Monotone Approximation for a Nonlinear Size and Class Age Structured Epidemic Model

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Abstract

In this paper, we study a nonautonomous size and class age structured epidemic model with nonlinear and nonlocal boundary conditions. We establish a comparison principle and construct convergent monotone sequences to prove the existence of solutions. Uniqueness of solutions is also established.

1 Introduction

In previous research literature, infectious disease dynamics are often modeled via systems of ordinary differential equations in which population structures such as age, size, sex, etc., are neglected. However, there are many cases in which incorporating some of the population structures into the model may provide additional and important information, and may be helpful in the understanding of disease dynamics. For example, it was suggested by the author in [16] that the age-structured seasonal interaction rates of measles that are due to school attendance may cause oscillations that are not described in corresponding models without age structure.

In this paper, we consider a three compartment model with a susceptible state, exposed state (latent state), and infectious state for the spread of a directly transmitted disease in a size-structured population. We assume that the disease under consideration is fatal so that no recovery occurs. We also assume that the disease is transmitted only horizontally, and that every newborn is susceptible to infection. Note that for many diseases there may be some delay between the initial latent state and infectious state. Furthermore, it is not biologically realistic to expect all the individuals to progress into the infectious state at a fixed time period after the initial latent infection. Additionally, individuals in the infectious state may have dramatically varying mortality rates depending on the times that they progress into the infectious state. Hence, in addition to size we also need to record the class age or residency time of the exposed and infectious states, i.e., the length of time the individuals have spent in their present state. Thus, the individuals in the exposed and infectious states are structured by two internal variables: size and class age.

To our knowledge, the paper [17] by Sinko and Streifer was the first appearing in the literature on structured population models with multiple internal variables in which the populations are categorized by chronological age and size. Although their simplified version with size only as the internal variable (the classical size-structured population model) has been studied extensively in past decades, the original version has not been as thoroughly investigated. We have found structured models with multiple internal variables for describing cell dynamics ([5, 10, 11, 12, 14]) and epidemic dynamics ([13, 18]). For example, the authors in [12] studied an autonomous linear chronological age and size structured cell population model that has both normal and quiescent individuals; the theory of positive operator semigroups was used to show that, under general assumptions about individual behavior, the age-size distribution of the population converges to a stable distribution. A proliferating cell population structured by chronological age and maturity was studied in [10] where existence and uniqueness of the solutions was established via the method of characteristics. This model was then extended to a nonlinear one in [11] with a nonlinear term corresponding to loss of individuals due to crowding; existence and asymptotic behavior of solutions were studied using semigroup operator theory. The authors in [5] investigated a linear population model describing cellular division with both quiescent and proliferating cell populations structured by chronological age and maturity of the cell in which a proliferating phase duration depends on the maturity of cells. The existence and uniqueness of solutions were established using semigroup theory by first summing over the chronological age to reduce the model to a system of size-structured delay partial differential equations with the time delay depending on maturity.

Modern analysis of structured epidemic models with multiple internal variables apparently began with the work by Hoppensteadt in [13] and Waltman in [18], where the population is structured by chronological age and class age. In [15] a general model of a population structured by several internal variables is formulated as a nonlinear integral equation, and existence, uniqueness and positivity of solutions is established using the method of characteristics, the contraction mapping theorem and semigroup theory.

The goal of this paper is to establish the existence and uniqueness of the solutions to the system (2.1) described below by defining upper and lower solutions, establishing a comparison principle and constructing monotone approximations. Using this approach, we replace the true solution in all the nonlinear and nonlocal terms with some previous estimate for the solution. We then solve the resulting linear model and obtain a new estimate for the solution. This iterative procedure yields the solution of the original problem by passing to the limit. The method also provides an explicit solution representation for each of the iterates. Thus, an efficient numerical scheme can be developed. The key step involved is a comparison principle between consecutive estimates [3]. This method has been used in both linear and nonlinear size-structured population models for a number of systems (e.g., [1, 2, 4]). However, we believe ours is the first paper which uses this method to study a structured model with multiple internal variables.

The models we consider here include as a special case the general models in [6]. These models were developed as part of a biomass vaccine production system wherein shrimp protein pathways are co-opted for rapid production via recombinant protein transfection. In [6] the authors used simulation studies to investigate the efficacy of such a production system as a first response countermeasure to toxic attacks. We provide here a well-posedness theoretical foundation for the systems developed in these applications.

The outline of this paper is organized as follows: We present our model in Section 2. We then establish a comparison principle and argue the uniqueness of solutions of our model in Section 3. Monotone sequences are constructed in Section 4 to establish existence of solutions for our model. We present concluding remarks in Section 5.

2 Size and Class Age Structured Epidemic Model

The model we consider here is given by

$$\begin{aligned}
&S_t(x,t) + (g^S(x,t)S(x,t))_x + m^S(x,t)S(x,t) = -S(x,t)\int_0^{x_{\max}}\int_0^{\infty}\lambda(y,\theta)I(y,\theta,t)d\theta dy\\
&g^S(0,t)S(0,t) = F(S(\cdot,t),E(\cdot,\cdot,t),I(\cdot,\cdot,t))\\
&S(x,0) = S^0(x)\\
&E_t(x,\theta,t) + (g^E(x,t)E(x,\theta,t))_x + E_\theta(x,\theta,t) + m^E(x,t)E(x,\theta,t) = -\rho^E(\theta)E(x,\theta,t)\\
&E(x,0,t) = S(x,t)\int_0^{x_{\max}}\int_0^{\infty}\lambda(y,\theta)I(y,\theta,t)d\theta dy\\
&E(0,\theta,t) = 0, \quad E(x,\theta,0) = E^0(x,\theta)\\
&I_t(x,\theta,t) + (g^I(x,t)I(x,\theta,t))_x + I_\theta(x,\theta,t) + m^I(\theta)I(x,\theta,t) = 0\\
&I(x,0,t) = \int_0^{\infty}\rho^E(\theta)E(x,\theta,t)d\theta\\
&I(0,\theta,t) = 0, \quad I(x,\theta,0) = I^0(x,\theta),
\end{aligned} \tag{2.1}$$

where $(x,\theta,t) \in [0,x_{\max}] \times [0,\infty) \times [0,T]$, and

$$\begin{aligned}
&F(S(\cdot,t),E(\cdot,\cdot,t),I(\cdot,\cdot,t))\\
&\quad = \int_0^{x_{\max}}\beta^S(x)S(x,t)dx + \int_0^{x_{\max}}\int_0^{\infty}\beta^E(x,\theta)E(x,\theta,t)d\theta dx + \int_0^{x_{\max}}\int_0^{\infty}\beta^I(x,\theta)I(x,\theta,t)d\theta dx.
\end{aligned}$$

Here $S(x,t)$ denotes the density of individuals in the susceptible state having size x at time t, $E(x,\theta,t)$ denotes the density of individuals having size x at time t that have spent θ time units in the latent (exposed) state, and $I(x,\theta,t)$ denotes the density of individuals having size x at time t that have spent θ time units in the infectious state. The functions $g^S(x,t)$, $g^E(x,t)$ and $g^I(x,t)$ are the growth rates of the individuals having size x at time t in the susceptible state, latent state, and infectious state, respectively. The functions $m^S(x,t)$ and $m^E(x,t)$ are the mortality rates of the individuals having size x at time t in the susceptible state, and latent state, respectively, and $m^I(\theta)$ is the mortality rate of individuals having spent θ time units in the infectious state. The function $\beta^S(x)$ represents the fecundity rate of individuals having size x in the susceptible state. The functions $\beta^E(x,\theta)$ and $\beta^I(x,\theta)$ denote the fecundity rates of the individuals having size x that have spent θ time units in the latent state, and infectious state, respectively. The function $\lambda(x,\theta)$ denotes the infection rate of individuals having size x that have spent θ time units in the infectious state. The function $\rho^E(\theta)$ represents the rate at which individuals having spent θ time units in the latent state become infectious.

For convenience, we will use the following notation throughout the paper:

$$\begin{aligned}
&D^S = (0,x_{\max}) \times (0,T), \quad D^{EI} = (0,x_{\max}) \times (0,\infty) \times (0,T),\\
&D^{EI}_{x\theta} = (0,x_{\max}) \times (0,\infty), \quad D^{EI}_n = (0,x_{\max}) \times (0,n) \times (0,T).
\end{aligned}$$

The space $C^1_{0,r}(D^{EI})$ is defined by:

$$\{\phi \in C^1(D^{EI}) |\text{ there exists some constant } \theta_\phi > 0 \text{ such that } \phi(x,\theta,t) = 0 \text{ for } \theta > \theta_\phi\},$$

and the space $C^1_{0,r}(D^{EI}_n)$ is given by

$$\{\phi \in C^1(D^{EI}_n)| \text{ there exists some constant } 0 < \theta_\phi < n \text{ such that } \phi(x,\theta,t) = 0 \text{ for } \theta > \theta_\phi\}.$$

The following standing hypotheses will be assumed throughout the paper:

(H1) The individual growth rates satisfy g^S, g^E and $g^I \in C^1(D^S)$. We further assume that $g^S(x,t) > 0$, $g^E(x,t) > 0$ and $g^I(x,t) > 0$ for $(x,t) \in [0, x_{\max}) \times [0,T]$ with $g^S(x_{\max}, t) = 0$, $g^E(x_{\max}, t) = 0$ and $g^I(x_{\max}, t) = 0$ for $t \in [0,T]$.

(H2) The mortality rates $m^S, m^E \in L^\infty(D^S)$ and $m^I \in L^\infty(0,\infty)$ are nonnegative functions.

(H3) The fecundity rates $\beta^S \in L^\infty(0, x_{\max})$ and $\beta^E, \beta^I \in L^\infty((0,x_{\max})\times(0,\infty))$ are all nonnegative functions.

(H4) The latent to infectious rate $\rho^E \in L^\infty(0,\infty)$ is a nonnegative function.

(H5) The infection rate $\lambda \in L^\infty((0,x_{\max}) \times (0,\infty))$ is a nonnegative function.

We now give the precise definition of the solution for the model (2.1).

Definition 2.1. $(S(x,t), E(x,\theta,t), I(x,\theta,t))$ is a solution of (2.1) if all of the following conditions hold:

(i) $S \in L^\infty((0,T); L^1(0,x_{\max}) \cap L^\infty(0,x_{\max}))$ and $E, I \in L^\infty((0,T); L^1(D^{EI}_{x\theta}) \cap L^\infty(D^{EI}_{x\theta}))$.

(ii) $S(x,0) = S^0(x)$ a.e. in $(0,x_{\max})$, $E(x,\theta,0) = E^0(x,\theta)$ and $I(x,\theta,0) = I^0(x,\theta)$ a.e. in $(0,x_{\max}) \times (0,\infty)$.

(iii) For every $t \in (0,T)$, every $\phi^S \in C^1(D^S)$, $\phi^E \in C^1_{0,r}(D^{EI})$ and $\phi^I \in C^1_{0,r}(D^{EI})$, we have

$$\begin{aligned}
&\int_0^{x_{\max}} S(x,t)\phi^S(x,t)dx \\
&\quad = \int_0^{x_{\max}} S(x,0)\phi^S(x,0)dx + \int_0^t \phi^S(0,s)F(S(\cdot,s),E(\cdot,\cdot,s),I(\cdot,\cdot,s))ds \\
&\qquad + \int_0^t\int_0^{x_{\max}} [\phi^S_s(x,s) + g^S(x,s)\phi^S_x(x,s) - m^S(x,s)\phi^S(x,s)]S(x,s)dxds \\
&\qquad - \int_0^t\int_0^{x_{\max}} S(x,s)\phi^S(x,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)I(y,\theta,s)d\theta dy\right]dxds,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\int_0^{x_{\max}}\int_0^\infty E(x,\theta,t)\phi^E(x,\theta,t)d\theta dx \\
&\quad = \int_0^{x_{\max}}\int_0^\infty E(x,\theta,0)\phi^E(x,\theta,0)d\theta dx \\
&\qquad + \int_0^t\int_0^{x_{\max}}\int_0^\infty [\phi^E_s(x,\theta,s) + g^E(x,s)\phi^E_x(x,\theta,s) + \phi^E_\theta(x,\theta,s)]E(x,\theta,s)d\theta dxds \\
&\qquad - \int_0^t\int_0^{x_{\max}}\int_0^\infty [m^E(x,s) + \rho^E(\theta)]E(x,\theta,s)\phi^E(x,\theta,s)d\theta dxds \\
&\qquad + \int_0^t\int_0^{x_{\max}} S(x,s)\phi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)I(y,\theta,s)d\theta dy\right]dxds,
\end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
&\int_0^{x_{\max}}\int_0^{\infty} I(x,\theta,t)\phi^I(x,\theta,t)d\theta dx \\
&= \int_0^{x_{\max}}\int_0^{\infty} I(x,\theta,0)\phi^I(x,\theta,0)d\theta dx + \int_0^t \int_0^{x_{\max}} \phi^I(x,0,s)\int_0^{\infty} \rho^E(\theta)E(x,\theta,s)d\theta dx ds \\
&\quad + \int_0^t \int_0^{x_{\max}}\int_0^{\infty} [\phi^I_s(x,\theta,s) + g^I(x,s)\phi^I_x(x,\theta,s) + \phi^I_\theta(x,\theta,s)]I(x,\theta,s)d\theta\, dx ds \\
&\quad - \int_0^t \int_0^{x_{\max}}\int_0^{\infty} m^I(\theta)I(x,\theta,s)\phi^I(x,\theta,s)d\theta dx ds.
\end{aligned}
\tag{2.4}
$$

3 Comparison Principle and Uniqueness

In this section, we give the definition of upper and lower solution of (2.1) and establish a comparison principle. Then we show that (2.1) has at most one solution.

Definition 3.1. A pair of functions $(\overline{S},\overline{E},\overline{I})$ and $(\underline{S},\underline{E},\underline{I})$ are called an upper solution and a lower solution of (2.1), respectively, if all of the following conditions hold:

(i) $\underline{S},\overline{S} \in L^\infty((0,T);L^1(0,x_{\max})\cap L^\infty(0,x_{\max}))$, $\underline{E},\overline{E} \in L^\infty((0,T);L^1(D^{EI}_{x\theta})\cap L^\infty(D^{EI}_{x\theta}))$, and $\underline{I},\overline{I} \in L^\infty((0,T);L^1(D^{EI}_{x\theta})\cap L^\infty(D^{EI}_{x\theta}))$.

(ii) $\underline{S}(x,0) \le S^0(x)$, $\overline{S}(x,0) \ge S^0(x)$ a.e. in $(0,x_{\max})$, $\underline{E}(x,\theta,0) \le E^0(x,\theta)$, $\overline{E}(x,\theta,0) \ge E^0(x,\theta)$, $\underline{I}(x,\theta,0) \le I^0(x,\theta)$ and $\overline{I}(x,\theta,0) \ge I^0(x,\theta)$ a.e. in $(0,x_{\max})\times(0,\infty)$.

(iii) For every $t \in (0,T)$, every nonnegative function $\psi^S \in C^1(D^S)$, $\psi^E \in C^1_{0,r}(D^{EI})$ and $\psi^I \in C^1_{0,r}(D^{EI})$, we have

$$
\begin{aligned}
&\int_0^{x_{\max}} \overline{S}(x,t)\psi^S(x,t)dx \\
&\ge \int_0^{x_{\max}} \overline{S}(x,0)\psi^S(x,0)dx + \int_0^t \psi^S(0,s)F(\overline{S}(\cdot,s),\overline{E}(\cdot,\cdot,s),\overline{I}(\cdot,\cdot,s))ds \\
&\quad + \int_0^t \int_0^{x_{\max}} [\psi^S_s(x,s) + g^S(x,s)\psi^S_x(x,s) - m^S(x,s)\psi^S(x,s)]\overline{S}(x,s)dx ds \\
&\quad - \int_0^t \int_0^{x_{\max}} \overline{S}(x,s)\psi^S(x,s)\left[\int_0^{x_{\max}}\int_0^{\infty} \lambda(y,\theta)\underline{I}(y,\theta,s)d\theta dy\right] dx ds.
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\int_0^{x_{\max}} \underline{S}(x,t)\psi^S(x,t)dx \\
&\le \int_0^{x_{\max}} \underline{S}(x,0)\psi^S(x,0)dx + \int_0^t \psi^S(0,s)F(\underline{S}(\cdot,s),\underline{E}(\cdot,\cdot,s),\underline{I}(\cdot,\cdot,s))ds \\
&\quad + \int_0^t \int_0^{x_{\max}} [\psi^S_s(x,s) + g^S(x,s)\psi^S_x(x,s) - m^S(x,s)\psi^S(x,s)]\underline{S}(x,s)dx ds \\
&\quad - \int_0^t \int_0^{x_{\max}} \underline{S}(x,s)\psi^S(x,s)\left[\int_0^{x_{\max}}\int_0^{\infty} \lambda(y,\theta)\overline{I}(y,\theta,s)d\theta dy\right] dx ds.
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&\int_0^{x_{\max}}\int_0^{\infty}\overline{E}(x,\theta,t)\psi^E(x,\theta,t)d\theta dx\\
&\quad\geq \int_0^{x_{\max}}\int_0^{\infty}\overline{E}(x,\theta,0)\psi^E(x,\theta,0)d\theta dx\\
&\qquad+\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[\psi_s^E(x,\theta,s)+g^E(x,s)\psi_x^E(x,\theta,s)+\psi_\theta^E(x,\theta,s)]\overline{E}(x,\theta,s)d\theta dxds\\
&\qquad-\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[m^E(x,s)+\rho^E(\theta)]\overline{E}(x,\theta,s)\psi^E(x,\theta,s)d\theta dxds\\
&\qquad+\int_0^t\int_0^{x_{\max}}\overline{S}(x,s)\psi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^{\infty}\lambda(y,\theta)\overline{I}(y,\theta,s)d\theta dy\right]dxds.
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&\int_0^{x_{\max}}\int_0^{\infty}\underline{E}(x,\theta,t)\psi^E(x,\theta,t)d\theta dx\\
&\quad\leq \int_0^{x_{\max}}\int_0^{\infty}\underline{E}(x,\theta,0)\psi^E(x,\theta,0)d\theta dx\\
&\qquad+\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[\psi_s^E(x,\theta,s)+g^E(x,s)\psi_x^E(x,\theta,s)+\psi_\theta^E(x,\theta,s)]\underline{E}(x,\theta,s)d\theta dxds\\
&\qquad-\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[m^E(x,s)+\rho^E(\theta)]\underline{E}(x,\theta,s)\psi^E(x,\theta,s)d\theta dxds\\
&\qquad+\int_0^t\int_0^{x_{\max}}\underline{S}(x,s)\psi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^{\infty}\lambda(y,\theta)\underline{I}(y,\theta,s)d\theta dy\right]dxds.
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&\int_0^{x_{\max}}\int_0^{\infty}\overline{I}(x,\theta,t)\psi^I(x,\theta,t)d\theta dx\\
&\quad\geq \int_0^{x_{\max}}\int_0^{\infty}\overline{I}(x,\theta,0)\psi^I(x,\theta,0)d\theta dx+\int_0^t\int_0^{x_{\max}}\psi^I(x,0,s)\int_0^{\infty}\rho^E(\theta)\overline{E}(x,\theta,s)d\theta dxds\\
&\qquad+\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[\psi_s^I(x,\theta,s)+g^I(x,s)\psi_x^I(x,\theta,s)+\psi_\theta^I(x,\theta,s)]\overline{I}(x,\theta,s)d\theta dxds\\
&\qquad-\int_0^t\int_0^{x_{\max}}\int_0^{\infty}m^I(\theta)\overline{I}(x,\theta,s)\psi^I(x,\theta,s)d\theta dxds.
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\int_0^{x_{\max}}\int_0^{\infty}\underline{I}(x,\theta,t)\psi^I(x,\theta,t)d\theta dx\\
&\quad\leq \int_0^{x_{\max}}\int_0^{\infty}\underline{I}(x,\theta,0)\psi^I(x,\theta,0)d\theta dx+\int_0^t\int_0^{x_{\max}}\psi^I(x,0,s)\int_0^{\infty}\rho^E(\theta)\underline{E}(x,\theta,s)d\theta dxds\\
&\qquad+\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[\psi_s^I(x,\theta,s)+g^I(x,s)\psi_x^I(x,\theta,s)+\psi_\theta^I(x,\theta,s)]\underline{I}(x,\theta,s)d\theta dxds\\
&\qquad-\int_0^t\int_0^{x_{\max}}\int_0^{\infty}m^I(\theta)\underline{I}(x,\theta,s)\psi^I(x,\theta,s)d\theta dxds.
\end{aligned}
\tag{3.6}
$$

Based on the above definition, we can establish the following comparison principle.

Theorem 3.2. *Suppose that all the hypotheses (H1)–(H5) hold. Let* $(\overline{S},\overline{E},\overline{I})$ *and* $(\underline{S},\underline{E},\underline{I})$ *be a nonnegative upper solution and a nonnegative lower solution of* (2.1)*, respectively. Then* $\overline{S}\geq\underline{S}$ *a.e. in* D^S, $\overline{E}\geq\underline{E}$ *and* $\overline{I}\geq\underline{I}$ *a.e. in* D^{EI}.

Proof. Let $u^S = \underline{S} - \overline{S}$, $u^E = \underline{E} - \overline{E}$ and $u^I = \underline{I} - \overline{I}$. We have $u^S(x,0) \le 0$ a.e. in $[0, x_{\max}]$, $u^E(x,\theta,0) \le 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$ and $u^I(x,\theta,0) \le 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$. Choose nonnegative functions $\psi^S \in C^1(D^S)$, $\psi^E \in C^1_{0,r}(D_n^{EI})$ and $\psi^I \in C^1_{0,r}(D_n^{EI})$. By (3.1), (3.2) and $u^S(x,0) \le 0$ a.e. in $[0,x_{\max}]$, we find that for every $t \in (0,T)$

$$\begin{aligned}
&\int_0^{x_{\max}} u^S(x,t)\psi^S(x,t)dx \\
&\quad \le \int_0^t \psi^S(0,s)F(u^S(\cdot,s),u^E(\cdot,\cdot,s),u^I(\cdot,\cdot,s))ds \\
&\qquad + \int_0^t\int_0^{x_{\max}} [\psi^S_s(x,s) + g^S(x,s)\psi^S_x(x,s) - m^S(x,s)\psi^S(x,s)]u^S(x,s)dxds \\
&\qquad - \int_0^t\int_0^{x_{\max}} u^S(x,s)\psi^S(x,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\overline{I}(y,\theta,s)d\theta dy\right]dxds \\
&\qquad + \int_0^t\int_0^{x_{\max}} \overline{S}(x,s)\psi^S(x,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)u^I(y,\theta,s)d\theta dy\right]dxds.
\end{aligned} \tag{3.7}$$

By (3.3), (3.4) and $u^E(x,\theta,0) \le 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$, we obtain that for every $t \in (0,T)$

$$\begin{aligned}
&\int_0^{x_{\max}}\int_0^\infty u^E(x,\theta,t)\psi^E(x,\theta,t)d\theta dx \\
&\quad \le \int_0^t\int_0^{x_{\max}}\int_0^\infty [\psi^E_s(x,\theta,s) + g^E(x,s)\psi^E_x(x,\theta,s) + \psi^E_\theta(x,\theta,s)]u^E(x,\theta,s)d\theta dxds \\
&\qquad - \int_0^t\int_0^{x_{\max}}\int_0^\infty [m^E(x,s) + \rho^E(\theta)]u^E(x,\theta,s)\psi^E(x,\theta,s)d\theta dxds \\
&\qquad + \int_0^t\int_0^{x_{\max}} u^S(x,s)\psi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\underline{I}(y,\theta,s)d\theta dy\right]dxds \\
&\qquad + \int_0^t\int_0^{x_{\max}} \overline{S}(x,s)\psi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)u^I(y,\theta,s)d\theta dy\right]dxds.
\end{aligned} \tag{3.8}$$

By (3.5), (3.6) and $u^I(x,\theta,0) \le 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$, we have that for every $t \in (0,T)$

$$\begin{aligned}
&\int_0^{x_{\max}}\int_0^\infty u^I(x,\theta,t)\psi^I(x,\theta,t)d\theta dx \\
&\quad \le \int_0^t\int_0^{x_{\max}} \psi^I(x,0,s)\int_0^\infty \rho^E(\theta)u^E(x,\theta,s)d\theta dxds \\
&\qquad + \int_0^t\int_0^{x_{\max}}\int_0^\infty [\psi^I_s(x,\theta,s) + g^I(x,s)\psi^I_x(x,\theta,s) + \psi^I_\theta(x,\theta,s)]u^I(x,\theta,s)d\theta dxds \\
&\qquad - \int_0^t\int_0^{x_{\max}}\int_0^\infty m^I(\theta)u^I(x,\theta,s)\psi^I(x,\theta,s)d\theta dxds.
\end{aligned} \tag{3.9}$$

Let $\psi^S(x,t) = \exp(\tau^S t)\varphi^S(x,t)$, where $\varphi^S \in C^1(D^S)$ and the constant τ^S are chosen so that

$\tau^S \geq \|m^S\|_\infty + \|\lambda\|_\infty \sup_{s\in[0,T]} \left\{ \int_0^{x_{\max}} \int_0^\infty \overline{I}(y,\theta,s) d\theta dy \right\}$. Then by (3.7), we have

$$
\begin{aligned}
&\int_0^{x_{\max}} u^S(x,t)\varphi^S(x,t)dx \\
&\quad \leq \int_0^t \varphi^S(0,s) F(u^S(\cdot,s), u^E(\cdot,\cdot,s), u^I(\cdot,\cdot,s))ds \\
&\qquad + \int_0^t \int_0^{x_{\max}} [\varphi^S_s(x,s) + g^S(x,s)\varphi^S_x(x,s)] u^S(x,s) dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} u^S(x,s)\varphi^S(x,s) \left[\tau^S - m^S(x,s) - \int_0^{x_{\max}} \int_0^\infty \lambda(y,\theta)\overline{I}(y,\theta,s)d\theta dy\right] dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} \overline{S}(x,s)\varphi^S(x,s) \left[\int_0^{x_{\max}} \int_0^\infty \lambda(y,\theta) u^I(y,\theta,s) d\theta dy\right] dxds.
\end{aligned}
\tag{3.10}
$$

Let $\psi^E(x,\theta,t) = \exp(\tau^E t)\varphi^E(x,\theta,t)$, where $\varphi^E \in C^1_{0,r}(D^{EI}_n)$ and the constant τ^E are chosen so that $\tau^E \geq \|m^E\|_\infty + \|\rho^E\|_\infty$. Then by (3.8), we have

$$
\begin{aligned}
&\int_0^{x_{\max}} \int_0^\infty u^E(x,\theta,t)\varphi^E(x,\theta,t) d\theta dx \\
&\quad \leq \int_0^t \int_0^{x_{\max}} \int_0^\infty [\varphi^E_s(x,\theta,s) + g^E(x,s)\varphi^E_x(x,\theta,s) + \varphi^E_\theta(x,\theta,s)] u^E(x,\theta,s) d\theta dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} \int_0^\infty [\tau^E - (m^E(x,s) + \rho^E(\theta))] u^E(x,\theta,s)\varphi^E(x,\theta,s) d\theta dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} u^S(x,s)\varphi^E(x,0,s) \left[\int_0^{x_{\max}} \int_0^\infty \lambda(y,\theta)\underline{I}(y,\theta,s)d\theta dy\right] dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} \overline{S}(x,s)\varphi^E(x,0,s) \left[\int_0^{x_{\max}} \int_0^\infty \lambda(y,\theta) u^I(y,\theta,s) d\theta dy\right] dxds.
\end{aligned}
\tag{3.11}
$$

Let $\psi^I(x,\theta,t) = \exp(\tau^I t)\varphi^I(x,\theta,t)$, where $\varphi^I \in C^1_{0,r}(D^{EI}_n)$ and the constant τ^I are chosen so that $\tau^I \geq \|m^I\|_\infty$. Then by (3.9), we have

$$
\begin{aligned}
&\int_0^{x_{\max}} \int_0^\infty u^I(x,\theta,t)\varphi^I(x,\theta,t) d\theta dx \\
&\quad \leq \int_0^t \int_0^{x_{\max}} \varphi^I(x,0,s) \int_0^\infty \rho^E(\theta) u^E(x,\theta,s) d\theta dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} \int_0^\infty [\varphi^I_s(x,\theta,s) + g^I(x,s)\varphi^I_x(x,\theta,s) + \varphi^I_\theta(x,\theta,s)] u^I(x,\theta,s) d\theta dxds \\
&\qquad + \int_0^t \int_0^{x_{\max}} \int_0^\infty [\tau^I - m^I(\theta)] u^I(x,\theta,s)\varphi^I(x,\theta,s) d\theta dxds.
\end{aligned}
\tag{3.12}
$$

We choose a nonnegative function $\varphi^S \in C^1(D^S)$ that satisfies the following problem

$$
\begin{aligned}
&\varphi^S_s(x,s) + g^S(x,s)\varphi^S_x(x,s) = 0, \quad 0 < s < t,\ 0 < x < x_{\max}, \\
&\varphi^S(x_{\max}, s) = 0, \quad 0 < s < t, \\
&\varphi^S(x,t) = \chi^S(x), \quad 0 \leq x \leq x_{\max},
\end{aligned}
\tag{3.13}
$$

where $\chi^S \in C_0^\infty((0,x_{\max}))$ and $0 \le \chi^S \le 1$. We choose a nonnegative function $\varphi^E \in C^1_{0,r}(D_n^{EI})$ that satisfies the following problem

$$\begin{aligned}
&\varphi^E_s(x,\theta,s) + g^E(x,s)\varphi^E_x(x,\theta,s) + \varphi^E_\theta(x,\theta,s) = 0, (x,\theta,s) \in (0,x_{\max}) \times (0,n) \times (0,t)\\
&\varphi^E(x_{\max},\theta,s) = 0, \quad 0 < s < t,\ 0 < \theta < n,\\
&\varphi^E(x,n,s) = 0, \quad 0 < s < t,\ 0 < x < x_{\max},\\
&\varphi^E(x,\theta,t) = \chi^E(x,\theta), \quad 0 \le x \le x_{\max},\ 0 \le \theta \le n,
\end{aligned} \tag{3.14}$$

where $\chi^E \in C_0^\infty((0,x_{\max}) \times (0,n))$ and $0 \le \chi^E \le 1$. We choose a nonnegative function $\varphi^I \in C^1_{0,r}(D_n^{EI})$ that satisfies the following problem

$$\begin{aligned}
&\varphi^I_s(x,\theta,s) + g^I(x,s)\varphi^I_x(x,\theta,s) + \varphi^I_\theta(x,\theta,s) = 0, (x,\theta,s) \in (0,x_{\max}) \times (0,n) \times (0,t),\\
&\varphi^I(x_{\max},\theta,s) = 0, \quad 0 < s < t,\ 0 < \theta < n,\\
&\varphi^I(x,n,s) = 0, \quad 0 < s < t,\ 0 < x < x_{\max},\\
&\varphi^I(x,\theta,t) = \chi^I(x,\theta), \quad 0 \le x \le x_{\max},\ 0 \le \theta \le n,
\end{aligned} \tag{3.15}$$

where $\chi^I \in C_0^\infty((0,x_{\max}) \times (0,n))$ and $0 \le \chi^I \le 1$. The existence of such $\varphi^S(x,s)$, $\varphi^E(x,\theta,s)$, and $\varphi^I(x,\theta,s)$ follows from the fact that all the above problems are linear with local boundary conditions. The boundary and initial conditions of the above problems imply that $0 \le \varphi^S, \varphi^E, \varphi^I \le 1$. Substituting φ^S that satisfies (3.13) into (3.10), we find

$$\begin{aligned}
&\int_0^{x_{\max}} u^S(x,t)\chi^S(x)dx\\
&\quad \le \mu^{S,S}\int_0^t\int_0^{x_{\max}} u^{S+}(x,s)dxds + \mu^{S,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty u^{E+}(x,\theta,s)d\theta dxds\\
&\qquad +\mu^{S,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty u^{I+}(x,\theta,s)d\theta dxds,
\end{aligned} \tag{3.16}$$

where the functions u^{S+}, u^{E+} and u^{I+} denote the positive parts of u^S, u^E and u^I, respectively, and $\mu^{S,S} = \|\beta^S\|_\infty + \tau^S - \inf_{(x,t)\in\overline{D^S}} m^S(x,t) - \inf_{(x,\theta)\in\overline{D^{EI}_{x\theta}}} \lambda(x,\theta) \inf_{s\in[0,T]}\left\{\int_0^{x_{\max}}\int_0^\infty \overline{I}(y,\theta,s)d\theta dy\right\}$, $\mu^{S,E} = \|\beta^E\|_\infty$ and $\mu^{S,I} = \|\beta^I\|_\infty + \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}} \overline{S}(y,s)dy\right\}$, where $\overline{D^S} = [0,x_{\max}] \times [0,T]$ and $\overline{D^{EI}_{x\theta}} = [0,x_{\max}] \times [0,\infty)$. Substituting φ^E that satisfies (3.14) into (3.11), we obtain

$$\begin{aligned}
&\int_0^{x_{\max}}\int_0^n u^E(x,\theta,t)\chi^E(x,\theta)d\theta dx\\
&\quad \le \mu^{E,S}\int_0^t\int_0^{x_{\max}} u^{S+}(x,s)dxds + \mu^{E,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty u^{E+}(x,\theta,s)d\theta dxds\\
&\qquad +\mu^{E,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty u^{I+}(x,\theta,s)d\theta dxds,
\end{aligned} \tag{3.17}$$

where $\mu^{E,S} = \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}}\int_0^\infty \underline{I}(y,\theta,s)d\theta dy\right\}$, $\mu^{E,E} = \tau^E - \inf_{(x,t)\in\overline{D^S}} m^E(x,t) - \inf_{\theta\in[0,\infty)} \rho^E(\theta)$ and $\mu^{E,I} = \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}} \overline{S}(y,s)dy\right\}$. Substituting φ^I that satisfies (3.15) into (3.12), we have

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^{n} u^I(x,\theta,t)\chi^I(x,\theta)d\theta dx\\&\quad\le \mu^{I,E}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{E+}(x,\theta,s)d\theta dxds+\mu^{I,I}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{I+}(x,\theta,s)d\theta dxds,\end{aligned}\tag{3.18}$$

with the constants $\mu^{I,E}=\|\rho^E\|_\infty$ and $\mu^{I,I}=\tau^I-\inf_{\theta\in[0,\infty)} m^I(\theta)$.

Since (3.16) holds for every χ^S, we can choose a sequence $\{\chi^{S,k}\}_{k=1}^\infty$ converging to $\overline{\chi}^S$, where

$$\overline{\chi}^S(x)=\begin{cases}1 & \text{if } u^S(x,t)>0\\ 0 & \text{otherwise}\end{cases}.$$

Substituting $\chi^{S,k}$ into (3.16) and using the dominated convergence theorem, we find

$$\begin{aligned}&\int_0^{x_{\max}} u^{S+}(x,t)dx\\&\quad\le \mu^{S,S}\int_0^t\int_0^{x_{\max}} u^{S+}(x,s)dxds+\mu^{S,E}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{E+}(x,\theta,s)d\theta dxds\\&\qquad+\mu^{S,I}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{I+}(x,\theta,s)d\theta dxds.\end{aligned}\tag{3.19}$$

Similarly we find (3.17) holds for every χ^E. Hence, we can choose a sequence $\{\chi^{E,k}\}_{k=1}^\infty$ converging to $\overline{\chi}^E$, where

$$\overline{\chi}^E(x,\theta)=\begin{cases}1 & \text{if } u^E(x,\theta,t)>0\\ 0 & \text{otherwise}\end{cases}.$$

Substituting such $\chi^{E,k}$ into (3.17), using the dominated convergence theorem, and using the fact that the constants $\mu^{E,S},\mu^{E,E}$ and $\mu^{E,I}$ are independent of n, we find

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^{\infty} u^{E+}(x,\theta,t)d\theta dx\\&\quad\le \mu^{E,S}\int_0^t\int_0^{x_{\max}} u^{S+}(x,s)dxds+\mu^{E,E}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{E+}(x,\theta,s)d\theta dxds\\&\qquad+\mu^{E,I}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{I+}(x,\theta,s)d\theta dxds.\end{aligned}\tag{3.20}$$

Similarly, we can find

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^{\infty} u^{I+}(x,\theta,t)d\theta dx\\&\quad\le \mu^{I,E}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{E+}(x,\theta,s)d\theta dxds+\mu^{I,I}\int_0^t\int_0^{x_{\max}}\int_0^{\infty} u^{I+}(x,\theta,s)d\theta dxds.\end{aligned}\tag{3.21}$$

By the fact $\int_0^\infty e^{-\theta}d\theta=1$ and (3.19)–(3.21), we have

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^{\infty}[e^{-\theta}u^{S+}(x,t)+u^{E+}(x,\theta,t)+u^{I+}(x,\theta,t)]d\theta dx\\&\quad\le \mu\int_0^t\int_0^{x_{\max}}\int_0^{\infty}[e^{-\theta}u^{S+}(x,s)+u^{E+}(x,\theta,s)+u^{I+}(x,\theta,s)]d\theta dxds,\end{aligned}$$

where $\mu = \max\{\mu^{S,S} + \mu^{E,S}, \mu^{S,E} + \mu^{E,E} + \mu^{I,E}, \mu^{S,I} + \mu^{E,I} + \mu^{I,I}\}$. By Gronwall's inequality, we have

$$\int_0^{x_{\max}}\int_0^\infty [e^{-\theta} u^{S+}(x,t) + u^{E+}(x,\theta,t) + u^{I+}(x,\theta,t)]d\theta dx = 0,$$

which implies that $u^S \leq 0$ a.e. in D^S, $u^E \leq 0$ and $u^I \leq 0$ a.e. in D^{EI} . □

Remark 3.3. From the proof of Theorem 3.2, it is easily seen that for any function $w^S \in L^\infty((0,T); L^1(0,x_{\max}) \cap L^\infty(0,x_{\max}))$ with $w^S(x,0) \leq 0$ a.e. in $[0,x_{\max}]$, if the following inequality holds for every $t \in (0,T)$ and every nonnegative function $\Psi^S \in C^1(D^S)$

$$\begin{aligned}\int_0^{x_{\max}} w^S(x,t)\Psi^S(x,t)dx \leq \int_0^t\int_0^{x_{\max}} [\Psi^S_s(x,s) + g(x,s)\Psi^S_x(x,s)]w^S(x,s)dxds \\ + \int_0^t\int_0^{x_{\max}} A^S(x,s)w^S(x,s)\Psi^S(x,s)dxds,\end{aligned} \tag{3.22}$$

with $A^S \in L^\infty(D^S)$, then $w^S(x,t) \leq 0$ a.e. in D^S.

Similarly, for any function $w^E \in L^\infty((0,T); L^1((0,x_{\max}) \times (0,\infty)) \cap L^\infty((0,x_{\max}) \times (0,\infty)))$ with $w^E(x,\theta,0) \leq 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$, if the following inequality holds for every $t \in (0,T)$ and every nonnegative function $\Psi^E \in C^1_{0,r}(D^{EI})$

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^\infty w^E(x,\theta,t)\Psi^E(x,\theta,t)d\theta dx \\ &\leq \int_0^t\int_0^{x_{\max}}\int_0^\infty [\Psi^E_s(x,\theta,s) + g(x,s)\Psi^E_x(x,\theta,s) + \Psi^E_\theta(x,\theta,s)]w^E(x,\theta,s)d\theta dxds \\ &\quad + \int_0^t\int_0^{x_{\max}}\int_0^\infty A^E(x,\theta,s)w^E(x,\theta,s)\Psi^E(x,\theta,s)d\theta dxds,\end{aligned} \tag{3.23}$$

with $A^E \in L^\infty(D^{EI})$, then $w^E(x,\theta,t) \leq 0$ a.e. in D^{EI}.

This remark will be used in Section 4. Next we will show that model (2.1) has at most one nonnegative solution.

Theorem 3.4. *Suppose that the assumptions (H1)–(H5) hold. Then there exists at most one nonnegative solution for the system of equations* (2.1).

Proof. Suppose that there exist two nonnegative solutions (S_1, E_1, I_1) and (S_2, E_2, I_2) for (2.1). Let $v^S = S_1 - S_2$, $v^E = E_1 - E_2$ and $v^I = I_1 - I_2$. We have $v^S(x,0) = 0$ a.e. in $[0,x_{\max}]$, $v^E(x,\theta,0) = 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$ and $v^I(x,\theta,0) = 0$ a.e. in $[0,x_{\max}] \times [0,\infty)$. We choose functions $\phi^S \in C^1(D^S)$, $\phi^E \in C^1_{0,r}(D^{EI}_n)$ and $\phi^I \in C^1_{0,r}(D^{EI}_n)$. By (2.2) and $v^S(x,0) = 0$ a.e. in $[0,x_{\max}]$, we find that for every $t \in (0,T)$

$$\begin{aligned}&\int_0^{x_{\max}} v^S(x,t)\phi^S(x,t)dx \\ &= \int_0^t \phi^S(0,s)F(v^S(\cdot,s), v^E(\cdot,\cdot,s), v^I(\cdot,\cdot,s))ds \\ &\quad + \int_0^t\int_0^{x_{\max}} [\phi^S_s(x,s) + g^S(x,s)\phi^S_x(x,s) - m^S(x,s)\phi^S(x,s)]v^S(x,s)dxds \\ &\quad - \int_0^t\int_0^{x_{\max}} v^S(x,s)\phi^S(x,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)I_1(y,\theta,s)d\theta dy\right]dxds \\ &\quad - \int_0^t\int_0^{x_{\max}} S_2(x,s)\phi^S(x,s)\left[\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)v^I(y,\theta,s)d\theta dy\right]dxds.\end{aligned} \tag{3.24}$$

By (2.3) and $v^E(x,\theta,0)=0$ a.e. in $[0,x_{\max}]\times[0,\infty)$, we have that for every $t\in(0,T)$

$$\begin{aligned}
&\int_0^{x_{\max}}\int_0^\infty v^E(x,\theta,t)\phi^E(x,\theta,t)d\theta dx \\
&\quad=\int_0^t\int_0^{x_{\max}}\int_0^\infty[\phi_s^E(x,\theta,s)+g^E(x,s)\phi_x^E(x,\theta,s)+\phi_\theta^E(x,\theta,s)]v^E(x,\theta,s)d\theta dxds \\
&\qquad-\int_0^t\int_0^{x_{\max}}\int_0^\infty[m^E(x,s)+\rho^E(\theta)]v^E(x,\theta,s)\phi^E(x,\theta,s)d\theta dxds \\
&\qquad+\int_0^t\int_0^{x_{\max}}v^S(x,s)\phi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^\infty\lambda(y,\theta)I_1(y,\theta,s)d\theta dy\right]dxds \\
&\qquad+\int_0^t\int_0^{x_{\max}}S_2(x,s)\phi^E(x,0,s)\left[\int_0^{x_{\max}}\int_0^\infty\lambda(y,\theta)v^I(y,\theta,s)d\theta dy\right]dxds.
\end{aligned}\tag{3.25}$$

By (2.4) and $v^I(x,\theta,0)=0$ a.e. in $[0,x_{\max}]\times[0,\infty)$, we obtain that for every $t\in(0,T)$

$$\begin{aligned}
&\int_0^{x_{\max}}\int_0^\infty v^I(x,\theta,t)\phi^I(x,\theta,t)d\theta dx \\
&\quad=\int_0^t\int_0^{x_{\max}}\phi^I(x,0,s)\int_0^\infty\rho^E(\theta)v^E(x,\theta,s)d\theta dxds \\
&\qquad+\int_0^t\int_0^{x_{\max}}\int_0^\infty[\phi_s^I(x,\theta,s)+g^I(x,s)\phi_x^I(x,\theta,s)+\phi_\theta^I(x,\theta,s)]v^I(x,\theta,s)d\theta dxds \\
&\qquad-\int_0^t\int_0^{x_{\max}}\int_0^\infty m^I(\theta)v^I(x,\theta,s)\phi^I(x,\theta,s)d\theta dxds.
\end{aligned}\tag{3.26}$$

We choose a function $\phi^S\in C^1(D^S)$ that satisfies the problem

$$\begin{aligned}
&\phi_s^S(x,s)+g^S(x,s)\phi_x^S(x,s)=0,\quad 0<s<t,\ 0<x<x_{\max},\\
&\phi^S(x_{\max},s)=0,\quad 0<s<t,\\
&\phi^S(x,t)=\Upsilon^S(x),\quad 0\le x\le x_{\max},
\end{aligned}\tag{3.27}$$

where $\Upsilon^S\in C_0^\infty((0,x_{\max}))$ and $-1\le\Upsilon^S\le 1$. We also choose a function $\phi^E\in C_{0,r}^1(D_n^{EI})$ that satisfies the problem

$$\begin{aligned}
&\phi_s^E(x,\theta,s)+g^E(x,s)\phi_x^E(x,\theta,s)+\phi_\theta^E(x,\theta,s)=0,(x,\theta,s)\in(0,x_{\max})\times(0,n)\times(0,t),\\
&\phi^E(x_{\max},\theta,s)=0,\quad 0<s<t,\ 0<\theta<n,\\
&\phi^E(x,n,s)=0,\quad 0<s<t,\ 0<x<x_{\max},\\
&\phi^E(x,\theta,t)=\Upsilon^E(x,\theta),\quad 0\le x\le x_{\max},\ 0\le\theta\le n,
\end{aligned}\tag{3.28}$$

where $\Upsilon^E\in C_0^\infty((0,x_{\max})\times(0,n))$ and $-1\le\Upsilon^E\le 1$. We further choose a function $\phi^I\in C_{0,r}^1(D_n^{EI})$ that satisfies the problem

$$\begin{aligned}
&\phi_s^I(x,\theta,s)+g^I(x,s)\phi_x^I(x,\theta,s)+\phi_\theta^I(x,\theta,s)=0,(x,\theta,s)\in(0,x_{\max})\times(0,n)\times(0,t),\\
&\phi^I(x_{\max},\theta,s)=0,\quad 0<s<t,\ 0<\theta<n,\\
&\phi^I(x,n,s)=0,\quad 0<s<t,\ 0<x<x_{\max},\\
&\phi^I(x,\theta,t)=\Upsilon^I(x,\theta),\quad 0\le x\le x_{\max},\ 0\le\theta\le n,
\end{aligned}\tag{3.29}$$

where $\Upsilon^I\in C_0^\infty((0,x_{\max})\times(0,n))$ and $-1\le\Upsilon^I\le 1$. The existence of such $\phi^S(x,s)$, $\phi^E(x,\theta,s)$, and $\phi^I(x,\theta,s)$ is guaranteed since all the above problems are linear with local boundary conditions.

The boundary and initial conditions of the above three problems imply that $-1 \leq \phi^S, \phi^E, \phi^I \leq 1$. Substituting ϕ^S that satisfies (3.27) into (3.24), we find

$$\begin{aligned} &\int_0^{x_{\max}} v^S(x,t)\Upsilon^S(x)dx \\ &\quad \leq \nu^{S,S}\int_0^t\int_0^{x_{\max}} |v^S(x,s)|dxds + \nu^{S,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,s)|d\theta dxds \\ &\quad +\nu^{S,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,s)|d\theta dxds, \end{aligned} \tag{3.30}$$

where $\nu^{S,S} = \|\beta^S\|_\infty + \|m^S\|_\infty + \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}}\int_0^\infty I_1(y,\theta,s)d\theta dy\right\}$, $\nu^{S,E} = \|\beta^E\|_\infty$ and $\nu^{S,I} = \|\beta^I\|_\infty + \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}} S_2(y,s)dy\right\}$. Substituting ϕ^E that satisfies (3.28) into (3.25), we have

$$\begin{aligned} &\int_0^{x_{\max}}\int_0^n v^E(x,\theta,t)\Upsilon^E(x,\theta)d\theta dx \\ &\quad \leq \nu^{E,S}\int_0^t\int_0^{x_{\max}} |v^S(x,s)|dxds + \nu^{E,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,s)|d\theta dxds \\ &\quad +\nu^{E,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,s)|d\theta dxds, \end{aligned} \tag{3.31}$$

where the constants $\nu^{E,S} = \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}}\int_0^\infty I_1(y,\theta,s)d\theta dy\right\}$, $\nu^{E,E} = \|m^E\|_\infty + \|\rho^E\|_\infty$ and $\nu^{E,I} = \|\lambda\|_\infty \sup_{s\in[0,T]}\left\{\int_0^{x_{\max}} S_2(y,s)dy\right\}$. Substituting ϕ^I that satisfies (3.29) into (3.26), we obtain

$$\begin{aligned} &\int_0^{x_{\max}}\int_0^n v^I(x,\theta,t)\Upsilon^I(x,\theta)d\theta dx \\ &\quad \leq \nu^{I,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,s)|d\theta dxds + \nu^{I,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,s)|d\theta dxds, \end{aligned} \tag{3.32}$$

where the constants $\nu^{I,E} = \|\rho^E\|_\infty$ and $\nu^{I,I} = \|m^I\|_\infty$.

Since (3.30) holds for every Υ^S, we can for each fixed t choose a sequence $\{\Upsilon^{S,k}\}_{k=1}^\infty$ converging to $\overline{\Upsilon}^S$, where

$$\overline{\Upsilon}^S(x) = \begin{cases} 1 & \text{if } v^S(x,t) > 0 \\ 0 & \text{if } v^S(x,t) = 0 \\ -1 & \text{if } v^S(x,t) < 0 \end{cases}.$$

Substituting $\Upsilon^{S,k}$ into (3.30) and using the dominated convergence theorem, we find

$$\begin{aligned} &\int_0^{x_{\max}} |v^S(x,t)|dx \\ &\quad \leq \nu^{S,S}\int_0^t\int_0^{x_{\max}} |v^S(x,s)|dxds + \nu^{S,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,s)|d\theta dxds \\ &\quad +\nu^{S,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,s)|d\theta dxds. \end{aligned} \tag{3.33}$$

We also note that (3.31) holds for every Υ^E, hence we can for each fixed t choose a sequence $\{\Upsilon^{E,k}\}_{k=1}^{\infty}$ converging to $\overline{\Upsilon}^E$, where

$$\overline{\Upsilon}^E(x,\theta) = \begin{cases} 1 & \text{if } v^E(x,\theta,t) > 0 \\ 0 & \text{if } v^E(x,\theta,t) = 0 \\ -1 & \text{if } v^E(x,\theta,t) < 0 \end{cases}.$$

Substituting such $\Upsilon^{E,k}$ into (3.31), using the dominated convergence theorem, and using the fact that the constants $\nu^{E,S}, \nu^{E,E}$ and $\nu^{E,I}$ are independent of n, we find

$$\begin{aligned} &\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,t)|d\theta dx \\ &\quad \le \nu^{E,S}\int_0^t\int_0^{x_{\max}} |v^S(x,s)|dxds + \nu^{E,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,s)|d\theta dxds \\ &\qquad + \nu^{E,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,s)|d\theta dxds. \end{aligned} \tag{3.34}$$

Similarly, we can find

$$\begin{aligned} &\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,t)|d\theta dx \\ &\quad \le \nu^{I,E}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^E(x,\theta,s)|d\theta dxds + \nu^{I,I}\int_0^t\int_0^{x_{\max}}\int_0^\infty |v^I(x,\theta,s)|d\theta dxds. \end{aligned} \tag{3.35}$$

By the fact $\int_0^\infty e^{-\theta}d\theta = 1$ and (3.33)–(3.35), we have

$$\begin{aligned} &\int_0^{x_{\max}}\int_0^\infty [e^{-\theta}|v^S(x,t)| + |v^E(x,\theta,t)| + |v^I(x,\theta,t)|]d\theta dx \\ &\quad \le \nu\int_0^t\int_0^{x_{\max}}\int_0^\infty [e^{-\theta}|v^S(x,s)| + |v^E(x,\theta,s)| + |v^I(x,\theta,s)|]d\theta dxds, \end{aligned}$$

where $\nu = \max\{\nu^{S,S}+\nu^{E,S}, \nu^{S,E}+\nu^{E,E}+\nu^{I,E}, \nu^{S,I}+\nu^{E,I}+\nu^{I,I}\}$. By Gronwall's inequality, we thus have

$$\int_0^{x_{\max}}\int_0^\infty [e^{-\theta}|v^S(x,t)| + |v^E(x,\theta,t)| + |v^I(x,\theta,t)|]d\theta dx = 0.$$

Hence, we have $v^S = 0$ a.e in D^S, $v^E = 0$ and $v^I = 0$ a.e. in D^{EI}. □

4 Monotone Approximation and Existence

Suppose that $(\overline{S}^0, \overline{E}^0, \overline{I}^0)$ and $(\underline{S}^0, \underline{E}^0, \underline{I}^0)$ are a pair of nonnegative upper and lower solution of (2.1), respectively, then by Theorem 3.2 we have

$$\begin{aligned} &\underline{S}^0(x,t) \le \overline{S}^0(x,t) \text{ a.e. in } D^S, \\ &\underline{E}^0(x,\theta,t) \le \overline{E}^0(x,\theta,t) \text{ a.e. in } D^{EI}, \\ &\underline{I}^0(x,\theta,t) \le \overline{I}^0(x,\theta,t) \text{ a.e. in } D^{EI}. \end{aligned} \tag{4.1}$$

We set up two sequences $\{\underline{S}^k, \underline{E}^k, \underline{I}^k\}_{k=1}^\infty$ and $\{\overline{S}^k, \overline{E}^k, \overline{I}^k\}_{k=1}^\infty$ by the following procedure:

$$\begin{aligned}
&\overline{S}_t^k(x,t) + (g^S(x,t)\overline{S}^k(x,t))_x = -\overline{S}^k(x,t)\left[m^S(x,t) + \int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\underline{I}^{k-1}(y,\theta,t)d\theta dy\right]\\
&g^S(0,t)\overline{S}^k(0,t) = F(\overline{S}^{k-1}(\cdot,t), \overline{E}^{k-1}(\cdot,\cdot,t), \overline{I}^{k-1}(\cdot,\cdot,t))\\
&\overline{S}^k(x,0) = S^0(x),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\underline{S}_t^k(x,t) + (g^S(x,t)\underline{S}^k(x,t))_x = -\underline{S}^k(x,t)\left[m^S(x,t) + \int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\overline{I}^{k-1}(y,\theta,t)d\theta dy\right]\\
&g^S(0,t)\underline{S}^k(0,t) = F(\underline{S}^{k-1}(\cdot,t), \underline{E}^{k-1}(\cdot,\cdot,t), \underline{I}^{k-1}(\cdot,\cdot,t))\\
&\underline{S}^k(x,0) = S^0(x),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\overline{E}_t^k(x,\theta,t) + (g^E(x,t)\overline{E}^k(x,\theta,t))_x + \overline{E}_\theta^k(x,\theta,t) = -[m^E(x,t) + \rho^E(\theta)]\overline{E}^k(x,\theta,t)\\
&\overline{E}^k(x,0,t) = \overline{S}^{k-1}(x,t)\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\overline{I}^{k-1}(y,\theta,t)d\theta dy\\
&\overline{E}^k(0,\theta,t) = 0, \quad \overline{E}^k(x,\theta,0) = E^0(x,\theta),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\underline{E}_t^k(x,\theta,t) + (g^E(x,t)\underline{E}^k(x,\theta,t))_x + \underline{E}_\theta^k(x,\theta,t) = -[m^E(x,t) + \rho^E(\theta)]\underline{E}^k(x,\theta,t)\\
&\underline{E}^k(x,0,t) = \underline{S}^{k-1}(x,t)\int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\underline{I}^{k-1}(y,\theta,t)d\theta dy\\
&\underline{E}^k(0,\theta,t) = 0, \quad \underline{E}^k(x,\theta,0) = E^0(x,\theta),
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&\overline{I}_t^k(x,\theta,t) + (g^I(x,t)\overline{I}^k(x,\theta,t))_x + \overline{I}_\theta^k(x,\theta,t) = -m^I(\theta)\overline{I}^k(x,\theta,t)\\
&\overline{I}^k(x,0,t) = \int_0^\infty \rho^E(\theta)\overline{E}^{k-1}(x,\theta,t)d\theta\\
&\overline{I}^k(0,\theta,t) = 0, \quad \overline{I}^k(x,\theta,0) = I^0(x,\theta),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\underline{I}_t^k(x,\theta,t) + (g^I(x,t)\underline{I}^k(x,\theta,t))_x + \underline{I}_\theta^k(x,\theta,t) = -m^I(\theta)\underline{I}^k(x,\theta,t)\\
&\underline{I}^k(x,0,t) = \int_0^\infty \rho^E(\theta)\underline{E}^{k-1}(x,\theta,t)d\theta\\
&\underline{I}^k(0,\theta,t) = 0, \quad \underline{I}^k(x,\theta,0) = I^0(x,\theta).
\end{aligned} \tag{4.7}$$

The existence of the solutions to the above problems (4.2)–(4.7) follows from standard results, given the fact that they are all linear problems with local boundary conditions for Sinko-Streifer type systems.

We first show that

$$\begin{aligned}
&\underline{S}^0(x,t) \le \underline{S}^1(x,t) \text{ a.e. in } D^S,\\
&\underline{E}^0(x,\theta,t) \le \underline{E}^1(x,\theta,t) \text{ a.e. in } D^{EI},\\
&\underline{I}^0(x,\theta,t) \le \underline{I}^1(x,\theta,t) \text{ a.e. in } D^{EI}.
\end{aligned} \tag{4.8}$$

Let $\underline{u}^S = \underline{S}^0 - \underline{S}^1$, $\underline{u}^E = \underline{E}^0 - \underline{E}^1$ and $\underline{u}^I = \underline{I}^0 - \underline{I}^1$, then by (4.3), and using the fact that $(\underline{S}^0, \underline{E}^0, \underline{I}^0)$ is a lower solution of (2.1), we have

$$\begin{aligned}&\int_0^{x_{\max}} \underline{u}^S(x,t)\psi^S(x,t)dx \\ &\quad \le \int_0^t \int_0^{x_{\max}} [\psi_s^S(x,s) + g^S(x,s)\psi_x^S(x,s) - m^S(x,s)\psi^S(x,s)]\underline{u}^S(x,s)dxds \\ &\qquad - \int_0^t \int_0^{x_{\max}} \underline{u}^S(x,s)\psi^S(x,s) \int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\overline{I}^0(y,\theta,s)d\theta\, dy\, dx\, ds.\end{aligned}$$

Hence, $\underline{u}^S(x,t)$ satisfies (3.22) with $A^S(x,t) = -m^S(x,t) - \int_0^{x_{\max}}\int_0^\infty \lambda(y,\theta)\overline{I}^0(y,\theta,t)d\theta dy$. Thus, $\underline{S}^0(x,t) \le \underline{S}^1(x,t)$. By (4.5), and using the fact that $(\underline{S}^0, \underline{E}^0, \underline{I}^0)$ is a lower solution of (2.1), we have

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^\infty \underline{u}^E(x,\theta,t)\psi^E(x,\theta,t)d\theta dx \\ &\quad \le \int_0^t \int_0^{x_{\max}}\int_0^\infty [\psi_s^E(x,\theta,s) + g^E(x,s)\psi_x^E(x,\theta,s) + \psi_\theta(x,\theta,s)]\underline{u}^E(x,\theta,s)d\theta dxds \\ &\qquad - \int_0^t \int_0^{x_{\max}}\int_0^\infty [m^E(x,s) + \rho^E(\theta)]\underline{u}^E(x,\theta,s)\psi^E(x,\theta,s)d\theta dxds.\end{aligned}$$

Hence, $\underline{u}^E(x,\theta,t)$ satisfies (3.23) with $A^E(x,\theta,t) = -m^E(x,t) - \rho^E(\theta)$. Thus, $\underline{E}^0(x,\theta,t) \le \underline{E}^1(x,\theta,t)$. By (4.7), and using the fact that $(\underline{S}^0, \underline{E}^0, \underline{I}^0)$ is a lower solution of (2.1), we have

$$\begin{aligned}&\int_0^{x_{\max}}\int_0^\infty \underline{u}^I(x,\theta,t)\psi^I(x,\theta,t)d\theta dx \\ &\quad \le \int_0^t \int_0^{x_{\max}}\int_0^\infty [\psi_s^I(x,\theta,s) + g^I(x,s)\psi_x^I(x,\theta,s) + \psi_\theta^I(x,\theta,s)]\underline{u}^I(x,\theta,s)d\theta dxds \\ &\qquad - \int_0^t \int_0^{x_{\max}}\int_0^\infty m^I(\theta)\underline{u}^I(x,\theta,s)\psi^I(x,\theta,s)d\theta dxds.\end{aligned}$$

Hence, $\underline{I}^0(x,\theta,t)$ satisfies (3.23) with $A^E(x,\theta,t) = -m^I(\theta)$. Thus, $\underline{I}^0(x,\theta,t) \le \underline{I}^1(x,\theta,t)$. Similarly, we can show that

$$\begin{aligned}&\overline{S}^0(x,t) \ge \overline{S}^1(x,t) \text{ a.e. in } D^S, \\ &\overline{E}^0(x,\theta,t) \ge \overline{E}^1(x,\theta,t) \text{ a.e. in } D^{EI}, \\ &\overline{I}^0(x,\theta,t) \ge \overline{I}^1(x,\theta,t) \text{ a.e. in } D^{EI}.\end{aligned} \tag{4.9}$$

By (4.2)–(4.9), we can see that $(\overline{S}^1, \overline{E}^1, \overline{I}^1)$ and $(\underline{S}^1, \underline{E}^1, \underline{I}^1)$ are a pair of upper and lower solutions of (2.1), respectively. Hence, by Theorem 3.2 we have

$$\begin{aligned}&\underline{S}^1(x,t) \le \overline{S}^1(x,t) \text{ a.e. in } D^S, \\ &\underline{E}^1(x,\theta,t) \le \overline{E}^1(x,\theta,t) \text{ a.e. in } D^{EI}, \\ &\underline{I}^1(x,\theta,t) \le \overline{I}^1(x,\theta,t) \text{ a.e. in } D^{EI}.\end{aligned} \tag{4.10}$$

We then assume that for some k, $(\overline{S}^k, \overline{E}^k, \overline{I}^k)$ and $(\underline{S}^k, \underline{E}^k, \underline{I}^k)$ are a pair of upper and lower solutions of (2.1), respectively. Proceeding analogously, we can first show that

$$\begin{aligned}
&\underline{S}^k(x,t) \le \underline{S}^{k+1}(x,t) \text{ and } \overline{S}^k(x,t) \ge \overline{S}^{k+1}(x,t) \text{ a.e. in } D^S,\\
&\underline{E}^k(x,\theta,t) \le \underline{E}^{k+1}(x,\theta,t) \text{ and } \overline{E}^k(x,\theta,t) \ge \overline{E}^{k+1}(x,\theta,t) \text{ a.e. in } D^{EI},\\
&\underline{I}^k(x,\theta,t) \le \underline{I}^{k+1}(x,\theta,t) \text{ and } \overline{I}^k(x,\theta,t) \ge \overline{I}^{k+1}(x,\theta,t) \text{ a.e. in } D^{EI},
\end{aligned} \tag{4.11}$$

and then by (4.2)–(4.7) and (4.11), we can claim that $(\overline{S}^{k+1}, \overline{E}^{k+1}, \overline{I}^{k+1})$ and $(\underline{S}^{k+1}, \underline{E}^{k+1}, \underline{I}^{k+1})$ are a pair of upper and lower solution of (2.1), respectively. Thus, we obtain two monotone sequences $\{\underline{S}^k, \underline{E}^k, \underline{I}^k\}$ and $\{\overline{S}^k, \overline{E}^k, \overline{I}^k\}$ which satisfy

$$\begin{aligned}
&\underline{S}^0 \le \underline{S}^1 \le \cdots \le \underline{S}^k \le \cdots \le \overline{S}^k \le \cdots \le \overline{S}^1 \le \overline{S}^0 \text{ a.e. in } D^S,\\
&\underline{E}^0 \le \underline{E}^1 \le \cdots \le \underline{E}^k \le \cdots \le \overline{E}^k \le \cdots \le \overline{E}^1 \le \overline{E}^0 \text{ a.e. in } D^{EI},\\
&\underline{I}^0 \le \underline{I}^1 \le \cdots \le \underline{I}^k \le \cdots \le \overline{I}^k \le \cdots \le \overline{I}^1 \le \overline{I}^0 \text{ a.e. in } D^{EI}.
\end{aligned}$$

From the monotonicity of the sequences $\{\underline{S}^k, \underline{E}^k, \underline{I}^k\}$ and $\{\overline{S}^k, \overline{E}^k, \overline{I}^k\}$ it follows that there exist functions $(\underline{S}, \underline{E}, \underline{I})$ and $(\overline{S}, \overline{E}, \overline{I})$ such that $(\underline{S}^k, \underline{E}^k, \underline{I}^k) \to (\underline{S}, \underline{E}, \underline{I})$ and $(\overline{S}^k, \overline{E}^k, \overline{I}^k) \to (\overline{S}, \overline{E}, \overline{I})$. Clearly, $\underline{S} \le \overline{S}$ a.e. in D^S, $\underline{E} \le \overline{E}$ and $\underline{I} \le \overline{I}$ a.e. in D^{EI}. On the other hand, by the dominated convergence theorem, we know that $(\underline{S}, \underline{E}, \underline{I})$ and $(\overline{S}, \overline{E}, \overline{I})$ are also an upper and lower solutions of (2.1), respectively. Hence, by Theorem 3.2, we have $\underline{S} \ge \overline{S}$ a.e. in D^S, $\underline{E} \ge \overline{E}$ and $\underline{I} \ge \overline{I}$ a.e. in D^{EI} . Thus, $\underline{S} = \overline{S}$ a.e. in D^S, $\underline{E} = \overline{E}$ and $\underline{I} = \overline{I}$ a.e. in D^{EI}. Defining this common limit by (S, E, I), then we see that it satisfies (2.2)–(2.4) for every nonnegative functions $\psi^S \in C^1(D^S)$, $\psi^E \in C^1_{0,r}(D^{EI})$ and $\psi^I \in C^1_{0,r}(D^{EI})$. By the property of mollifiers and the dominated convergence theorem, we can easily show that the limit satisfies (2.2)–(2.4) for every function $\phi^S \in C^1(D^S)$, $\phi^E \in C^1_{0,r}(D^{EI})$ and $\phi^I \in C^1_{0,r}(D^{EI})$. Hence, (S, E, I) is the solution of (2.1).

Based on the above discussion and Theorem 3.4, we have the following existence-uniqueness result.

Theorem 4.1. *Suppose that all the assumptions (H1)–(H5) hold. We assume that* $(\overline{S}^0, \overline{E}^0, \overline{I}^0)$ *and* $(\underline{S}^0, \underline{E}^0, \underline{I}^0)$ *are a pair of nonnegative upper and lower solution of* (2.1)*, respectively. Then, there exist monotone sequences* $\{\underline{S}^k, \underline{E}^k, \underline{I}^k\}$ *and* $\{\overline{S}^k, \overline{E}^k, \overline{I}^k\}$ *which converge to the unique solution* (S, E, I) *of* (2.1)*.*

Remark 4.2. For the initial data $S^0(x) = \eta^S \exp(-\alpha^S x)$, $E^0(x,\theta) = \eta^E \exp(-\alpha^E x)\exp(-\theta)$ and $I^0(x,\theta) = \eta^I \exp(-\alpha^I x)\exp(-\theta)$ with some positive constants α^S, α^E, α^I, η^S, η^E and η^I, we can construct the nonnegative lower and upper solution as follows:

$$\begin{aligned}
&(\underline{S}(x,t), \underline{E}(x,\theta,t), \underline{I}(x,\theta,t)) = (0,0,0),\\
&\overline{S}(x,t) = a^S \exp(b^S t)\exp(-d^S x),\\
&\overline{E}(x,\theta,t) = \frac{1}{1+\theta^2} a^E \exp(b^E t)\exp(-d^E x),\\
&\overline{I}(x,\theta,t) = \frac{1}{1+\theta^2} a^I \exp(b^I t)\exp(-d^I x),
\end{aligned}$$

with positive constants a^S, a^E, a^I, b^S, b^E, b^I, d^S, d^E and d^I which are to be determined. First we choose positive constants b^E and b^I large enough such that

$$\begin{aligned} b^E &\geq 1 + \max_{(x,t)\in D} |g_x^E(x,t)| + \alpha^E \max_{(x,t)\in D} g^E(x,t), \\ b^I &\geq 1 + \max_{(x,t)\in D} |g_x^I(x,t)| + \alpha^I \max_{(x,t)\in D} g^I(x,t), \end{aligned}$$

where $D = [0,1] \times [0,T]$. Fixing these constants b^E and b^I, then we choose b^S sufficiently large so that

$$b^S \geq \max\left\{ \max_{(x,t)\in D} |g_x^S(x,t)| + \alpha^S \max_{(x,t)\in D} g^S(x,t),\ b^E,\ b^I \right\}.$$

The positive constant a^E is chosen so that $a^E \geq \eta^E$, and a^I is chosen so that

$$a^I \geq \max\left\{ \eta^I, \frac{\pi}{2} \|\rho^E\|_\infty \exp(b^E T) a^E \right\}.$$

The positive constant a^S is then chosen to satisfy $a^S \geq \{\eta^S, a^E, a^I\}$. We then fix $x_{\max}$ sufficiently small so that

$$\begin{aligned} x_{\max} \leq \min \Bigg\{ 1,\ & \frac{2\min_{t\in[0,T]} g^S(0,t)}{9\|\beta^S\|_\infty}, \frac{4\min_{t\in[0,T]} g^S(0,t)}{9\pi\|\beta^E\|_\infty}, \\ & \frac{4\min_{t\in[0,T]} g^S(0,t)}{9\pi\|\beta^I\|_\infty}, \frac{4a^E}{3\pi\|\lambda\|_\infty a^S a^I} \exp(-b^S - b^I)T) \Bigg\}, \end{aligned}$$

where the L_∞ norm of β^S is defined on $(0,1)$, the L_∞ norms of β^E, β^I and λ are all defined on $(0,1) \times (0,\infty)$. Note that since $\lim_{d^S \to 0+} \frac{1 - \exp(-d^S x_{\max})}{d^S} = x_{\max}$, we can choose d^S sufficiently small so that

$$d^S \leq \alpha^S \text{ and } \frac{1 - \exp(-d^S x_{\max})}{d^S} \leq \frac{3}{2} x_{\max}.$$

Similarly, d^E is chosen sufficiently small enough so that

$$\frac{1 - \exp(-d^E x_{\max})}{d^E} \leq \frac{3}{2} x_{\max} \text{ and } d^E \leq \min\{\alpha^E, d^S\}.$$

Fixing these constants for d^S and d^E, then we choose d^I such that

$$\frac{1 - \exp(-d^I x_{\max})}{d^I} \leq \frac{3}{2} x_{\max} \text{ and } d^I \leq \min\{\alpha^I, d^E\}.$$

With all these positive constants fixed as above, we can easily show that $(\underline{S}(x,t), \underline{E}(x,\theta,t), \underline{I}(x,\theta,t))$ and $(\bar{S}(x,t), \bar{E}(x,\theta,t), \bar{I}(x,\theta,t))$ are a pair of nonnegative lower and upper solutions of (2.1), respectively.

5 Concluding Remarks

In the above presentation we developed a comparison principle and constructed monotone sequences to establish the existence and uniqueness of solutions for a class of SEI epidemic models in which individuals in the exposed state and infectious state are structured by size and class age (or residency time). A special case of the general class is the biomass/viral infection model of [6]. We believe that this method can be applied to a more generalized version of this model in which mortality rates and birth rates depend on the total population or total biomass of each state; this would yield results under even more reasonable and realistic assumptions. Note that we defined coupled upper and lower solutions for the equation in the susceptible state because of the nonlinear term in this equation. If we wish to apply this method to a more general case, we may need to define totally coupled upper and lower solutions for the equations of all the states. Because of the complexity of our model, there are other issues we have not considered in this paper such as calculating the basic reproduction ratio to determine if the disease persists or dies out. Nor have we considered control of the population environment to yield maximum (for the case of the biomass/vaccine production problem) or minimum (for the case of epidemic prevention among healthy populations) infected biomass. These and other issues are the subject of current efforts.

There are other methods available in the literature that can be used to prove existence and uniqueness of solutions to problems such as (2.1). We have discussed some of these in the introductory section. Weak solutions and semigroup theory have been used in [7, 8] to prove existence and uniqueness for a classical size structured population model by computing the existence of the solution semigroup in an extrapolation space. In [9] a classical size structured population model that includes non-observable characteristics responsible for variations in growth rates for individuals of the same size is investigated. Well-posedness of this model and a nonlinear perturbation of it is proved. Our future endeavors will include investigation of the use of weak formulations and nonlinear semigroup theory to establish existence, uniqueness and continuous dependence of solutions to problems such as (2.1). A sensitivity analysis framework is also under development.

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