \operatorname{trace}(A),$$

where $\sum_{i=1}^{n} \lambda_i = \operatorname{trace}(A)$. Now bound $\sum_{i=1}^{n} \frac{\lambda_i^2}{1+\lambda_i} \leq \frac{n\rho(A)^2}{1+\lambda_{\min}}$ and exponentiate the inequalities.

When $\operatorname{trace}(A) = 0$ then $\exp(\operatorname{trace}(A)) = 1$. □

LEMMA 4.3. *If λ is a complex scalar with $|\lambda| < 1$ then*

$$\left|\ln(1+\lambda) - \sum_{p=1}^{m} \frac{(-1)^{p-1}}{p} \lambda^p\right| \leq -|\lambda|^m\ \ln(1-|\lambda|).$$

Proof. For $|\lambda| < 1$ one can use the series expansion [2, 4.1.24]

$$\ln(1+\lambda) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} \lambda^p.$$

Hence

$$|\ln(1+\lambda)| \leq \sum_{p=1}^{\infty} \frac{1}{p} |\lambda|^p = -\ln(1-|\lambda|),$$

see also [2, 4.1.38]. Therefore

$$\left|\ln(1+\lambda) - \sum_{p=1}^{m} \frac{(-1)^{p-1}}{p} \lambda^p\right| \leq \sum_{p=m+1}^{\infty} \frac{1}{p} |\lambda|^p = |\lambda|^m \sum_{p=1}^{\infty} \frac{1}{p+m} |\lambda|^p \leq -|\lambda|^m\ \ln(1-|\lambda|).$$

□

Lemma 4.4. *Define*

$$D_m \equiv \exp\left(\sum_{p=1}^{m} \frac{(-1)^{p-1}}{p}\,\mathrm{trace}(A^p)\right).$$

If $\rho(A) < 1$ *then*

$$\frac{|\det(I+A) - D_m|}{|D_m|} \le c\rho(A)^m\, e^{c\rho(A)^m}, \qquad \text{where} \quad c \equiv -n\ln(1-\rho(A)).$$

If also $c\rho(A)^m < 1$ *then*

$$\frac{|\det(I+A) - D_m|}{|D_m|} \le \frac{7}{4} c\,\rho(A)^m.$$

Proof. Since $\rho(A) < 1$, Lemma 4.1 implies

$$\det(I+A) = \exp\left(\sum_{i=1}^{n} \ln(1+\lambda_i)\right),$$

where for simplicity $\lambda_i = \lambda_i(A)$. Hence $\det(I+A) = D_m\, e^z$, where

$$z \equiv \sum_{i=1}^{n} \left\{ \ln(1+\lambda_i) - \sum_{p=1}^{m} \frac{(-1)^{p-1}}{p} \lambda_i^p \right\}$$

and

$$\frac{|\det(I+A) - D_m|}{|D_m|} = |e^z - 1|.$$

From [2, 4.2.39] $|e^z - 1| \le |z| e^{|z|}$ and, if $0 < |z| < 1$ then [2, 4.2.38] $|e^z - 1| \le \frac{7}{4}|z|$. It remains to bound $|z|$.

The triangle inequality and Lemma 4.3 imply

$$|z| \le -\sum_{i=1}^{n} |\lambda_i|^m \,\ln(1 - |\lambda_i|) \le -n\rho(A)^m \,\ln(1-\rho(A)).$$

Therefore

$$\frac{|\det(I+A) - D_m|}{|D_m|} \le |e^z - 1| \le |z|e^{|z|} \le c\rho(A)^m\, e^{c\rho(A)^m},$$

and if $c\rho(A)^m < 1$ then

$$\frac{|\det(I+A) - D_m|}{|D_m|} \le |e^z - 1| \le \frac{7}{4}|z| \le \frac{7}{4} c\,\rho(A)^m.$$

□

Lemma 4.5. *Define*

$$D_m \equiv \exp\left(\sum_{p=1}^{m} \frac{(-1)^{p-1}}{p}\,\mathrm{trace}(A^p)\right).$$

If A has real eigenvalues, $\rho(A) < 1$, and m is odd then then

$$0 \leq \frac{D_m - \det(I + A)}{D_m} \leq 1 - e^{\frac{n}{m+1}\rho(A)^{m+1}}.$$

If also $\frac{2n}{m+1}\rho(A)^{m+1} < 1$ then

$$0 \leq \frac{D_m - \det(I + A)}{D_m} \leq \frac{2n}{m+1}\rho(A)^{m+1}.$$

Proof. Abbreviate $\lambda_i \equiv \lambda_i(A)$. Lemma 4.1 implies

$$\det(I + A) = \exp\left(\sum_{i=1}^{n} \ln(1 + \lambda_i)\right).$$

Hence $\det(I + A) = D_m \, e^z$, where

$$z \equiv \sum_{i=1}^{n} \left\{ \ln(1 + \lambda_i) - \sum_{p=1}^{m} \frac{(-1)^{p-1}}{p} \lambda_i^p \right\}$$

and

$$\frac{D_m - \det(I + A)}{D_m} = 1 - e^z.$$

Let's bound $1 - e^z$. For $-1 < \lambda < 1$ one can use the series expansion [2, 4.1.24]

$$\ln(1 + \lambda) = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} \lambda^p.$$

Thus

$$\ln(1 + \lambda) - \sum_{p=1}^{m} \frac{(-1)^{p-1}}{p} \lambda^p = \sum_{p=m+1}^{\infty} \frac{(-1)^{p-1}}{p} \lambda^p.$$

When m is odd, i.e. $m = 2k + 1$ for some $k \geq 0$,

$$\ln(1 + \lambda) - \sum_{p=1}^{2k+1} \frac{(-1)^{p-1}}{p} \lambda^p = \sum_{p=2k+2}^{\infty} \frac{(-1)^{p-1}}{p} \lambda^p < 0,$$

because $-1 < \lambda < 1$. Hence $z \leq 0$, $e^z \leq 1$ and

$$\frac{D_m - \det(I + A)}{D_m} = 1 - e^z \geq 0,$$

which proves the lower bound.

As for the upper bound, when $m = 2k + 1$ for some $k \geq 0$ then

$$z \geq -\sum_{i=1}^{n} \frac{\lambda^{2k+2}}{2k+2} \geq -\frac{n}{2k+2}\rho(A)^{2k+2} = -\frac{n}{m+1}\rho(A)^{m+1},$$

again because $-1 < \lambda_i < 1$. Therefore

$$\frac{D_m - \det(I + A)}{D_m} = 1 - e^z \leq 1 - e^{-\frac{n}{m+1}\rho(A)^{m+1}}.$$

If $\frac{2n}{m+1}\rho(A)^{m+1} < 1$ then we can apply [2, 4.2.38] $|e^x - 1| \leq \frac{7}{4}|x|$ for $|x| < 1$ to get

$$\frac{D_m - \det(I+A)}{D_m} \leq 1 - e^{-\frac{n}{m+1}\rho(A)^{m+1}} \leq \frac{7n}{4(m+1)}\rho(A)^{m+1} \leq \frac{2n}{m+1}\rho(A)^{m+1}.$$

□

5. Proofs of the Main Results in §2. We use the determinantal inequalities in §4 to derive the approximations and their bounds in §2.

Let M be a complex square matrix, and partition $M = M_0 + M_E$, where M_0 is non-singular. Denote by λ_j the eigenvalues of $M_0^{-1}M_E$.

THEOREM 5.1. *If* $\det(M)$ *is real,* M_0 *is non-singular with* $\det(M_0)$ *real,* $\text{trace}(M_0^{-1}M_E) = 0$ *and all eigenvalues* λ_j *of* $M_0^{-1}M_E$ *are real with* $\lambda_j > -1$ *then*

$$0 < \det(M) \leq \det(M_0) \qquad \text{or} \qquad \det(M_0) \leq \det(M) < 0.$$

Proof. Since M_0 is non-singular we can write $M = M_0(I + M_0^{-1}M_E)$. Then $\det(M) = \det(M_0)\det(I + M_0^{-1}M_E)$, and Lemma 4.2 implies $0 < \det(I + M_0^{-1}M_E) \leq 1$. This means $0 < \det(M) \leq \det(M_0)$ when $\det(M_0) > 0$, and $\det(M_0) \leq \det(M) < 0$ when $\det(M_0) < 0$. □

Proof of Theorem 2.1. Follows from Theorem 5.1 with $M_0 = M_D$ being block-diagonal and $M_E = M_{\text{off}}$ having a zero block diagonal. Hence $M_D^{-1}M_{\text{off}}$ also has a zero block-diagonal and $\text{trace}(M_D^{-1}M_{\text{off}}) = 0$.

Proof of Corollary 2.2. In the case of Hadamard's inequality choose k to be the dimension of M and M_D a diagonal matrix with scalar entries $M_{jj} \equiv m_{jj}$. Hence $\det(M_D) = \prod_j m_{jj}$. For Fischer's inequality set $k = 2$ and

$$M_D = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix},$$

a 2×2 block diagonal matrix. Hence $\det(M_D) = \det(M_{11})\det(M_{22})$.

Both inequalities assume that M is Hermitian positive-definite. This means all principal submatrices M_{jj} are Hermitian positive-definite and have Hermitian square roots $M_{jj}^{1/2}$ [5, Theorem 7.2.6]. The matrix

$$M_D^{1/2} \equiv \begin{pmatrix} M_{11}^{1/2} & & \\ & \ddots & \\ & & M_{kk}^{1/2} \end{pmatrix}$$

is a Hermitian square root of M_D. Decompose $M = M_D^{1/2}(I+B)M_D^{1/2}$ where $B \equiv M_D^{-1/2}M_{\text{off}}M_D^{-1/2}$ is a Hermitian matrix with zero diagonal blocks, hence $\text{trace}(B) = 0$. Moreover $I + B$ has the same inertia as M, i.e. all eigenvalues of $I + B$ are positive. Hence $\lambda_j(B) > -1$ for all eigenvalues of B. Lemma 4.2 implies $0 < \det(I+B) \leq 1$. Therefore

$$0 < \det(M) = \det(M_D^{1/2})\det(I+B)\det(M_D^{1/2}) \leq \det(M_D).$$

Proof of Theorem 2.3. Write $\det(M) = \det(M_D)\det(I{+}A)$, where $A = M_D^{-1} M_{\rm off}$ and $\text{trace}(M_D^{-1} M_{\rm off}) = 0$. Apply Lemma 4.2 to $\det(I + A)$. Then

$$0 \le 1 - \det(I + A) \le 1 - e^{-\frac{n\rho^2}{1+\lambda_{\min}}}.$$

Now multiply top and bottom of $1 - \det(I + A)$ by $\det(M_D)$.

Proof of Theorem 2.4. This is a special case of Theorem 2.9 with $m = 1$, $M_0 = M_D$, $M_E = M_{\rm off}$ and $\text{trace}(M_0^{-1} M_E) = 0$.

Proof of Theorem 2.7. This is the special case of Theorem 2.8 with $m = 1$, $M_0 = M_D$, $M_E = M_{\rm off}$ and $\text{trace}(M_0^{-1} M_E) = 0$.

Proof of Theorem 2.8. Write $\det(M) = \det(M_0)\det(I + A)$, where $A = M_0^{-1} M_E$. Apply Lemma 4.4 to $\det(I + A)$ and set $\Delta_m = \det(M_0)\, D_m$.

Proof of Theorem 2.9. Write $\det(M) = \det(M_0)\det(I + A)$, where $A = M_0^{-1} M_E$. Apply Lemma 4.5 to $\det(I + A)$ and set $\Delta_m = \det(M_0)\, D_m$.

LEMMA 5.2. *Let T be block tridiagonal with zero block diagonal. Then the odd powers of T also have a zero block diagonal.*

Proof. Let L_k be any matrix that has only non-zero elements in the kth subdiagonal, $k \ge 1$, and $L_0 = 0$. Similarly, U_k is any matrix that has only non-zero elements on the kth superdiagonal, $k \ge 1$, and $U_0 = 0$. D denotes any matrix with non-zero elements only on the block diagonal.

With this notation, we have the representations, $T^0 = D$, $T = L_1 + U_1$, $T^2 = L_2 + D + U_2$, $T^3 = L_3 + L_1 + U_1 + U_3$, etc. Induction shows that for even k we have $T^k = D + \sum_{i=0,i \text{ even}}^{k} (L_i + U_i)$, and for odd k we have $T^k = \sum_{i=0,i \text{ odd}}^{k} (L_i + U_i)$. Since for odd k the powers of T^k do not contain the summand D, their block diagonal is zero. □

Proof of Theorem 2.10. Lemma 5.2 implies that $\text{trace}\left((T_D^{-1} T_{\rm off})^m\right) = 0$ for m odd. Hence for the approximations in Theorem 2.8 we get $\Delta_m = \Delta_{m-1}$ for m odd. For m even

$$\Delta_m = \Delta_{m-2} * \exp\left(\frac{(-1)^{m-1}}{m}\ \text{trace}\left((T_D^{-1} T_{\rm off})^m\right)\right) = \Delta_{m-2}/\exp\left(\frac{\text{trace}\left((T_D^{-1} T_{\rm off})^m\right)}{m}\right).$$

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