Homogenization of linear spatially periodic electronic circuits

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Abstract: In this paper we establish a simplified model of general spatially periodic linear electronic analog networks. It has a two-scale structure. At the macro level it is an algebro-differential equation and a circuit equation at the micro level. Its construction is based on the concept of two-scale convergence, introduced by the author in the framework of partial differential equations, adapted to vectors and matrices. Simple illustrative examples are detailled by hand calculation and a numerical simulation is reported.

1 Introduction

It is well known that when the size of an analog electronic network increases too much, the size of the unknown vectors, namely the voltages, the currents and the electric node′s voltage, become very large and the system of equation becomes impossible to solve on existing computers. In this paper, we are concerned by such large systems of electronic equations arising in the case of spatially periodic architectures of analog electronic circuits. Among the applications that we have in mind, some of them are for purely analog electronic systems or for Micro-Electro-Mechanical Systems (MEMS) arrays which have always a periodic structure and include or will include in a near future an electronic network. The MEMS arrays are used for a wide range of applications in various scientific or technological areas as biology, medicine, communications, aeronautics, etc... . Due to the small place available in those architectures, analog circuits are preferred in comparison with digital circuits. Other motivations of using arrays of analog circuits are their good computing power per unit area (when moderated resolutions are required) accompanied with a low energy consumption. Some applications to Smart Structures may also be found in the cases where the actuators and sensors are numerous and distributed in a periodic way in their host structure, see for example [7] and [6].

The method for the simplified model derivation that we present here refers to the general homogenization which has been intensively developed in mechanics for composite materials

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modelling. Various approaches have been investigated under various denominations. We will not make a comparison of them, we only mention that the more general and rigorous one was based on an asymptotic expansion with respect to the vanishing cells size (or equivalently to the number of cells that is assumed to tend to infinity). It was introduced by E. Sanchez-Palencia and then widely developed in the reference book [11]. This theory has been rigorously justified in [2] and later its domain of applications has been expanded and the proofs significantly simplified by the introduction of the two-scale convergence in [1] and later by the introduction of the two-scale transform and a new two-scale convergence in [8]. This last improvement has allowed the treatment of network equations which was not encompassed by the other approaches. Furthermore, it has led to a so simple and natural technique that later it has been rediscovered independently by two other groups [3] and [5] in the context of partial differential equations.

In our first works on the electronic networks homogenization [8], [9] and [10], we have formulated the electric network equations under the form of partial differential equations under variational form. Its well posedness has been proved by a combination of functional analysis arguments commonly used in the field of partial differential equations and some graph theory properties. Then, the two-scale limit of the transposed incidence matrix, which was expressed as a spatial derivative along the network, has been carefully formulated. This was the corner-stone of the two-scale models construction from which the homogenized models have been built. This program has been achieved for general network topologies but limited to static problems and to some particular linear devices, passive devices in [8], passive devices plus linear VCVS in [9] and passive devices plus linear VCCS in [10].

Let us turn to this paper contributions. First, the two-scale transform and convergence which was formulated in the context of functions and partial differential equations are now rewritten for vectors and matrices which is the usual framework in electronics. It is the first time that the fundamental properties of the two-scale transform of vector and of matrices are stated and proved. Second, the asymptotics of the Kirchhoff voltage law is carefully analyzed. This is the more difficult and technical part. The technic of this proof is new. It is more general and adaptable that the former thus it may be easier to extend to complex systems including electronics as well as thermal or mechanical effects for instance. Third, this paper covers general linear multi-port devices under the condition that all their ports belong to a same cell. We say that they are local. Fourth, the condition under which the model is justified and its solution exists are made in details. Fifth, three illustrative examples are presented. They have been chosen very simple so that to allow hand calculations with the hope that they are sufficiently illustratives. The solution of the third example was numerically simulated so we report a comparison between the complete solution and the solution of our simplified model. Through this example we also underline the interest of the simplified model in term of computing time.

The paper is organized as follows. In the second section, the problem is introduced and the three examples are stated. The third section if devoted to the statement of the assumptions and of the simplified model itself. Then the definition and the properties of the two-scale transform for vectors and for matrices are stated and partially proved in the fourth section. Some technical points are postponed in annex. The derivation of the model is detailed in the fifth section. Finally, in the sixth section, the simplified model is applied to the three examples and the numerical results are reported.

2 Presentation of the problem

In this section we start by introducing standard circuit equations in § 2.1, then we describe what is called a periodic circuit § 2.2 and we end by three examples of such circuits in § 2.3-2.5.

2.1 Circuit equations

A graph associated with an electrical circuits is denoted by $\mathcal{G} = (\mathcal{E}, \mathcal{N})$ where $\mathcal{E}$ is the branch set and $\mathcal{N}$ the node set. We denote by $\varphi \in \mathbb{R}^{|\mathcal{N}|}$, $v \in \mathbb{R}^{|\mathcal{E}|}$ and $i \in \mathbb{R}^{|\mathcal{E}|}$ the nodal voltages (or electric potential), the branch voltages and the currents where $|Z|$ represents the number of elements belonging to a set Z. The circuit equations used in this paper are:

- the Kirchhoff voltage law:

$$v = \mathcal{A}^T \varphi, \tag{1}$$

- the Tellegen theorem:

$$i^T \mathcal{A}^T \psi = 0 \text{ for all } \psi \in \Psi, \tag{2}$$

- the branch equations characterizing the circuit devices:

$$\mathcal{M}v + \mathcal{R}i = u_s, \tag{3}$$

- and the ground node equations:

$$\varphi_i = 0 \text{ for all node } n_i \in \mathcal{N}_0 \tag{4}$$

where $\mathcal{A} \in \mathbb{R}^{|\mathcal{N}|} \times \mathbb{R}^{|\mathcal{E}|}$ is the incidence matrix, $\mathcal{N}_0 \subset \mathcal{N}$ is the subset of ground nodes,

$$\Psi = \{\psi \in \mathbb{R}^{|\mathcal{N}|} \text{ such that } \psi_i = 0 \text{ for all } n_i \in \mathcal{N}_0\}$$

is the set of admissible potentials, $\mathcal{M}$ and $\mathcal{R}$ are two square matrices with $|\mathcal{E}|$ rows and columns and $u_s \in \mathbb{R}^{|\mathcal{E}|}$ represents voltages and currents sources regrouped in a single vector.

In the following, we reformulate this set of equations in a condensed form:

$$\begin{gathered} \varphi \in \Psi, \, v = \mathcal{A}^T \varphi, \\ \mathcal{M}\mathcal{A}^T \varphi + \mathcal{R}i = u_s, \\ \text{and } i^T \mathcal{A}^T \psi = 0 \text{ for all } \psi \in \Psi. \end{gathered} \tag{5}$$

These equations may take into account general multi-port linear devices in statics. Linear circuit equations of evolution may also be written on this form when applying the Laplace transform.

2.2 Periodic circuit

Now let us consider the class of circuits that are distributed in $d \geq 1$ space directions so that their graph is periodic in all these directions. Electrical devices are assumed to be periodically distributed excepted on the boundary where specific devices may be installed so that to realize specific boundary conditions. Each branch is assumed to belong entirely to one and only one cell. If it is not the case, the circuit must be rearranged in a convenient manner.

We assume that the circuit is confined in a bounded set $\Omega \subset \mathbb{R}^d$ and that the number of its periods is large in all the d directions. For simplicity, it is assumed that Ω is an unit square $\Omega = (0,1)^d$ and that in all directions the period lengths are equal to an identical small parameter ε.

A unit graph is built by picking one cell of the complete graph, expanding it by a factor $1/\varepsilon$ and shifting it so that to occupy the unit cell $Y = (-\frac{1}{2}, \frac{1}{2})^d$. The unit graph is denoted by $G = (E, N)$. From the above assumption, it turns out that E is a set of entire branches. Because $\mathcal{N}$ is $\varepsilon Y-$periodic, each node $n \in N$ located on the boundary of Y has its counterpart n' on the opposite side. We assume that n and n' are linked by at least one path (a sequence of connected branches) that does not include any ground node. Such a path is called a crossing path. Let us introduce the subset E_C of ″crossing branches″.

Criterion 1 *The subset $E_C \subset E$ is constituted of all branches of some of the crossing paths. For each n and n' defined as above the branches of at least one crossing path linking n to n' among many must belong to E_C.*

It may be noticed that the criterion 1 do not determine totally E_C. A complementary criterion is given in the remark 6. The complementary set $E - E_C$ is denoted by E_{NC}. The subset E_C is partitioned in its n_c connected components $E_C = \cup_{k=1}^{n_c} E_{Ck}$.

The subsets N_C and N_{NC} of N are defined as the set of nodes involved in at least one of the branch of E_C and E_{NC} respectively. It is worth pointing out that these two subsets are not a partition of N because in general $N_C \cap N_{NC} \neq \varnothing$ as soon as E_C have E_{NC} common nodes.

The set $\mathcal{N}_0$ of ground nodes is shared in two parts, the first $\mathcal{N}_{0\Gamma}$ referring to ground nodes located on the boundary Γ of the whole domain Ω and the other being distributed periodically in the graph. The corresponding set of this later in N is denoted by N_0. The ground nodes in $\mathcal{N}_{0\Gamma}$ correspond to some nodes in N located on the cell boundary. Therefore they belong to N_C which have been separated in many connected components N_{Ck} which in turn define a partition of $\mathcal{N}_{0\Gamma} = \cup_{k=1}^{n_c} \mathcal{N}_{0\Gamma k}$. We denote by Γ_{0k} the part of Γ where the nodes $\mathcal{N}_{0\Gamma k}$ are distributed.

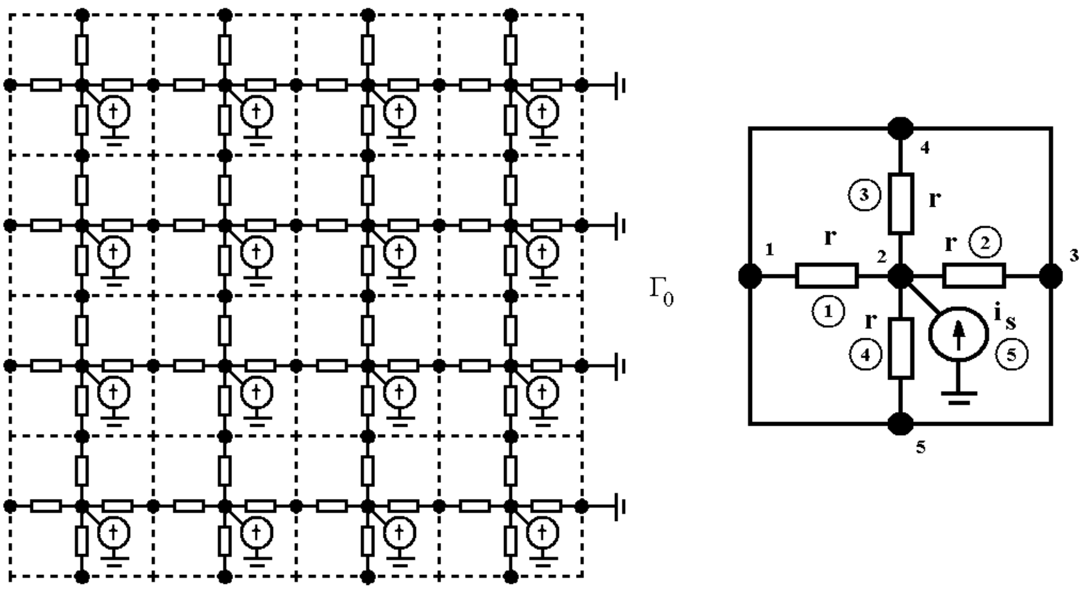
The solution of the simplified model introduced in this paper realizes an approximation of the solution of 5 for small values of ε ($\varepsilon << 1$). It is derived as a limit of the latter when the cells length ε diminishes towards zero.

2.3 Example 1: A regular grid of resistors

The first example of periodic circuit has been extensively studied in the literature. It is a two-dimensional regular mesh of resistors. The elementary cell is made of four resistor (with the same resistance in all cells) that realize two crossing paths in the two directions and one source of current that may vary from one cell to the other. Thus E_C is made of the resistors and E_{NC} of the current source. The nodes located on the part Γ_0 of the boundary are connected to the earth. The complementary part of the boundary is denoted by Γ_1. Making an adapted choice of resistance $r = \varepsilon r_0$ and of current sources $i_s = \varepsilon i_s^0$, this circuit realizes the discretization by the finite differences method of the Laplace equation with mixed (Dirichlet and Neumann) boundary conditions:

$$-\Delta\varphi^0 = f \text{ in } \Omega$$
$$\varphi^0 = 0 \text{ on } \Gamma_0 \text{ and } \nabla\varphi^0.n = 0 \text{ on } \Gamma_1$$

where $f = 2r_0 i_s^0$. It turns out that the components of the nodal voltage φ at the center of the cells are a approximations of φ^0 at those points. The model presented in this paper recover this result and in addition provides the expressions of the currents and voltages in all branches of the circuit. Evidently, our model is very general so it encompasses much more general situations.



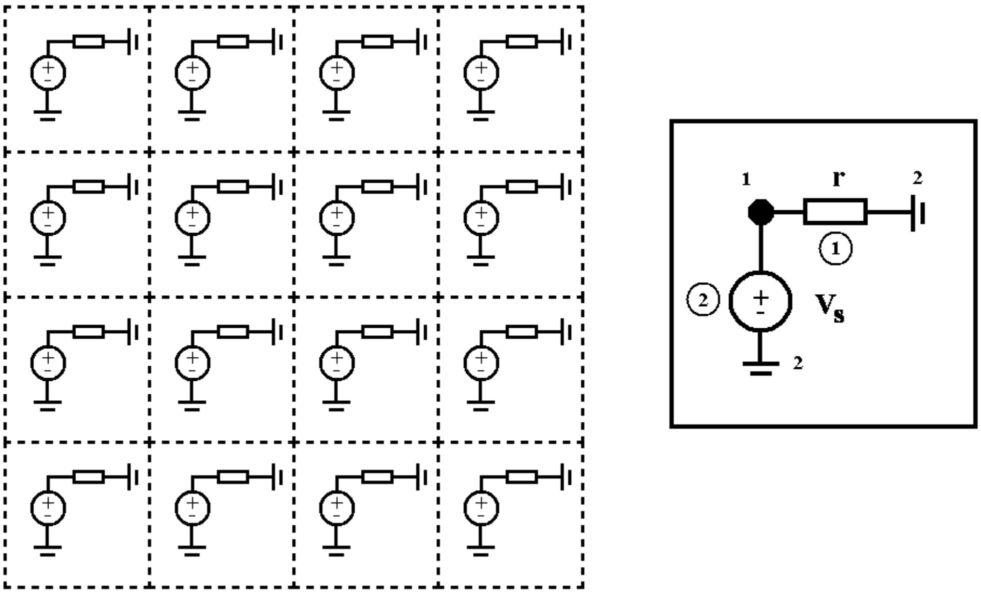
Example 1

2.4 Example 2: Disconnected circuits

In that example the sub-circuits of a cell is disconnected from the sub-circuit of the other so $E_C = \varnothing$. The reference cell is made of a voltage source and of a resistor. The voltage source

v_s may take different values in different cells but not the resistor. The circuit equations can be solved independently in each cell, it comes that

$$\varphi = -v_s \text{ and } i = -\frac{v_s}{r}.$$

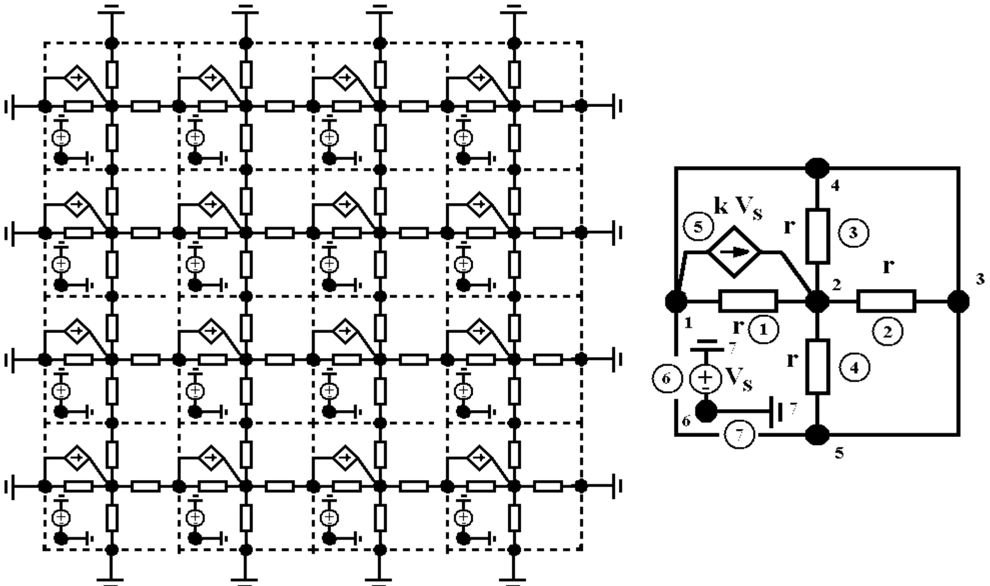


Example 2

If the vector v_s is an approximation of a continuous field v_s^0 for $\varepsilon << 1$ then the vectors φ and i are some approximations of the continuous fields $-v_s^0$ and $-\frac{v_s^0}{R}$. This trivial result is encompassed by our model that can represent general periodic disconnected circuits.

2.5 Example 3: Active and passive devices

The first two examples are elementary illustrations of crossing and non crossing circuits with passive devices. It could be possible to choose more complex circuits to illustrate the interest of the simplified model presented hereafter. But in this paper we prefer to stay simple as much as we can. From that simple examples, the interested reader will be able to foresee more complex applications. So the third example is also elementary and is made of passive and active devices, of crossing sub-circuits and non crossing sub-circuits, see the figure. Here E_C is made of resistors and of a actively controlled current source when E_{NC} is constituted of a passive voltage source and of the ampli&er′s input.



Example 3

3 Statement of the simplified model

Before to state the model in § 3.3 we introduce in § 3.1 the concept of two-scale transform and in § 3.2 the assumption on which the model is justified.

3.1 Two-scale transform

The multi-integer $\mu = (\mu_1, .., \mu_d)$ enumerates all the cells Y^ε_μ in Ω and takes its values in $\{1, .., m\}^d$. The center of a cell Y^ε_μ is denoted by x^ε_μ. We define the concept of two-scale transform relatively to a set $\mathcal{Z}$ of objects being distributed $\varepsilon Y-$periodically in Ω. It must be understood that $\mathcal{Z}$ may represent either $\mathcal{N}$ or $\mathcal{E}$. Similarly, Z represents either N or E. The objects of $\mathcal{Z}$ are indexed by $\mathcal{I} \in \{1, .., |\mathcal{Z}|\}$ and those of Z by $j \in \{1, ..|Z|\}$. Each object is referenced by an unique index $\mathcal{I}$, but it can also be referred by a multi-integer μ referring to the cell which it belongs and by an index j in Z. This correspondence is denoted by $\mathcal{I} \sim (\mu, j)$ and is not one to one in general. Using this correspondence, for each vector $u \in \mathbb{R}^{|\mathcal{Z}|}$ one may define a unique tensor $U_{\mu j}$ with $(\mu, j) \in \{1, .., m\}^d \times \{1, .., |Z|\}$ by $U_{\mu j} = u_{\mathcal{I}}$ for $\mathcal{I} \sim (\mu, j)$.

By another way, we introduce the set $\mathbb{P}^0(\Omega)$ of piecewise constant functions on each cell of Ω: $f(x) = \sum_i \chi_{Y^\varepsilon_\mu}(x) f_i$, each f_i being a scalar coefficient and $\chi_{Y^\varepsilon_\mu}(x)$ being the characteristic function of the set Y^ε_μ equal to 1 when $x \in Y^\varepsilon_\mu$ and 0 otherwise. We denote by $\mathbb{P}^0(\Omega)^{|Z|}$ the set of vectors having $|Z|$ components, each of them being in $\mathbb{P}^0(\Omega)$. It is easy to verify that $\mathbb{P}^0(\Omega)$ is included in $L^2(\Omega) = \{f \text{ such that } \int_\Omega f^2(x)\ dx < +\infty\}$ the set of square integrable functions in Ω.

Definition 2 *The two-scale transform of a vector* $u \in \mathbb{R}^{|\mathcal{Z}|}$ *is the vector of functions* $\widehat{u} \in \mathbb{P}^0(\Omega)^{|Z|}$ *defined by*

$$\widehat{u}_j(x) = \sum_{\mu \in \{1,..,m\}^d} \chi_{Y_\mu^\varepsilon}(x) U_{\mu j} \text{ for all } x \in \Omega \text{ and } j \in \{1,..,|Z|\}$$

where $U_{\mu j} = u_{\mathcal{I}}$ *with* $\mathcal{I} \sim (\mu, j)$. *The linear map* $u \mapsto \widehat{u}$ *from* $\mathbb{R}^{|\mathcal{Z}|}$ *to* $\mathbb{P}(\Omega)^{|Z|} \subset L^2(\Omega)^{|Z|}$ *is denoted by* T_Z.

Let us illustrate this concept on the example 2 where $d = 2$, $|E| = 2$ and $|\mathcal{E}| = 2m^2$. The components of the two-scale transform $\widehat{v}(x) = (\widehat{v}_1(x), \widehat{v}_2(x)) \in \mathbb{P}^0(\Omega)^2$ of branch voltages $v \in \mathbb{R}^{2m^2}$ have the form $\widehat{v}_j(x) = \sum_{\mu \in Y_\mu^\varepsilon} \chi_{Y_\mu^\varepsilon}(x) V_{\mu j}$ for $j = 1, 2$ where $V_{\mu j}$ represent the voltages in the 2 branches of the cell Y_μ^ε.

In the following, we will constantly refer to the concept of local matrices $\mathcal{B} \in \mathbb{R}^{|\mathcal{Z}_1|} \times \mathbb{R}^{|\mathcal{Z}_2|}$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ being two periodic sets, which transform a vector having its non vanishing components in one cell into a vector having also its non vanishing components in the same cell.

Definition 3 *(i)* $\mathcal{B}$ *is said to be local if* $\mathcal{B}_{\mathcal{I}\mathcal{J}} = 0$ *for all* $\mathcal{I} \sim (\mu, j)$ *and* $\mathcal{J} \sim (\lambda, l)$ *when* $\mu \neq \lambda$.

(ii) Let $\mathcal{B} \in \mathbb{R}^{|\mathcal{Z}_1|} \times \mathbb{R}^{|\mathcal{Z}_2|}$ *be a local matrix, if there exist a matrix* $B \in \mathbb{R}^{|Z_1|} \times \mathbb{R}^{|Z_2|}$ *such that* $\mathcal{B}_{\mathcal{I}\mathcal{J}} = \delta_{\mu\lambda} B_{jl}$ *for all* $\mathcal{I} \sim (\mu, j)$ *and* $\mathcal{J} \sim (\lambda, l)$ *then* $\mathcal{B}$ *is said to be local and* $\varepsilon Y-$*periodic. The matrix* B *is called the reduced matrix of* $\mathcal{B}$.

Example 4 *Since we have assumed that each branch belong to only one cell it comes that the incidence matrix* $\mathcal{A}$ *is local and* $\varepsilon Y-$*periodic. Its reduced matrix is the incidence matrix of the graph* $G = (E, N)$ *denoted by* A. *The transpose* $\mathcal{A}^T$ *is also local and* $\varepsilon Y-$*periodic with reduced matrix* A^T.

Here $\delta_{\mu\lambda}$ is the Kronecker symbol equal to 1 when $\mu = \lambda$ and equal to zero otherwise. In other words, a local $\varepsilon Y-$periodic matrix is a bloc diagonal matrix so that all its blocs are identical.

The linear space $L^2(\Omega)^{|Z|}$ admits a scalar product and a norm

$$(u, v) = \int_{\Omega \times Z} u_j(x) v_j(x)\ dx \text{ and } ||u|| = (u, u)^{1/2}$$

where we use the notation

$$\int_{\Omega \times Z} f_j(x)\ dx = \sum_{j / z_j \in Z} \int_\Omega f_j(x)\ dx$$

z_j describing an element of Z. This notation is constantly used in this paper for Z being E, N or one of their parts. The proposition 8 shows that for $\mathcal{Z} = \mathcal{E}$ and $Z = E$ the two-scale transform preserves the norm,

$$\varepsilon^d u^T.u = \varepsilon^d \sum_{\mathcal{I}} |u_{\mathcal{I}}|^2 = ||\widehat{u}||^2 \text{ for all } u \in \mathbb{R}^{|\mathcal{E}|}. \tag{6}$$

The linear space $L^2(\Omega)^{|Z|}$ being normed and ε being a parameter tending to zero, one will say that a sequence $u^\varepsilon \in L^2(\Omega)^{|Z|}$ indexed by ε converges strongly in $L^2(\Omega)^{|Z|}$ towards a limit u^0, which necessarily belongs to $L^2(\Omega)^{|Z|}$, if $||u^\varepsilon - u^0||$ vanishes when ε tends to zero. The sequence is said to be weakly convergent in $L^2(\Omega)^{|Z|}$ towards u^0 if the scalar product $(u^\varepsilon - u^0, v)$ vanishes when ε tends to zero for all $v \in L^2(\Omega)^{|Z|}$, see [12] for more details. The strong convergence implies the weak convergence but the converse is generally false. For example, the sequence $\sin(\frac{x}{\varepsilon}) \in L^2(\Omega)$ is bounded in $L^2(\Omega)$, it is weakly convergent towards 0, but it does not converge strongly towards any limit in $L^2(\Omega)$.

The weak convergence plays an important role in our approach because the model is stated on the weak limits of the voltage′s and current′s two-scale transforms. The existence of such weak limits comes from the following lemma (see [12]).

Lemma 5 *From any bounded sequence in $L^2(\Omega)$ one may extract a subsequence that is weakly convergent in $L^2(\Omega)$.*

3.2 Assumptions

Before to state further assumptions, let us summarize those made in the past sections.

(H0) A branch $e \in \mathcal{E}$ can intersect the boundary of a cell only with its tips.

(H1) Each opposite nodes n and n' are linked by at least one crossing path that do not come across the ground. Furthermore, they do not belong to any corner of the cell.

The next assumptions state that not only the graph is periodic but also the distribution of devices in the circuit as well as their coefficients.

(H2) The matrices $\mathcal{M}$ and $\mathcal{R}$ are local and $\varepsilon Y-$periodic. Their reduced matrices are denoted by M and R.

The next assumption says that the voltages and the currents are respectively of the order of ε and 1 in E_C and of the order of 1 and ε in E_{NC}. We formulate this by using the scaling matrices S_v, S_c and S_s applied to the two-scale transforms

$$\widehat{i^\varepsilon} = S_c\widehat{i},\ \widehat{v^\varepsilon} = S_v\widehat{v},\ \widehat{u_s}^\varepsilon = S_s\widehat{u_s},\ \widehat{\varphi}^\varepsilon = \widehat{\varphi}. \tag{7}$$

(H3) The norms $||\widehat{i^\varepsilon}||$, $||\widehat{v^\varepsilon}||$, $||\widehat{\varphi}^\varepsilon||$, $||\widehat{u_s}^\varepsilon||$ are bounded and the data $\widehat{u_s}^\varepsilon$ converges weakly in $L^2(\Omega)^{|E|}$ towards a limit u_s^0.

The $|E| \times |E|$ scaling matrices are

$$S_v = \varepsilon^{-1} I_{E_C} + I_{E_{NC}}, S_c = I_{E_C} + \varepsilon^{-1} I_{E_{NC}} \text{ and } S_s = \Pi_c S_c + \Pi_v S_v$$

where for any subset E_1 of E the $|E| \times |E|$ matrix I_{E_1} is the projector on E_1:

$$\begin{aligned} (I_{E_1})_{jk} &= \delta_{jk} \text{ if } e_j \in E_1 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Each branch equation in (3) being homogeneous or to a current or to a voltage, from this distinction we deduce a partition of E in two subsets. The $|E| \times |E|$ matrices Π_c and Π_v (for currents and voltages respectively) are defined as the projectors on these two subsets.

The reduced matrices M and R of $\mathcal{M}$ and $\mathcal{R}$ are scaled in a consistent manner

$$M^\varepsilon = S_s M S_v^{-1} \text{ and } R^\varepsilon = S_s R S_c^{-1}. \quad (8)$$

(H4) The scaled reduced matrices M^ε and R^ε converge towards some limit M^0 and R^0.

Remark 6 *As indicated in the criterion 1, E_C is made of all the branches of some crossing paths and for each couple (n, n') at least one crossing path linking n and n' must be part of E_C. In the case where many crossing paths are linking n and n' the designer is free to decide which are included in E_C and which are not, with regard to the assumption (H2).*

Let us introduce the so-called cell problem (or problem micro). For two given vectors $\eta \in \mathbb{R}^{n_c}$, $u_s \in \mathbb{R}^{|E|}$ and a given matrix $\theta \in \mathbb{R}^d \times \mathbb{R}^{n_c}$ the vectors i, $v \in \mathbb{R}^{|E|}$ and $(\varphi_C, \varphi_{NC}) \in \Psi^m(\eta)$ are solutions of the cell problem

$$\begin{gathered} v = I_{E_C} A^T \varphi_C + I_{E_{NC}} A^T \varphi_{NC} \\ R^0 i + M^0 v = u_s - M^0(\tau\theta + I^0\eta) \\ \text{and } i^T w = 0 \end{gathered} \quad (9)$$

for all vector $w = I_{E_C} A^T \psi_C + I_{E_{NC}} A^T \psi_{NC}$ with $(\psi_C, \psi_{NC}) \in \Psi^m$, the admissible nodal voltage set for the cell problem being

$$\Psi^m = \{(\psi_C, \psi_{NC}) \in \mathbb{R}^{|N|}_{per} \times \mathbb{R}^{|N|} \text{ such that } I_{N_C^0 \cup N - N_C} \psi_C = 0,\ I_{N_C \cup N_0} \psi_{NC} = 0\}.$$

The tensor τ is defined by

$$\tau_{lkp} = \sum_{j=1}^{|N|} y_k(n_j) A_{jl} I^0_{jp}, \quad (10)$$

where $y(n) \in \mathbb{R}^d$ is the coordinates vector of a node $n \in N$. For computational purpose, it may be remarked that $\tau_{lkp} = 0$ for $e_l \notin E_{Cp}$ because $A_{jl} = 0$ when $n_j \in N_{Cp}$ and $e_l \notin E_{Cp}$. Then

$$\begin{aligned} \tau_{lkp} &= \sum_{j \text{ s.t. } n_j \in N_{Cp}} y_k(n_j) A_{jl} \text{ for } e_l \in E_{Cp} \\ &= 0 \text{ otherwise.} \end{aligned}$$

Throughout this paper, we will use the tensorial product notation

$$(\tau\theta)_l = \sum_{k=1}^{d} \sum_{p=1}^{n_c} \tau_{lkp} \theta_{kp}. \quad (11)$$

Moreover, I^0 is a matrix in $\mathbb{R}^{|N|} \times \mathbb{R}^{n_c}$ defined by

$$\begin{aligned} I^0_{jp} &= 1 \text{ if } n_j \in N_{Cp} \\ &= 0 \text{ otherwise,} \end{aligned} \quad (12)$$

N_C^0 is the set of n_c nodes constituted of one arbitrary node of each connected component N_{Cp}. Finally

$$\mathbb{R}^{|N|}_{per} = \{\phi \in \mathbb{R}^{|N|} \text{ such that } \phi_j = \phi_{j'} \text{ for all couple } (n_j, n_{j'}) \text{ of opposite nodes}\}. \tag{13}$$

(H5) For each $\eta \in \mathbb{R}^{n_c}$, $\theta \in \mathbb{R}^d \times \mathbb{R}^{n_c}$ and $u_s \in \mathbb{R}^{|E|}$ the cell problem (9) has a unique solution $(\varphi_C, \varphi_{NC}, i, v)$.

From (H5) and the map $(\eta, \theta, u_s) \mapsto (i, v, \varphi_{NC})$ being linear there exists some matrices $\mathcal{L}_x$, $\mathcal{H}_x$ and a third order tensor $\mathcal{P}_x$ such that

$$\mathcal{L}_i\eta + \mathcal{P}_i\theta + \mathcal{H}_iu_s = i,\ \mathcal{L}_\varphi\eta + \mathcal{P}_\varphi\theta + \mathcal{H}_\varphi u_s = \varphi_{NC} \text{ and } \mathcal{L}_v\eta + \mathcal{P}_v\theta + \mathcal{H}_v u_s = v \tag{14}$$

where $\mathcal{P}_x\theta$ is defined according to (11).

3.3 The model

From the assumption (H3) and the lemma 5 there exists at least one extracted subsequence of $(\widehat{i^\varepsilon}, \widehat{v^\varepsilon}, \widehat{\varphi}^\varepsilon)$ that is weakly converging towards some limits (i^0, v^0, φ^0) in L^2. The model satisfied by the latter is stated in this subsection. It constitutes the main result of the paper.

Theorem 7 *(i) If the assumptions (H0-H3) are fulfilled then $I_{E_C}A^T\varphi^0 = 0$ or equivalently there exist $\varphi_C^0 \in L^2(\Omega)^{n_c}$ such that*

$$\varphi^0 = I^0\varphi_C^0 + \varphi_{NC}^0 \tag{15}$$

where $\varphi_{NC}^0 := I_{N-N_C}\varphi^0$. Moreover there exists $\varphi_C^1 \in L^2(\Omega; \mathbb{R}^{|E|}_{per})$ such that

$$v^0 = \partial_\tau\varphi_C^0 + I_{E_C}A^T\varphi_C^1 + I_{E_{NC}}A^T\varphi^0 \text{ and } I_{N-N_C}\varphi_C^1 = 0. \tag{16}$$

This is the asymptotic Kirchhoff Voltage Law.

(ii) Furthermore, if the assumptions (H4) and (H5) are satisfied then $\varphi_C^0 \in \Psi^H$ is solution of the algebro-differential equation, so-called homogenized circuit equations,

$$Q^H\nabla\varphi_C^0 + S^H\varphi_C^0 = F^Hu_s^0 \text{ and } A^H(\mathcal{P}_i\nabla\varphi_C^0 + \mathcal{L}_i\varphi_C^0) = -A^H\mathcal{H}_iu_s^0 \text{ in } \Omega \tag{17}$$

with the boundary conditions

$$\varphi_{Cp}^0 = 0 \text{ on } \Gamma_{0p} \text{ and } (\mathcal{P}_i\nabla\varphi_C^0 + \mathcal{L}_i\varphi_C^0)n_\tau = 0 \text{ on } \Gamma - \Gamma_{0p}.$$

(iii) Finally $(\varphi_C^1, \varphi_{NC}^0, i^0, v = I_{E_C}A^T\varphi_C^1 + I_{E_{NC}}A^T\varphi_{NC}^0)$ is solution of the cell problem (9) with $(\eta, \theta, u_s) = (\varphi_C^0, \nabla\varphi_C^0, u_s^0)$.

The homogenized matrices Q^H, S^H, F^H and operator A^H are defined by

$$\begin{aligned} Q^H &= R^0\mathcal{P}_i + M^0(\tau + \mathcal{L}_v),\ S^H = R^0\mathcal{L}_i + M^0(I_{E_C}A^TI^0 + \mathcal{L}_v), \\ F^H &= I - R^0\mathcal{H}_i - M^0\mathcal{H}_v \text{ and } A^H = -\partial_{\tau^*} + I^{0T}AI_{E_{NC}} \end{aligned}$$

where $\partial_{\tau^*} i = \tau^* \nabla i$ with $\tau^*_{pkl} = \tau_{lkp}$ and the use of notation (11). The derivative $\partial_\tau \varphi^0_C$ and the normal n_τ are defined by

$$\partial_\tau \varphi^0_C = \tau \nabla \varphi^0_C \text{ and } (n_\tau)_{lp} = \sum_{k=1}^{d} \tau_{lkp} n_k$$

∇ being the gradient $(\partial_{x_k})_{k=1..d}$ and $n = (n_k)_{k=1..d}$ being the outward normal vector to the boundary Γ of Ω.

4 Properties of the two-scale transform

We prove the fundamental properties of the two-scale transform which are useful for the model derivation.

4.1 Adjoint of T_E and norm preservation

First the adjoint T^*_E of the two-scale transform T_E is established. Then the relationship between the scalar product $[.,.]$ and the norm $|.|$ in $\mathbb{R}^{|\mathcal{E}|}$, defined by

$$[u, v] = \varepsilon^{-d} u^T.v \text{ and } |v| = [v, v]^{1/2} \text{ for all } u, v \in \mathbb{R}^{|\mathcal{E}|},$$

and the scalar product and the norm in $L^2(\Omega)^{|E|}$ is derived.

Proposition 8 *(i) Under the assumption (H0) the adjoint T^*_E is equal to*

$$(T^*_E u)_{\mathcal{I}} = \varepsilon^{-d} \int_{Y^\varepsilon_\mu} u_j(x)\ dx \text{ for } \mathcal{I} \sim (\mu, j) \tag{18}$$

for all $u \in L^2(\Omega)^{|E|}$.

(ii) Furthermore, the restriction T^*_E to $\mathbb{P}(\Omega)^{|E|}$ is

$$(T^*_E u)_{\mathcal{I}} = U_{\mu j} \text{ for } \mathcal{I} \sim (\mu, j)$$

for all $u \in \mathbb{P}(\Omega)^{|E|}$ so that $u_j(x) = \sum_{\mu \in \{1,..,m\}^d} \chi_{Y^\varepsilon_\mu}(x) U_{\mu j}$.

(iii) $T^*_E T_E = I_{\mathcal{E}}$ on $\mathbb{R}^{|\mathcal{E}|}$.

(iv) $T_E T^*_E = I_E$ on $\mathbb{P}^0(\Omega)^{|E|}$.

(v) T_E is one to one from $\mathbb{R}^{|\mathcal{E}|}$ to $\mathbb{P}^0(\Omega)^{|E|}$ and T^*_E is its inverse.

(vi) The scalar product as well as the norm are conserved through the two-scale transform

$$(T_E u, T_E v) = [u, v] \text{ and } ||T_E u|| = |u| \text{ for all } u, v \in \mathbb{R}^{|\mathcal{E}|}. \tag{19}$$

Proof. (i) For $u \in L^2(\Omega)^{|E|}$ $T^*_E u$ is defined through the equality $[T^*_E u, v] = (T_E v, u)$ for all $v \in \mathbb{R}^{|\mathcal{E}|}$. But $(T_E v, u) = \int_\Omega (T_E v).u(x)\ dx = \varepsilon^d \sum_{\mu \in \{1,..,m\}^d} \sum_{j=1}^{|E|} \varepsilon^{-d} \int_{Y^\varepsilon_\mu} u_j(x)\ dx\ V_{\mu j}$ which leads to the characterization of T^*_E because the correspondence $\sim$ is one to one.

The point (ii) is a straightforward consequence of the point (i).

(iii) Let $u \in \mathbb{R}^{|\mathcal{E}^\varepsilon|}$ and $\mathcal{I} \sim (\mu, j)$ $(T_E^* T_E u)_\mathcal{I} = T_\mathcal{I}^*(\sum_{\lambda\in\{1,..,m\}^d} U_{\lambda j}\chi_{Y_\lambda^\varepsilon}(x)) = \varepsilon^{-d}\int_{Y_\mu^\varepsilon}\sum_{\lambda\in\{1,..,m\}^d}\chi_{Y_\lambda^\varepsilon}(x)$ $dx\ U_{\lambda j} = U_{\mu j} = u_\mathcal{I}$.

(iv) Let $u \in \mathbb{P}^0(\Omega;\mathbb{R}^{|E|})$ so that $u_j(x) = \sum_{\mu\in\{1,..,m\}^d} U_{\mu j}\chi_{Y_\mu^\varepsilon}(x)$ then $(T_E T_E^* u)_j(x) = (T_E(\varepsilon^{-d}\int_{Y_\mu^\varepsilon} u_j(x')dx'))_j(x) = \sum_{\mu\in\{1,..,m\}^d}\varepsilon^{-d}\int_{Y_\mu^\varepsilon} u_j(x')dx'\chi_{Y_\mu^\varepsilon}(x)$. Replacing u by its expression yields

$= \sum_{\mu\in\{1,..,m\}^d}\sum_{\lambda\in\{1,..,m\}^d}\varepsilon^{-d}\int_{Y_\mu^\varepsilon}\chi_\lambda(x')\,dx'\,U_{\lambda j}\chi_{Y_\mu^\varepsilon}(x)$. Finally we use the fact that $\varepsilon^{-d}\int_{Y_\mu^\varepsilon}\chi_\lambda(x')$ $dx' = \delta_{\mu\lambda}$ to conclude that $(T_E T_E^* u)_j(x) = \sum_{\mu\in\{1,..,m\}^d} U_{\mu j}\chi_{Y_\mu^\varepsilon}(x) = u_j(x)$.

(v) is just a consequence of (iii) and of (iv).

The proof of (vi) is straightforward $(T_E u, T_E v) = [T_E^* T_E u, v] = [u, v]$ from which the equality of norms follows by posing $v = u$. ■

4.2 Two-scale transform of matrices

We start this section by providing the definition of the two-scale transform of a matrix operating on $\mathbb{R}^{|\mathcal{E}|}$ providing that the assumption H0 holds. We continue by stating some of its properties in the particular case of $|\mathcal{E}| \times |\mathcal{E}|$ matrices. Since we wish to apply the two-scale transform to the incidence matrix $\mathcal{A}^T$ which operate on $\mathbb{R}^{|\mathcal{N}|}$ we end this section by defining the two-scale transform of general local $\varepsilon Y-$periodic matrices which evidently applies to $\mathcal{A}^T$.

Definition 9 *Assuming that H0 holds, then the two-scale transform of a matrix $\mathcal{B} \in \mathbb{R}^{|\mathcal{Z}|}\times\mathbb{R}^{|\mathcal{E}|}$ is the linear operator defined from $L^2(\Omega)^{|E|}$ to $\mathbb{P}^0(\Omega)^{|Z|} \subset L^2(\Omega)^{|Z|}$ by*

$$\widehat{\mathcal{B}} = T_Z \mathcal{B} T_E^*.$$

Let us focus on matrices $\mathcal{B} \in \mathbb{R}^{|\mathcal{E}|}\times\mathbb{R}^{|\mathcal{E}|}$. Its two-scale transform $\widehat{\mathcal{B}} = T_E\mathcal{B}T_E^*$ is a linear operator from $L^2(\Omega)^{|E|}$ to $\mathbb{P}^0(\Omega)^{|E|} \subset L^2(\Omega)^{|E|}$, however, in the following statement we consider only its restriction defined from $\mathbb{P}^0(\Omega)^{|E|}$ to itself. The norm of such a matrix is $|\mathcal{B}| = \sup_{u\in\mathbb{R}^{|\mathcal{E}|}}\frac{|\mathcal{B}u|}{|u|}$.

Proposition 10 *For $\mathcal{B}$ a $|\mathcal{E}|\times|\mathcal{E}|$ matrix and $u, v \in \mathbb{R}^{|\mathcal{E}|}$ the following properties hold:*

(i) $\widehat{(\mathcal{B}u)} = \widehat{\mathcal{B}}\widehat{u}$;

(ii) $(\widehat{\mathcal{B}}\widehat{u}, \widehat{v}) = [\mathcal{B}u, v]$;

(iii) $\widehat{\mathcal{B}^T} = \widehat{\mathcal{B}}^*$;

(iv) $||\widehat{\mathcal{B}}|| = |\mathcal{B}|$.

Proof. (i) $\widehat{(\mathcal{B}u)} = T_E\mathcal{B}u = T_E\mathcal{B}T_E^*T_E u = \widehat{\mathcal{B}}\widehat{u}$.

(ii) $(\widehat{\mathcal{B}}T_E u, T_E v) = (T_E\mathcal{B}T_E^*T_E u, T_E v) = [\mathcal{B}u, T_E^*T_E v] = [\mathcal{B}u, v]$.

(iii) $\widehat{\mathcal{B}^T} = T_E\mathcal{B}^T T_E^* = (T_E\mathcal{B}T_E^*)^* = \widehat{\mathcal{B}}^*$.

(iv) $||\widehat{\mathcal{B}}|| = \sup_{u\in\mathbb{P}^0(\Omega;\mathbb{R}^{|E|})}\dfrac{||\widehat{\mathcal{B}}u||}{||u||_E}$, $||\widehat{\mathcal{B}}u|| = ||T_E\mathcal{B}T_E^*u|| = |\mathcal{B}T_E^*u|$ and $||u||_E = |T_E^*u|$ yield $||\widehat{\mathcal{B}}|| = \sup_{u\in\mathbb{P}^0(\Omega;\mathbb{R}^{|E|})}\dfrac{|\mathcal{B}T_E^*u|}{|T_E^*u|} = \sup_{v\in\mathbb{R}^{|\mathcal{E}|}}\dfrac{|\mathcal{B}v|}{|v|} = |\mathcal{B}|$ because T_E^* is one to one from $\mathbb{P}^0(\Omega;\mathbb{R}^{|E|})$ to $\mathbb{R}^{|\mathcal{E}|}$. ■

We cannot define the two-scale transform of a general matrix $\mathcal{B} \in \mathbb{R}^{|\mathcal{Z}_1|}\times\mathbb{R}^{|\mathcal{Z}_2|}$ but we can do it for local $\varepsilon Y-$periodic matrices providing that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are two periodic sets.

Definition 11 *If* $\mathcal{B} \in \mathbb{R}^{|\mathcal{Z}_1|} \times \mathbb{R}^{|\mathcal{Z}_2|}$ *is a local* $\varepsilon Y-$*periodic matrix and* B *is its reduced matrix then the two-scale transform* $\widehat{\mathcal{B}}$ *of* $\mathcal{B}$ *is defined by*

$$(\widehat{\mathcal{B}}\phi)_j(x) = \sum_{k=1}^{|Z_2|} B_{jk}\phi_k(x) \text{ for all } \phi \in \mathbb{P}^0(\Omega)^{|Z_2|}.$$

Based on this definition the following property holds.

Lemma 12 *If* $\mathcal{B} \in \mathbb{R}^{|\mathcal{Z}_1|} \times \mathbb{R}^{|\mathcal{Z}_2|}$ *is a local* $\varepsilon Y-$*periodic matrix then*

$$\widehat{\mathcal{B}\phi} = \widehat{\mathcal{B}}\widehat{\phi}.$$

4.3 Two-scale convergence of Kirchhoff Voltage Law

This section is devoted to the derivation of the point (i) of the main theorem 7. The proof is a little technical, so it has been decomposed. One part requiring detailed explanation regarding the two-scale transform for nodes as well as tricky operations is postponed in annex.

Let us recall that the sets $\mathcal{N}$ and $\mathcal{E}$ of nodes and of branches depend on the number of cells in the circuit or equivalently depend on the parameter ε. For a given ε we consider a vector of nodal voltages $\varphi \in \mathbb{R}^{|\mathcal{N}|}$ and $v = \mathcal{A}^T\varphi \in \mathbb{R}^{|\mathcal{E}|}$ the branch voltages. By doing so, φ and v are also depending on ε and all together constitute a sequence indexed by ε. The same thing can be said about their scaled two-scale transforms $\widehat{\varphi}^\varepsilon = \widehat{\varphi}$ and $\widehat{v}^\varepsilon = S_v\widehat{v}$ where the dependence on ε is made more visible. When their norms $||\widehat{\varphi}^\varepsilon||$ and $||\widehat{v}^\varepsilon||$ are bounded, thanks to the lemma 5, one may extract a subsequence of the couple still denoted by $(\widehat{\varphi}^\varepsilon, \widehat{v}^\varepsilon)$ which converges weakly in $L^2(\Omega)^{|N|} \times L^2(\Omega)^{|E|}$ towards a limit (φ^0, v^0).

Lemma 13 *The weak limits* (φ^0, v^0) *satisfy* $I_{E_C}A^T\varphi^0 = 0$ *or equivalently there exists* $\varphi^0_C \in \mathbb{R}^{n_c}$ *such that*

$$I_{N_C}\varphi^0 = I^0\varphi^0_C$$

then

$$\varphi^0 = I^0\varphi^0_C + \varphi^0_{NC} \text{ where } \varphi^0_{NC} = I_{N-N_C}\varphi^0$$

and there exists $\varphi^1_C \in L^2(\Omega; \mathbb{R}^{|E|}_{per})$ *such that*

$$v^0 = \partial_\tau\varphi^0_C + I_{E_C}A^T\varphi^1_C + I_{E_{NC}}A^T(\varphi^0_{NC} + I^0\varphi^0_C).$$

Proof. (i) We start by proving that $I_{N_C}\varphi^0 = I^0\varphi^0_C$. The fact that $||\widehat{v}^\varepsilon||$ is bounded and the lemma 12 imply together that $\varepsilon^{-1}||I_{E_C}A^T\widehat{\varphi}^\varepsilon||$ is bounded and by passing to the limit in $(I_{E_C}A^T\widehat{\varphi}^\varepsilon, w) \leq C\varepsilon||w||$ for all $w \in L^2(\Omega)^{|E|}$ that $I_{E_C}A^T\varphi^0 = 0$. This is equivalent to say that φ^0 is constant on each connected component E_{Ck} of E_C or equivalently that there exists a vector $\varphi^0_C \in L^2(\Omega)^{|n_c|}$ such that $I_{N_C}\varphi^0 = I^0\varphi^0_C$.

(ii) Let us establish that $I_{E_{NC}}v^0 = I_{E_{NC}}A^T(\varphi^0_{NC} + I^0\varphi^0_C)$ where $\varphi^0_{NC} = I_{N-N_C}\varphi^0$. The fact that $\widehat{\varphi}^\varepsilon$ and $\widehat{v}^\varepsilon$ converge weakly towards φ^0 and v^0 implies that the equality $I_{E_{NC}}\widehat{v}^\varepsilon = I_{E_{NC}}A^T\widehat{\varphi}^\varepsilon$ converges weakly towards $I_{E_{NC}}A^Tv^0 = I_{E_{NC}}A^T\varphi^0 = I_{E_{NC}}A^T(\varphi^0_{NC} + I^0\varphi^0_C)$.

The end of the proof is devoted to the derivation of the expression of v^0 in E_C :

$$I_{E_C}v^0 = \partial_\tau\varphi^0 + I_{E_C}A^T\varphi^1_C.$$

(iii) We prove that the two following statements are equivalent:

(A) $v = A^T\phi$ with $\phi \in \mathbb{R}^{|N|}_{per}$;

(B) $v \in \mathbb{R}^{|E|}$, $(v, \mu) = 0$ for all $\mu \in \mathbb{R}^{|E|}$ such that $I_{N-\partial N}A\mu = 0$ and $A\mu \in \mathbb{R}^{|N|}_{per}$.

Let us introduce the matrix $B \in \mathbb{R}^{|N|} \times \mathbb{R}^{|E|}$ defined by

$$\begin{aligned}(B\mu)_j &= (A\mu)_j - (A\mu)_{j'} \text{ for } n_j \in \partial N \\ &= (A\mu)_j \text{ for for } n_j \in N - \partial N.\end{aligned}$$

The statement (B) is equivalent to $(v, \mu) = 0$ for all $\mu \in KerB$ which means that v is orthogonal to $KerB = \operatorname{Im} B^T$. But

$$(B^T\phi)_j = \sum_{k=1}^{|N|} A_{kj}((I_{N-\partial N}\phi)_k + (I_{\partial N}\phi)_k + (I_{\partial N}\phi)_{k'})$$

which is equivalent to (A).

(iv) We prove that $I_{E_C}v = \partial_\tau\phi_C + I_{E_C}A^T\varphi^1$ with $\varphi^1 \in \mathbb{R}^{|N|}_{per}$ and $I_{N_{NC}}\varphi^1 = 0$. From the lemma 14 we know that $v = I_{E_C}v^0 - \partial_\tau\varphi^0_C$ satisfies the statement (B) or equivalently the statement (A) and we pose $\phi = \varphi^1 \in \mathbb{R}^{|N|}_{per}$. This ends the proof. ■

The subset of N of nodes belonging to the boundary of the cell Y is denoted by ∂N. Consider $\mu \in \mathcal{C}^1(\Omega)^{|E|}$ such that $A\mu(x) \in \mathbb{R}^{|N|}_{per}$, $\mu(x)$ vanishes in E_{NC} ($I_{E_{NC}}\mu = 0$) for all $x \in \Omega$ and $I_{N-\partial N}A\mu = 0$ where ∂N is the subset of nodes belonging to the boundary of the cell (remark that $\partial N \subset N_C$).

Lemma 14 *If $\mu \in \mathcal{C}^1(\Omega)^{|E|}$ satisfies*

$$A\mu(x) \in \mathbb{R}^{|N|}_{per},\ I_{E_{NC}}\mu(x) = 0 \text{ and } I_{N-\partial N}A\mu = 0 \text{ for all } x \in \Omega$$

then

$$(I_{E_C}v^0, \mu) = (\partial_\tau\varphi^0_C, \mu).$$

Proof. For $\psi = A\mu$,

$$(I_{E_C}v^0, \mu) = \lim_{\varepsilon\to 0}(I_{E_C}\widehat{v}^\varepsilon, \mu).$$

Noticing that $(I_{E_C}\widehat{v}^\varepsilon, \mu) = (\widehat{v}^\varepsilon, \mu) = \frac{1}{\varepsilon}(\widehat{\varphi}^\varepsilon, A\mu) = \frac{1}{\varepsilon}(\widehat{\varphi}^\varepsilon, \psi)_{\partial N}$ and using the lemma 20,

$$\begin{aligned}(I_{E_C}v^0, \mu) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega\times\partial N}\widehat{\varphi}^\varepsilon_j(x)\psi_j(x)\ dx = \lim_{\varepsilon\to 0} -\int_{\Omega\times\partial N}\widehat{\varphi}^\varepsilon_j(x)(y.\nabla_x\psi)_j(x)\ dx + b(\widehat{\varphi}^\varepsilon, \psi) \\ &= -\int_{\Omega\times\partial N}\varphi^0_j(x)(y.\nabla_x\psi)_j(x)\ dx + \int_{\partial(\Omega,N)}\varphi^0_j(x)\psi_j(x)\ ds(x)\end{aligned}$$

where

$$\partial(\Omega, N) = \{(x, n) \in \Gamma \times \partial N \text{ such that } n_Y(n) = n_\Omega(x)\}.$$

(i) Let us prove that

$$\int_{\Omega \times \partial N} \varphi_j^0(x)\ (y.\nabla_x \psi)_j(x)\ dx = (\varphi_C^0, \partial_\tau^* \mu)$$

with

$$(\partial_\tau^* \mu)_p(x) = \sum_{k=1}^{d} \sum_{l=1}^{|E|} \tau_{lkp} \partial_{x_k} \mu_l(x) \text{ for } p \in \{1, .., n_c\}.$$

From $\partial N \subset N_C$ and $I_{N_C} \varphi^0 = I^0 \varphi_C^0$ comes

$$\int_{\Omega \times \partial N} \varphi_j^0(x)\ (y.\nabla_x \psi)_j(x)\ dx = \int_{\Omega \times \partial N} (I^0 \varphi_C^0)_j(x) (y.\nabla_x \psi)_j(x)\ dx.$$

Combined with the facts that $I_{N_C - \partial N}(\tau.\nabla_x \psi) = 0$ and $I_{N - N_C} I^0 \varphi_C^0 = 0$ yields

$$= (I^0 \varphi_C^0, y.\nabla_x \psi) = (\varphi_C^0, I^{0T}(y.\nabla_x A\mu)) = (\varphi_C^0, \partial_\tau^* \mu).$$

(ii) For all $\nu \in \mathbb{R}^{|N|}_{antiper}$ and $x \in \Gamma$ let us prove that:

$$\sum_{j/(x,n_j) \in \partial(\Omega,N)} \nu_j = \sum_{j/n_j \in \partial N} \nu_j y(n_j) n_\Omega(x)$$

where

$$\mathbb{R}^{|N|}_{antiper} = \{\phi \in L^2(\Omega; \mathbb{R}^{|N|}) \text{ such that } \phi_j(x) = -\phi_{j'}(x) \text{ a.e. } x \in \Omega \text{ for all } n_j \in \partial Y\}.$$

We remark that if $(x, n_j) \in \partial(\Omega, N)$ then $n_Y(n_j).n_\Omega(x) = 1$ and $\nu_j = \nu_j n_Y(n_j).n_\Omega(x)$. Moreover $n_Y(n_{j'}) = -n_Y(n_j)$ and $\nu_{j'} = -\nu_j$ imply that $\nu_j n_Y(n_j) = \nu_{j'} n_Y(n_{j'})$. Thus $\nu_j = \frac{1}{2}(\nu_j n_Y(n_j) + \nu_{j'} n_Y(n_{j'})) n_\Omega(x) = (\nu_j y(n_j) + \nu_{j'} y(n_{j'})) n_\Omega(x)$ and

$$\sum_{j/(x,n_j) \in \partial(\Omega,N)} \nu_j = \sum_{j/(x,n_j) \in \partial(\Omega,N)} (\nu_j y(n_j) + \nu_{j'} y(n_{j'})) n_\Omega(x) = \sum_{j/n_j \in \partial N} \nu_j y(n_j) n_\Omega(x).$$

(iii) Let us derive the formula:

$$\int_{\partial(\Omega,N)} \varphi_j^0(x) \psi_j(x)\ ds(x) = \int_{\Gamma \times N} (I^0 \varphi_C^0)_j (y.n_\Omega A\mu)_j(x)\ ds(x).$$

with $(y.n_\Omega A\mu)_j(x) = \sum_{k=1}^{d} \sum_{l=1}^{|E|} y_k(n_j) n_{\Omega k}(x) A_{jl} \mu_l(x)$. Since $I_{E_C} \varphi^0 = I^0 \varphi_C^0$

$$\int_{\partial(\Omega,N)} \varphi_j^0(x) \psi_j(x)\ ds(x) = \sum_{k=1}^{n_c} \int_\Gamma \varphi_{Ck}^0(x) \sum_{j\ /\ (x,n_j) \in \partial(\Omega,N)} I_{jk}^0 \psi_j(x)\ ds(x).$$

But (ii) with $\nu_j = I^0_{jk}\psi_j(x)$ (k being frozen), providing that $\psi = A\mu$, says that

$$\sum_{j\ /\ (x,n_j)\in\partial(\Omega,N)} I^0_{jk}\psi_j(x) = \sum_{j/n_j\in\partial N} I^0_{jk}(A\mu)_j(x)y(n_j).n_\Omega(x).$$

Thus

$$\int_{\partial(\Omega,N)} \varphi^0_j(x)\psi_j(x)\ ds(x) = \int_{\Gamma\times\partial N}(I^0\varphi^0_C)_j(x)(y.n_\Omega A\mu)_j(x)\ ds(x).$$

A reasoning similar to this made in (i) yields

$$= \int_{\Gamma\times N}(I^0\varphi^0_C)_j(y.n_\Omega A\mu)_j(x)\ ds(x).$$

(iv) The end of the proof is done using (i), (iii) and the Green like formula:

$$-\sum_{p=1}^{n_c}\int_\Omega \varphi^0_{Cp}(x)(\partial^*_\tau\mu)_p(x)\ dx + \sum_{j=1}^{|N|}\int_\Gamma (I^0\varphi^0_C)_j(x)(y.n_\Omega A\mu)_j(x)\ ds(x)$$
$$= \sum_{l=1}^{|E|}\int_\Omega (\partial_\tau\varphi^0_C)_l(x)\mu_l(x)\ dx.$$

■

4.4 Convergence of test functions

Let us introduce the set of admissible two-scale potentials

$$\Psi = \{(\psi^0_C,\psi^1_C,\psi^0_{NC}) \in L^2(\Omega)^{n_c}\times L^2(\Omega;\mathbb{R}^{|N|}_{per})\times L^2(\Omega)^{|N|} \text{ s.t. } \partial_\tau\psi^0_C \in L^2(\Omega)^{|E|}, \quad (20)$$
$$I_{N-N_C}\psi^1_C = 0,\ I_{N_C\cup N_0}\psi^0_{NC} = 0,\ \psi^0_{Cp}(x) = 0\ \forall x\in\Gamma_{0p} \text{ for all } p = 1..n_c\}.$$

For $(\psi^0_C,\psi^1_C,\psi^0_{NC}) \in \Psi\cap\mathcal{C}^1(\Omega)^{n_c}\times\mathcal{C}^1(\Omega;\mathbb{R}^{|N|}_{per})\times\mathcal{C}^1(\Omega)^{|N|}$ let us define ψ^0 and ψ^1 in $\mathbb{R}^{|\mathcal{N}|}$ by

$$\begin{aligned}
\psi^0_\mathcal{I} &= \psi^0_{Cp}(x^\varepsilon_\mu + \varepsilon y(n_j)) \text{ for } n_j\in N_{Cp} \text{ for } p = 1,..,n_c\\
&= \psi^0_{NCj}(x^\varepsilon_\mu) \text{ for } n_j \in N - N_C,\\
\psi^1_\mathcal{I} &= \psi^1_{Cj}(x^\varepsilon_\mu) \text{ for } n_j\in N-\partial N\\
&= \frac{1}{2}(\psi_{Cj}(x^\varepsilon_\mu) + \psi_{Cj'}(x^\varepsilon_{\mu'})) \text{ for } n_j\in\partial N
\end{aligned}$$

where $\mathcal{I}\sim(\mu,j)$ and (μ',j'), (see in annex for details regarding (μ',j')).

Lemma 15 *(i)* $\widehat{\psi}^0 = I^0\psi^0_C + \psi^0_{NC} + O(\varepsilon)$.
(ii) $S_v A^T\widehat{\psi}^0 = \partial_\tau\psi^0_C + I_{E_{NC}}A^T(\psi^0_{NC} + I^0\psi^0_C) + O(\varepsilon)$.
(iii) $\widehat{\psi}^1 = \psi^1_C + O(\varepsilon)$.
(iv) $I_{E_C}A^T\widehat{\psi}^1 = I_{E_C}A^T\psi^1_C + O(\varepsilon)$.

Proof. (i) Let us prove successively that $I_{N_C}\widehat{\psi}^0(x) = I^0\psi^0_C(x) + O(\varepsilon)$ and $I_{N-N_C}\widehat{\psi}^0 = \psi^0_{NC} + O(\varepsilon)$. Let us start with $I_{N_C}\widehat{\psi}^0$. For $n_j \in N_{Cp}$,

$$\begin{aligned}\widehat{\psi}^0_j(x) &= \sum_\mu \psi^0_{Cp}(x^\varepsilon_\mu + \varepsilon y(n_j))\chi_{Y^\varepsilon_\mu}(x) = \sum_\mu \psi^0_{Cp}(x^\varepsilon_\mu)\chi_{Y^\varepsilon_\mu}(x) + O(\varepsilon)\\ &= \sum_\mu \psi^0_{Cp}(x)\chi_{Y^\varepsilon_\mu}(x) + O(\varepsilon) = \psi^0_{Cp}(x) + O(\varepsilon).\end{aligned}$$

Now we continue with $I_{N-N_C}\widehat{\psi}^0$. For $n_j \in N - N_C$:

$$(I_{N-N_C}\widehat{\psi}^0)_j(x) = \sum_\mu \psi^0_{NCj}(x^\varepsilon_\mu)\chi_{Y^\varepsilon_\mu}(x) = \sum_\mu \psi^0_{NCj}(x)\chi_{Y^\varepsilon_\mu}(x) + O(\varepsilon) = \psi^0_{NCj}(x) + O(\varepsilon).$$

(ii) Let us establish successively that $I_{E_C}S_vA^T\widehat{\psi}^0 = \partial_\tau\psi^0_C + O(\varepsilon)$ and $I_{E_{NC}}S_vA^T\widehat{\psi}^0 = I_{E_{NC}}A^T(\psi^0_{NC} + I^0\psi^0_C) + O(\varepsilon)$. Using $I_{E_C}S_v = \frac{1}{\varepsilon}I_{E_C}$, for $e_l \in E_C$:

$$\begin{aligned}(I_{E_C}S_vA^T\widehat{\psi}^0)_l(x) &= \frac{1}{\varepsilon}\sum_{j=1}^{|N|} A_{jl}\widehat{\psi}^0_j(x)\\ &= \frac{1}{\varepsilon}\sum_\mu\sum_{p=1}^{n_c}\sum_{j=1 \text{ s.t. } n_j\in N_{Cp}}^{|N|} A_{jl}\psi^0_j(x^\varepsilon_\mu + \varepsilon y(n_j))\chi_{Y^\varepsilon_\mu}(x).\end{aligned}$$

But $\psi^0_j(x^\varepsilon_\mu + \varepsilon y(n_j)) = \psi^0_j(x^\varepsilon_\mu) + \sum_{k=1}^d \partial_{x_k}\psi^0_j(x^\varepsilon_\mu)\varepsilon y_k(n_j) + \varepsilon O(\varepsilon)$ then

$$= \frac{1}{\varepsilon}\sum_\mu\sum_{p=1}^{n_c}\sum_{j=1 \text{ s.t. } n_j\in N_{Cp}}^{|N|} A_{jl}\psi^0_j(x^\varepsilon_\mu)\chi_{Y^\varepsilon_\mu}(x) + \sum_{k=1}^d A_{jl}\partial_{x_k}\psi^0_j(x^\varepsilon_\mu)\varepsilon y_k(n_j)\chi_{Y^\varepsilon_\mu}(x) + \varepsilon O(\varepsilon).$$

Since $\sum_{j=1 \text{ s.t. } n_j\in N_{Cp}}^{|N|} A_{jl} = 1 - 1 = 0$ for all l and $\partial_{x_k}\psi^0_j(x^\varepsilon_\mu) = \partial_{x_k}\psi^0_j(x) + O(\varepsilon)$ for $x \in Y^\varepsilon_\mu$ it remains

$$\frac{1}{\varepsilon}\sum_{p=1}^{n_c}\sum_{j=1 \text{ s.t. } n_j\in N_{Cp}}^{|N|}\sum_{k=1}^d A_{jl}\partial_{x_k}\psi^0_j(x)\varepsilon y_k(n_j) + \varepsilon O(\varepsilon) = (\partial_\tau\psi^0_C)_l + O(\varepsilon).$$

Now, $I_{E_{NC}}S_vA^T\widehat{\psi}^0 = I_{E_{NC}}A^T\widehat{\psi}^0 = I_{E_{NC}}A^T\widehat{\psi}^0 = I_{E_{NC}}A^TI_{N_{NC}}\widehat{\psi}^0$. But

$$\begin{aligned}(I_{N_{NC}}\widehat{\psi}^0)_j(x^\varepsilon_\mu) &= \psi^0_{NCj}(x^\varepsilon_\mu) \text{ for } n_j \in N - N_C\\ &= \psi^0_{Cp}(x^\varepsilon_\mu + \varepsilon y(n_j)) \text{ for } n_j \in N_{Cp}\cap N_{NC}.\end{aligned}$$

But $\psi^0_{Cp}(x^\varepsilon_\mu + \varepsilon y(n_j)) = \psi^0_{Cp}(x^\varepsilon_\mu) + O(\varepsilon)$ $(I_{N_{NC}}\widehat{\psi}^0)(x^\varepsilon_\mu) = \psi^0_{NC}(x^\varepsilon_\mu) + I^0\psi^0_C + O(\varepsilon)$. Thus

$$(I_{E_{NC}}S_vA^T\widehat{\psi}^0)(x) = I_{E_{NC}}S_vA^T(\psi^0_{NC} + I^0\psi^0_C)(x) + O(\varepsilon).$$

This complete the proof of (ii).

(iii) For $n_j \in N_C - \partial N$

$$\widehat{\psi}^1_j(x) = \sum_\mu \psi^1_{Cj}(x^\varepsilon_\mu)\chi_{Y^\varepsilon_\mu}(x) = \sum_\mu \psi^1_{Cj}(x)\chi_{Y^\varepsilon_\mu}(x) + O(\varepsilon) = \psi^1_{Cj}(x) + O(\varepsilon).$$

For $n_j \in \partial N$

$$\widehat{\psi}_j^1(x) = \sum_\mu \frac{1}{2}(\psi^1_{Cj}(x^\varepsilon_\mu) + \psi^1_{Cj'}(x^\varepsilon_{\mu'}))\chi_{Y^\varepsilon_\mu}(x)$$

but $\psi^1_{Cj'}(x^\varepsilon_{\mu'}) = \psi^1_{Cj'}(x^\varepsilon_\mu) + O(\varepsilon) = \psi^1_{Cj}(x^\varepsilon_\mu) + O(\varepsilon)$ due to periodicity. Then

$$= \sum_\mu \psi^1_{Cj}(x^\varepsilon_\mu)\chi_{Y^\varepsilon_\mu}(x) + O(\varepsilon) = \psi^1_{Cj}(x) + O(\varepsilon).$$

The global result $\widehat{\psi}_j^1 = \psi^1_{Cj} + O(\varepsilon)$ follows.

(iv) comes from (iii) by applying $I_{E_C}A^T$ on each side of the equality. ■

5 Proof of the theorem 7

The point (i) has been established in the lemma 13. In order to state (ii) and (iii), we establish the so called two-scale model which is posed on both the cell circuit and the macroscopic domain Ω. From (i) we know that φ^0 and of v^0 can be expressed with respect to the fields φ^0_C, φ^1_C and φ^0_{NC} so that they satisfy the expression (15) and (16).

Lemma 16 *Under the assumptions (H0-H4), $(\varphi^0_C, \varphi^1_C, \varphi^0_{NC}) \in \Psi$ and $i^0 \in L^2(\Omega)^{|E|}$ are solutions of the two-scale circuit equations*

$$\begin{aligned} R^0 i^0(x) + M^0 v^0(x) &= u^0_s(x) \text{ for all } x \in \Omega \\ (i^0, \partial_\tau \psi^0_C + I_{E_C}A^T\psi^1_C + I_{E_{NC}}A^T(\psi^0_{NC} + I^0\psi^0_C) &= 0 \text{ for all } (\psi^0_C, \psi^1_C, \psi^0_{NC}) \in \Psi. \end{aligned} \quad (21)$$

with Ψ defined in (20).

In order to prove (iii), we replace v^0 by its expression and pose $\psi^0_C = 0$:

$$v = I_{E_C}A^T\varphi^1_C + I_{E_{NC}}A^T\varphi^0_{NC}$$
$$R^0 i^0(x) + M^0 v(x) = u^0_s(x) - M^0(\partial_\tau\varphi^0_C + I_{E_{NC}}A^T I^0\varphi^0_C)(x) \text{ for all } x \in \Omega$$
$$(i^0, I_{E_C}A^T\psi^1_C + I_{E_{NC}}A^T\psi^0_{NC}) = 0.$$

This proves that $(\varphi^1_C, \varphi^0_{NC}, i^0, v)$ is solution of the cell problem (9) with $(\eta, \theta, u_s) = (\varphi^0_C, \nabla\varphi^0_C, u^0_s)(x)$ at a given x and $(\psi_C, \psi_{NC}) := (\psi^1_C, \psi^0_{NC})$. Remark that $I_{N_C \cup N_0}\psi^0_{NC} = 0$ has been replaced by $I_{N^0_C \cup N - N_C}\psi_C = 0$ for the sake of uniqueness of φ_C.

(ii) Thanks to the assumption (H5) and to (iii) we know that

$$i^0 = \mathcal{L}_i\varphi^0_C + \mathcal{P}_i\nabla\varphi^0_C + \mathcal{H}_i u^0_s,$$
$$\varphi_{NC} = \mathcal{L}_\varphi\varphi^0_C + \mathcal{P}_\varphi\nabla\varphi^0_C + \mathcal{H}_\varphi u^0_s,$$
$$\text{and } v = \mathcal{L}_v\varphi^0_C + \mathcal{P}_v\nabla\varphi^0_C + \mathcal{H}_v u^0_s.$$

Replacing in the two-scale branch equations leads to

$$(R^0\mathcal{P}_i + M^0(\tau + \mathcal{L}_v))\nabla\varphi_C^0 + (R^0\mathcal{L}_i + M^0(I_{E_{NC}}A^T I^0 + \mathcal{L}_v))\varphi_C^0 = (I - R^0\mathcal{H}_i - M^0\mathcal{H}_v)u_s^0$$

or equivalently to $Q^H\nabla\varphi_C^0 + S^H\varphi_C^0 = F^H u_s^0$. Now, posing $\psi_C^1 = \psi_{NC}^0 = 0$ it follows that

$$\begin{aligned} &\int_{\Omega\times E} (\mathcal{P}_i\nabla\varphi_C^0 + \mathcal{L}_i\varphi_C^0)_j(x)(\partial_\tau\psi_C^0 + I_{E_{NC}}A^T I^0\psi_C^0)_j(x)\ dx \\ = &- \int_{\Omega\times E} (\mathcal{H}_i u_s^0)_j(x)(\partial_\tau\psi_C^0 + I_{E_{NC}}A^T I^0\psi_C^0)_j(x)\ dx \text{ for all } \psi_C^0 \in \Psi^H. \end{aligned}$$

Applying standard argument in related to variational formulations of partial differential equations yields to the partial differential equation (17_2) and its associated boundary conditions.

It remains to prove the lemma 16.

Proof. The fact that $(\varphi_C^0, \varphi_C^1, \varphi_{NC}^0) \in \Psi$ comes from the lemma 13. It remains to derive the equations (21). We start from the circuit equations (5). Let us apply the two-scale transform and the lemma 10 to the first equation and the scalar product preservation (19) and the lemma 12 to the second equation:

$$M\widehat{v} + R\widehat{i} = \widehat{u_s} \text{ and } (\widehat{i}, A^T\widehat{\psi}) = 0.$$

Introducing the scaled two-scale transforms (7) and (8) of vectors and matrices

$$M^\varepsilon S_v A^T\widehat{\varphi}^\varepsilon + R^\varepsilon\widehat{\widehat{i}^\varepsilon} = \widehat{u_s}^\varepsilon \text{ and } (\widehat{i}^\varepsilon, S_v A^T\widehat{\psi}) = 0.$$

The scalar product between the first equation and a test function $j \in L^2(\Omega)^{|E|}$ yields

$$(M^\varepsilon S_v A^T\widehat{\varphi}^\varepsilon, j) + (R^\varepsilon\widehat{\widehat{i}^\varepsilon}, j) = (\widehat{u_s}^\varepsilon, j) \text{ and } (\widehat{i}^\varepsilon, S_v A^T\widehat{\psi}) = 0$$

or equivalently

$$(S_v A^T\widehat{\varphi}^\varepsilon, M^{\varepsilon T}j) + (\widehat{i}^\varepsilon, R^{\varepsilon T}j) = (\widehat{u_s}^\varepsilon, j) \text{ and } (\widehat{i}^\varepsilon, S_v A^T\widehat{\psi}) = 0.$$

Thanks to (H3) and (H4) and the lemma 15 one may pass to the limit $\varepsilon \to 0$

$$(v^0, M^{0T}j) + (i^0, R^{0T}j) = (u_s^0, j) \text{ and } (i^0, w^0) = 0.$$

The first equation being valid for all $j \in L^2(\Omega)^{|E|}$ is also equivalent to $R^0 i^0 + M^0 v^0 = u_s^0$. According to the lemma 15 for each $(\psi_C^0, \psi_C^1, \psi_{NC}^0) \in \Psi$, there exists such a w^0 with

$$w^0 = \partial_\tau\psi_C^0 + I_{E_C}A^T\psi_C^1 + I_{E_{NC}}A^T(I^0\psi_C^0 + \psi_{NC}^0).$$

Plugging this expression in the second equation ends the proof. ■

6 Examples

Let us establish in detail the homogenized models for the three examples.

6.1 Example 1

The nodes and branches are numbered according to the figure, $n_c = 1$, $E_C = \{e_1, e_2, e_3, e_4\}$, $E_{NC} = \{e_5\}$, $N_C = \{n_1, n_2, n_3, n_4, n_5\}$, $N_{NC} = \{n_2, n_6\}$, $N_0 = \{n_6\}$, $N_C^0 = \{n_2\}$ (arbitrary choice in N_C). The local matrices are

$$\begin{aligned}
R &= \begin{pmatrix} rI_4 & 0_{4\times1} \\ 0_{1\times4} & 1 \end{pmatrix},\ M = M^\varepsilon = M^0 = \begin{pmatrix} -I_4 & 0_{4\times1} \\ 0_{1\times4} & 0 \end{pmatrix},\ \widehat{u}_s = \begin{pmatrix} 0_4 \\ \widehat{i}_s \end{pmatrix}, \\
S_v &= \begin{pmatrix} \frac{1}{\varepsilon}I_4 & 0_{4\times1} \\ 0_{1\times4} & 1 \end{pmatrix},\ S_c = \begin{pmatrix} I_4 & 0_{4\times1} \\ 0_{1\times4} & \frac{1}{\varepsilon} \end{pmatrix},\ \Pi_c = \begin{pmatrix} 0_{4\times4} & 0_{4\times1} \\ 0_{1\times4} & \frac{1}{\varepsilon} \end{pmatrix}, \\
\Pi_v &= \begin{pmatrix} I_4 & 0_{4\times1} \\ 0_{1\times4} & 0 \end{pmatrix},\ S_s = \frac{1}{\varepsilon}I_5,\ R^\varepsilon = \begin{pmatrix} \frac{1}{\varepsilon}rI_4 & 0_{4\times1} \\ 0_{1\times4} & 1 \end{pmatrix},\ \widehat{u}_s^\varepsilon = \begin{pmatrix} 0_4 \\ \frac{1}{\varepsilon}\widehat{i}_s \end{pmatrix}.
\end{aligned}$$

So we pose $r = \varepsilon r_0$ and $\widehat{i}_s = \varepsilon(i_s^0 + O(\varepsilon))$ then $R^0 = \begin{pmatrix} r_0 I_4 & 0_{4\times1} \\ 0_{1\times4} & 1 \end{pmatrix}$, $u_s^0 = \begin{pmatrix} 0_4 \\ i_s^0 \end{pmatrix}$. The incidence matrix is

$$A^T = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Here $\psi_{NC} = 0$, then $\Psi^m = \{\psi_C \in \mathbb{R}^6 \,/\, \psi_C = J\psi_C^* \text{ where } \psi_C^* \in \mathbb{R}^2\}$ with $J = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}^T$.

Moreover $y(n) = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$, $\tau = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}^T$, $I^0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \end{pmatrix}^T$ $u_s^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & i_s^0 \end{pmatrix}^T$. The problem micro has the form $K(i, \varphi_C^*)^T = L(\theta, \eta, i_s)^T$ (here we prefer to work with i_s^0 in place of the whole u_s^0). An explicit calculation shows that $G = K^{-1}L = \begin{pmatrix} G_{11} & 0_{4\times2} \\ 0_{3\times2} & G_{22} \end{pmatrix}$ with $G_{11} = -\frac{1}{2r_0}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^T$ and $G_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^T$. Thus $\mathcal{L}_i$, $\mathcal{P}_v$ and $\mathcal{H}_v$ vanish, $\mathcal{P}_i = \frac{1}{2r_0}\tau$, $\mathcal{H}_i = \mathcal{L}_v = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T$. Then, Q^H, S^H and F^H vanish and φ_C^0 is governed by the Laplace equation

$$\Delta\varphi_C^0 = 2r_0 i_s^0 \text{ in } \Omega$$

with the boundary conditions $\varphi_C^0 = 0$ on Γ_0 and $\nabla\varphi_C^0.n = 0$ on $\Gamma - \Gamma_0$. Finally, the two-scale current and voltages are given by

$$\begin{aligned}
i^0 &= \begin{pmatrix} \partial_{x_1}\varphi_C^0 & \partial_{x_1}\varphi_C^0 & \partial_{x_2}\varphi_C^0 & \partial_{x_2}\varphi_C^0 & i_0^s \end{pmatrix}^T \\
v^0 &= \begin{pmatrix} \partial_{x_1}\varphi_C^0 & \partial_{x_1}\varphi_C^0 & \partial_{x_2}\varphi_C^0 & \partial_{x_2}\varphi_C^0 & \varphi_C^0 \end{pmatrix}^T.
\end{aligned}$$

6.2 Example 2

Here $n_c = 0$, $E_C = \varnothing$, $E_{NC} = \{e_1, e_2\}$, $N_C = \varnothing$, $N_{NC} = \{n_1, n_2\}$. The local matrices are

$$R = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \; M = M^\varepsilon = M^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; \widehat{u}_s = \widehat{u}_s^\varepsilon = \begin{pmatrix} 0 \\ \widehat{v}_s \end{pmatrix},$$

$$S_v = I_2, \; S_c = \frac{1}{\varepsilon} I_2, \; \Pi_c = 0, \; \Pi_v = I_2, \; S_s = I_2, \; R^\varepsilon = \begin{pmatrix} \varepsilon r & 0 \\ 0 & 0 \end{pmatrix}, \; A^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

So we pose $r = \frac{1}{\varepsilon} r_0$ and $\widehat{v}_s = v_s^0 + O(\varepsilon)$ then $R^0 = \begin{pmatrix} r_0 & 0 \\ 0 & 0 \end{pmatrix}$, $u_s^0 = \begin{pmatrix} 0 \\ v_s^0 \end{pmatrix}$. Since $E_C = \varnothing$ there is no macroscopic model and (i^0, v, φ_{NC}^0) solves only the cell problem with $\Psi^m = \{\psi_{NC} \in \mathbb{R}^2 \,/\, \psi_{NC} = J \psi_{NC}^* \text{ where } \psi_{NC}^* \in \mathbb{R}\}$ with $J = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So The problem micro has the form $K(i, \varphi_{NC}^{0*})^T = L(v_s)^T$ which leads to $i^0 = -\frac{v_s^0}{r_0}(1,1)^T$ $\varphi_{NC}^0 = -v_s^0$ and $v = v_s^0(-1,1)^T$.

6.3 Example 3

Here $n_c = 1$, $E_C = \{e_1, e_2, e_3, e_4, e_5\}$, $E_{NC} = \{e_6, e_7\}$, $N_C = \{n_1, n_2, n_3, n_4, n_5\}$, $N_{NC} = \{n_6, n_7\}$, $N_0 = \{n_7\}$, $N_C^0 = \{n_2\}$ (arbitrary choice in N_C),

$$R = \begin{pmatrix} rI_4 & 0_{4\times3} \\ 0_{3\times4} & \delta_{11} + \delta_{33} \end{pmatrix}, \; M = M^\varepsilon = \begin{pmatrix} -I_4 & 0_{4\times3} \\ 0_{3\times4} & k\delta_{13} + \delta_{22} \end{pmatrix}, \; \widehat{u}_s = \begin{pmatrix} 0_5 \\ \widehat{v}_s \\ 0 \end{pmatrix},$$

$$S_v = \begin{pmatrix} \frac{1}{\varepsilon} I_5 & 0_{5\times2} \\ 0_{2\times5} & I_2 \end{pmatrix}, \; S_c = \begin{pmatrix} I_5 & 0_{5\times2} \\ 0_{2\times5} & \frac{1}{\varepsilon} I_2 \end{pmatrix}, \; \Pi_c = \begin{pmatrix} 0_{4\times4} & 0_{4\times2} \\ 0_{2\times4} & \frac{1}{\varepsilon}(\delta_{11} + \delta_{33}) \end{pmatrix},$$

$$\Pi_v = \begin{pmatrix} I_4 & 0_{4\times3} \\ 0_{3\times4} & \delta_{22} \end{pmatrix}, \; S_s = \begin{pmatrix} \frac{1}{\varepsilon} r I_4 & 0_{4\times3} \\ 0_{3\times4} & \delta_{11} + \delta_{22} + \frac{1}{\varepsilon}\delta_{33} \end{pmatrix},$$

$$R^\varepsilon = \begin{pmatrix} \frac{1}{\varepsilon} r I_4 & 0_{4\times3} \\ 0_{3\times4} & \delta_{11} + \delta_{33} \end{pmatrix}, \; \widehat{u}_s^\varepsilon = \begin{pmatrix} 0_5 \\ \widehat{v}_s \\ 0 \end{pmatrix}$$

where we used the submatrix δ_{ij} having all its entries vanishing excepted the entry (i,j). The size of such a submatrix is known by its surrounding submatrices. So we pose $r = \varepsilon r_0, k = k_0$ and $\widehat{v}_s = v_s^0 + O(\varepsilon)$ then $R^0 = \begin{pmatrix} r_0 I_4 & 0_{4\times3} \\ 0_{3\times4} & \delta_{11} + \delta_{33} \end{pmatrix}$, $u_s^0 = (0_5, v_s^0, 0)^T$. The incidence matrix is $A^T = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ where

$$X_{11} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}, \; X_{12} = 0_{4\times2}, \; X_{22} = \delta_{11} - \delta_{12}, \; X_{22} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$y(n)$ is the same than in example 1, $\tau = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}^T$ and $I^0 = I_{N_C}$. Here $\Psi^m = \{(\psi_C, \psi_{NC}) = (J_C\psi_C^*, J_{NC}\psi_{NC}^*)$ where $(\psi_C^*, \psi_{NC}^*) \in \mathbb{R}^2 \times \mathbb{R}\}$ with $J_C = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}^T$ and $J_{NC} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$. The problem micro has the form $K(i, \varphi_C^*, \varphi_{NC}^*)^T = L(\theta, \eta, v_s^0)^T$ (here we prefer to work with v_s^0 in place of the whole u_s^0). An explicit calculation shows that $G = K^{-1}L = \begin{pmatrix} G_{11} & G_{12} \\ 0_{6\times 2} & G_{22} \end{pmatrix}$ with $G_{11} = -\frac{1}{2r_0}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^T$, $G_{12} = \frac{1}{2k_0}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$ and $G_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ k_0 & 0 & 0 & -\frac{1}{2}k_0 r_0 & 0 & 1 \end{pmatrix}$. Since

$$i = \mathcal{T}_i(\theta, \eta, v_s)^T,$$
$$v = I_{E_C}A^T J_C \varphi_C^* + I_{E_{NC}}A^T J_{NC}\varphi_{NC}^* = \mathcal{T}_v(\theta, \eta, v_s)^T,$$
$$\text{and } \varphi_{NC} = J_{NC}\varphi_{NC}^* = \mathcal{T}_\varphi(\theta, \eta, v_s)^T$$

with $\mathcal{T}_i = [K^{-1}L]_{\{1,..,7\}\times .}$, $\mathcal{T}_v = I_{E_C}A^T J_C[K^{-1}L]_{\{8,9\}\times .} + I_{E_{NC}}A^T J_{NC}[K^{-1}L]_{\{10\}\times .}$, $\mathcal{T}_\varphi = J_{NC}[K^{-1}L]_{\{10\}\times .}$. Then $\mathcal{P}_i = [\mathcal{T}_i]_{.\times\{1,2\}} = -\frac{1}{2r_0}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}^T$, $\mathcal{L}_i = [\mathcal{T}_i]_{.\times\{3\}} = 0$, $\mathcal{H}_i = [\mathcal{T}_i]_{.\times\{4\}} = \frac{k_0}{2}(-1,1,0,0,2,0,0)^T$, $\mathcal{P}_v = [\mathcal{T}_v]_{.\times\{1,2\}} = 0$, $\mathcal{L}_v = [\mathcal{T}_v]_{.\times\{3\}} = 0$ and

$\mathcal{H}_v = [\mathcal{T}_v]_{.\times\{4\}} = (-\frac{r_0k_0}{2}, \frac{r_0k_0}{2}, 0, 0, \frac{r_0k_0}{2}, 1, -1)^T$, $\mathcal{P}_\varphi = [\mathcal{T}_\varphi]_{.\times\{1,2\}} = 0$, $\mathcal{L}_\varphi = [\mathcal{T}_\varphi]_{.\times\{3\}} = 0$ and $\mathcal{H}_\varphi = [\mathcal{T}_\varphi]_{.\times\{4\}} = (0,0,0,0,0,1,0)^T$. Q^H, S^H and F^H vanishes and φ_C^0 is governed by the Laplace equation

$$-\Delta\varphi_C^0 = r_0\partial_{x_1}(k_0 v_s^0) \text{ in } \Omega$$

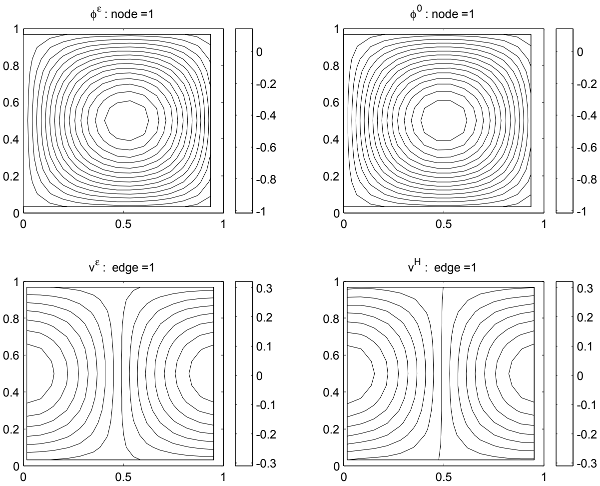
with the boundary conditions $\varphi_C^0 = 0$ on Γ_0 and $\nabla\varphi_C^0.n = 0$ on $\Gamma - \Gamma_0$. Finally, the two-scale current and voltages are given by

$$i^0 = -\frac{1}{2}(\frac{\partial_{x_1}\varphi_C^0}{r_0} + \frac{k_0 v_s^0}{2}, \frac{\partial_{x_1}\varphi_C^0}{r_0} - \frac{k_0 v_s^0}{2}, \frac{\partial_{x_2}\varphi_C^0}{r_0}, \frac{\partial_{x_2}\varphi_C^0}{r_0}, 2k_0, 0, 0)^T,$$
$$v^0 = -\frac{1}{2}(\partial_{x_1}\varphi_C^0 - r_0k_0v_s^0, \partial_{x_1}\varphi_C^0 - r_0k_0v_s^0, \partial_{x_2}\varphi_C^0, \partial_{x_2}\varphi_C^0, \partial_{x_1}\varphi_C^0 - r_0k_0, 1, -1)^T,$$
$$\text{and } \varphi_{NC}^0 = (0,0,0,0,0,1,0)^T v_s^0.$$

6.4 Numerical validation

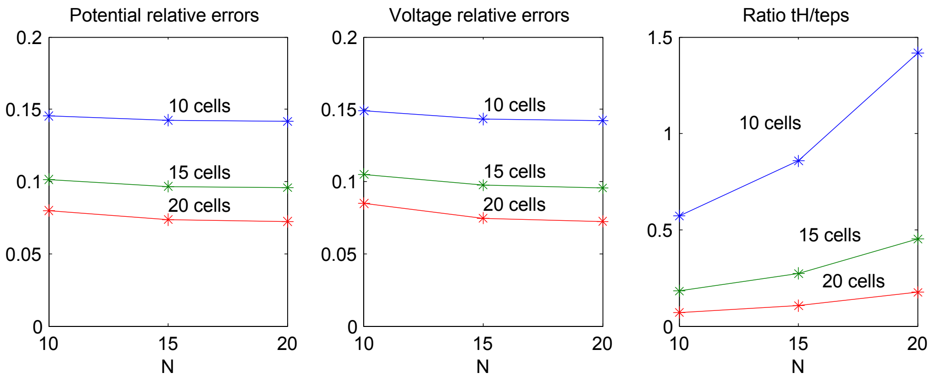
Let us report the result of our simulation in the third example. First, we compare the two solutions computed on the one hand on the periodic network of 15×15 cells and on the other hand by using the homogenized model. The calculation have been carried out for the values $r_0 = k_0 = 1$. The distribution of voltage source $v_s^0(x_1, x_2) = -2\pi\cos(\pi x_1)\sin(\pi x_2)$ when its counterpart $\widehat{v}_s^\varepsilon$ for the periodic circuit is taken equal to $v_s(x_i^\varepsilon)$, the x_i^ε being the centres of the cells . The node′s voltage φ_C^0 is computed using a P^1 finite elements method with 15×15 elements.

The first component of the two-scale transform $\widehat{\varphi}_1^\varepsilon(x_i^\varepsilon)$ is compared with the first component of the approximation, with the finite elements method, of the limit φ^0 at the point x_i^ε.The results are presented respectively to the left and to the right of the first row of the figure below. The second figure presents the first component of the two-scale transform of the branch s voltages $\widehat{v}_1^\varepsilon(x_i^\varepsilon)$ and the approximation of the first component of the limit v_1 at the point x_i^ε. They are placed respectively to the left and to the right of the second row. The results show a good qualitative agreement between the two models.



Comparison of the complete model and the two-scale model

Quantitative comparisons are detailed on the next figure. Global relative errors, in L^2 norm, for node′s voltages and branch s voltages are compared when the periodic network has 10, 15 or 20 cells in each direction and when the finite elements methods is with $N = 10$, 15 or 20 elements in each direction. It shows that, in this case, the errors diminish with the increase of the cell number but is not so much in! uenced by the number of finite elements. The observation of the ratio $\frac{t^H}{t^\varepsilon}$ of the simulation times of the two models yields to the conclusion that the homogenized model presents a great interest on this point of view. This is particularly true for large number of cells. Moreover, we have observed that more the complexity of the circuit increases, more the ratio is favourable to the homogenized model.



Errors and simulation times

7 Annex

The proof of the lemma 14 necessitates the fundamental lemma 20 stated and proved hereafter. It requires additional results on two-scale transform for nodes that we establish at first. The proof includes quite long calculations that we do not want to see in the core of the paper. However, we must emphasis that it constitutes an important part of our work.

7.1 Properties of the two-scale transform T_N

Let us recall that the set $\mathcal{N}$ is made of nodes $n_{\mathcal{I}}$ with I$\in \{1,..,|\mathcal{N}|\}$ and the set N of nodes n_j with $j \in \{1,..,|N|\}$. The subset of N of nodes belonging to the boundary of the cell Y is denoted by ∂N. Because $\mathcal{N}$ is $\varepsilon Y-$periodic, it turns out that each node $n \in \partial N$ has its counterpart $n' \in \partial N$ on the opposite side. If the former's index is j then the latter's one is denoted by j'. The outward normal vector to the boundary of Y at n being denoted by $n_Y(n)$, it turns out that $n_Y(n') = -n_Y(n)$. For a given multi-integer $\mu \in \{1,..,m\}^d$ we define the multi-integer

$$\mu' = \mu + n_Y(n)$$

associated to n and μ. Let I and (μ, j) be linked through the relation I$\sim (\mu, j)$. If $n_{\mathcal{I}}$ belongs to only one cell then there exists an unique such (μ, j). If $n_{\mathcal{I}}$ is located at the interface between two cells then I is associated to two couples (μ, j) and (μ', j') with μ' and j' derived as above from μ and j. In short we say that I$\sim (\mu, j)$ and I$\sim (\mu', j')$. Conversely if two couples (μ, j) and (λ, l) correspond to the same I then j is the index of a node located on the boundary of the cell and $(\lambda, l) = (\mu', j')$. These statements are condensed in the next proposition.

Proposition 17 *Two couples (μ, j) and (λ, l) come from a same index I if and only if $n_j \in \partial N$ and $(\lambda, l) = (\mu', j')$.*

The map that send a vector $u \in \mathbb{R}^{|\mathcal{N}^\varepsilon|}$ towards a tensor $U \in \mathbb{R}^{m^d} \times \mathbb{R}^{|N|}$ has been well defined in § 3.1. From the above discussion, it is clear that it is not onto. There exist some $U \in \mathbb{R}^{m^d} \times \mathbb{R}^{|N|}$ that does not have a counter-image $u \in \mathbb{R}^{|\mathcal{N}^\varepsilon|}$.

Proposition 18 *A tensor* $U \in \mathbb{R}^{m^d} \times \mathbb{R}^{|N|}$ *is the image of* $u \in \mathbb{R}^{|\mathcal{N}^\varepsilon|}$ *if and only if* $U_{\mu j} = U_{kl}$ *for all couples* (μ, j), (λ, l) *such that* $n_j \in \partial N$ *and* $(\lambda, l) = (\mu', j')$.

Proof. We must prove that $U_{\mu j} = U_{kl}$ for all couples (μ, j) and (λ, l) associated a same index I. The proposition 17 yields the conclusion. ∎

It becomes clear that if $\partial N \neq \varnothing$ then T_N is not onto in $\mathbb{P}^0(\Omega)^{|N|}$. Let us state the compatibility conditions on $v \in \mathbb{P}^0(\Omega)^{|N|}$ insuring that it has a counter-image by T_E. For a given $x \in \Omega$ and node $n \in \partial N$ we define

$$x' = x + \varepsilon n_Y(n).$$

Proposition 19 *A function* $v \in \mathbb{P}^0(\Omega; \mathbb{R}^{|N|})$ *is the two-scale transform of a vector* $\mathbb{R}^{|\mathcal{N}^\varepsilon|}$ *if and only if*

$$v_{j'}(x') = v_j(x)$$

for all $(x, n_j) \in \Omega \times \partial N$ *and for* $x' = x + \varepsilon n_Y(n_j)$.

Proof. Since $v \in \mathbb{P}^0(\Omega)^{|N|}$ it may be written $v_j(x) = \sum_{\mu \in \{1,..,m\}^d} V_{\mu j} \chi_{Y^\varepsilon_\mu}(x)$. From the proposition 17 there exists $u \in \mathbb{R}^{|\mathcal{N}^\varepsilon|}$ such that $U_{\mu j} = V_{\mu j}$ if and only if $V_{\mu j} = V_{\mu' j'}$ for all $\mu \in \{1, .., m\}^d$ and $j \in \partial N$. In other words $v_j(x^\varepsilon_\mu) = v_{j'}(x^\varepsilon_{\mu'})$ or equivalently $v_j(x^\varepsilon_\mu) = v_{j'}(x^\varepsilon_\mu + \varepsilon n_Y(n_j))$ because $x^\varepsilon_{\mu'} = x^\varepsilon_\mu + \varepsilon n_Y(n_j)$. The result follows remarking that v is piecewise constant with respect to x. ∎

For a given node $n \in \partial N$, the largest subset of $x \in \Omega$ such that $x' \in \Omega$ is denoted by $\Omega(n)$:

$$\Omega(n) = \{x \in \Omega \text{ such that } x' \in \Omega\}.$$

Because $n_Y(n') = -n_Y(n)$ one may observe that $x = x' + n_Y(n')$, so $x \in \Omega(n)$ if and only if $x' \in \Omega(n')$.

The outward normal vector to the boundary Γ of Ω in a point $x \in \Gamma$ is denoted by $n_\Omega(x)$ and the subset of couples $(x, n) \in \Gamma \times \partial N$ having the same normal $n_Y(n)$ and $n_\Omega(x)$ is denoted by

$$\partial(\Omega, N) = \{(x, n) \in \Gamma \times \partial N \text{ such that } n_Y(n) = n_\Omega(x)\}.$$

A straightforward characterization of the complementary set $\Omega - \Omega(n)$ of $\Omega(n)$ follows:

$$\Omega - \Omega(n) = \{x = \overline{x} - \varepsilon\theta n_Y(n) \text{ where } \theta \in (0, 1) \text{ and } (\overline{x}, n) \in \partial(\Omega, N)\}. \tag{22}$$

7.2 Fundamental lemma

Lemma 20 *For* $\phi \in \mathbb{P}^0(\Omega; \mathbb{R}^{|N|})$ *belonging to the range of* T_N *and* $\psi \in \mathcal{C}^1(\Omega; \mathbb{R}^{|N|}_{antiper})$ *then*

$$\frac{1}{\varepsilon} \int_{\Omega \times \partial N} \phi_j(x)\psi_j(x)\ dx = -\int_{\Omega \times \partial N} \phi_j(x)(y.\nabla_x \psi)_j(x)\ dx + b(\phi, \psi) + O(\varepsilon)$$

more precisely

$$|\frac{1}{\varepsilon}\int_{\Omega\times\partial N}\phi_j(x)\psi_j(x)\ dx+\int_{\Omega\times\partial N}\phi_j(x)(y.\nabla_x\psi)_j(x)\ dx-b(\phi,\psi)|$$
$$\leq\varepsilon(e_1(\phi,\psi)+e_2(\phi,y.\nabla_x\psi)+e_2(\phi,\psi)).$$

Here

$$\begin{aligned}
y.\nabla_x\psi &\in \mathcal{C}^0(\Omega)^{|N|},\ (y.\nabla_x\psi)_j(x)=\sum_{l=1}^{d}y_l(n_j)\partial_{x_l}\psi_j(x),\\
b(\phi,\psi) &= \sum_{j\in\partial N}\int_\Gamma \chi_{\partial(\Omega,N)}(\overline{x},j)\phi_j(\overline{x})\int_0^1\psi(\overline{x}-\varepsilon\theta n_Y(n_j))d\theta\ d\overline{x}\\
e_1(\phi,\psi) &= \frac{1}{2}||\phi||_{\partial N}(||\frac{\Delta_{\varepsilon n_Y}\psi-n_Y.\nabla_x\psi}{\varepsilon}||_{\partial N}+||(y-\frac{n_Y}{2})\nabla_x\Delta_{\varepsilon n_Y}\psi||_{\partial N})\\
e_2(\phi,\psi) &= (\int_{\partial(\Omega,N)}\phi_j^2(\overline{x})\ d\overline{x})^{1/2}(\int_{\partial(\Omega,N)}\int_0^1\psi_j^2(\overline{x}-\varepsilon\theta n_Y)\ d\theta d\overline{x})^{1/2}
\end{aligned}$$

where $n_Y(n)$ is set to zero for $n\notin\partial N$, $(\Delta_{\varepsilon n_Y}\psi)_j(x)=\dfrac{\psi_j(x+\varepsilon n_Y(n_j))-\psi_j(x)}{\varepsilon}$,

$$\mathbb{R}^{|N|}_{antiper}=\{\phi\in L^2(\Omega;\mathbb{R}^{|N|})\text{ such that }\phi_j(x)=-\phi_{j'}(x)\text{ a.e. }x\in\Omega\text{ for all }n_j\in\partial N\}$$

and for shortness we have used the notations

$$(\phi,\psi)_{\partial N}=\int_{\Omega\times\partial N}\phi_j(x)\psi_j(x)\ dx\text{ and }||\phi||_{\partial N}=(\phi,\phi)^{1/2}_{\partial N}.$$

Proof. For each $n\in\partial N$ we use the partition of Ω in $\Omega(n)$ and its complementary so that

$$(\phi,\psi)_{\partial N}=a_{int}(\phi,\psi)+a_b(\phi,\psi)$$

where

$$a_{int}(\phi,\psi)=\sum_{n_j\in\partial N}\int_{\Omega(n_j)}\phi_j(x)\psi_j(x)dx\text{ et }a_b(\phi,\psi)=\sum_{n_j\in\partial N}\int_{\Omega-\Omega(n_j)}\phi_j(x)\psi_j(x)dx.$$

(i) The characterization (22) of $\Omega-\Omega(n)$ yields $|a_b(\phi,\psi)|=|\varepsilon b(\phi,\psi)|\leq\varepsilon e_2(\phi,\psi)$.
(ii) Let us prove that

$$a_{int}(\phi,\psi)=-a_{int}(\phi,\psi(x+\varepsilon n_Y(n))).$$

In the one side ϕ belongs to the range of T_E and the proposition 19 tell us that $\phi_j(x)=\phi_{j'}(x')$ and in the other side $\psi_j=-\psi_{j'}$. Then

$$a_{int}(\phi,\psi)=-\sum_{n_j\in\partial N}\int_{\Omega(n_j)}\phi_{j'}(x')\ \psi_{j'}(x)\ dx.$$

For a given j let us first apply the variable change $x \to x' = x + \varepsilon n_Y(n_j)$ which maps $\Omega(n_j)$ to $\Omega(n_{j'})$ and in a second step let us replace the numbering by j with a numbering by j' it comes

$$= - \sum_{n_{j'} \in \partial N} \int_{\Omega(n_{j'})} \phi_{j'}(x')\ \psi_{j'}(x)\ dx' = -a_{int}(\phi, \psi(x')) = -a_{int}(\phi, \psi(x + \varepsilon n_Y(n))).$$

(iii) Let us deduce that

$$|\frac{1}{\varepsilon} a_{int}(\phi, \psi) + a_{int}(\phi, (y.\nabla_x)\psi)| \leq \varepsilon e_1(\phi, \psi). \tag{23}$$

Thanks to (ii),

$$\frac{1}{\varepsilon} a_{int}(\phi, \psi) = \frac{a_{int}(\phi, \psi) - a_{int}(\phi, \psi(x'))}{2\varepsilon} = -\frac{1}{2} a_{int}(\phi, \Delta_{\varepsilon n_Y} \psi). \tag{24}$$

For $n \in \partial N$, we make use of the decomposition $y(n) = [y(n)] + \{y(n)\}$ in its periodic part $\{y(n)\} = (y(n) + y(n'))/2$ and its counter-periodic parts $[y(n)] = (y(n) - y(n'))/2$. For $n \in N - \partial N$, $[y(n)]$ and $\{y(n)\}$ are set to 0. From the triangular inequality,

$$\begin{aligned} |\frac{1}{\varepsilon} a_{int}(\phi, \psi) + a_{int}(\phi, y.\nabla_x \psi)| \ &\leq \ |\frac{1}{\varepsilon} a_{int}(\phi, \psi) + a_{int}(\phi, [y].\nabla_x \psi)| \\ &\quad + |a_{int}(\phi, [y].\nabla_x \psi) - a_{int}(\phi, y.\nabla_x \psi)| \end{aligned}$$

combined with (24) and the fact that $n_Y(y) = 2[y]$:

$$\leq |\frac{1}{2} a_{int}(\phi, \Delta_{\varepsilon n_Y} \psi - n_Y.\nabla_x \psi)| + |a_{int}(\phi, \{y\}.\nabla_x \psi)|.$$

Applying (24) with $\psi_j := \{y(n_j)\} \nabla_x \psi_j$:

$$a_{int}(\phi, \{y\} \nabla_x \psi) = -\frac{\varepsilon}{2} a_{int}(\phi, \{y\}.\nabla_x \Delta_{\varepsilon n_Y} \psi) \leq \frac{\varepsilon}{2} ||\phi||_{\partial N} ||\{y\}.\nabla_x \Delta_{\varepsilon n_Y} \psi||_{\partial N}$$

thus

$$\leq \frac{\varepsilon}{2} ||\phi||_{\partial N} (||\frac{\Delta_{\varepsilon n_Y} \psi - n_Y.\nabla_x \psi}{\varepsilon}||_{\partial N} + ||\{y\}.\nabla_x \Delta_{\varepsilon n_Y} \psi||_{\partial N})$$

which is the wanted result (23).

(iv) Thus

$$|\frac{1}{\varepsilon} a_{int}(\phi, \psi) + (\phi, y.\nabla_x \psi)_{\partial N}| \leq \varepsilon (e_1(\phi, \psi) + e_2(\phi, y.\nabla_x \psi)).$$

after remarking that

$$(\phi, y.\nabla_x \psi)_{\partial N} - a_{int}(\phi, y.\nabla_x \psi) = a_b(\phi, y.\nabla_x \psi)$$

and by using (i).

(v) The conclusion comes from

$$|\frac{1}{\varepsilon}(\phi, \psi)_{\partial N} + (\phi, y.\nabla_x \psi)_{\partial N} - b(\phi, \psi)| \leq |\frac{1}{\varepsilon} a_{int}(\phi, \psi) + (\phi, y.\nabla_x \psi)_{\partial N}| + |\frac{1}{\varepsilon} a_b(\phi, \psi) - b(\phi, \psi)|$$

and by using (i) and (iv). ■

Conclusion: A two-scale model of spatially periodic linear electronic circuit have been stated, proved and illustrated by few simple examples. Its statement and derivation are based on the concept of two-scale transform and convergence of vector and matrices also introduced in this paper. The numerical results prove the interest of the method in terms of computation cost. We think that this kind of model which gives a global view of the whole system could also be very usefull in a process of circuit design.

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