WELL-POSEDNESS OF INVERSE PROBLEMS FOR SYSTEMS WITH TIME DEPENDENT PARAMETERS

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Abstract. In this paper we investigate the abstract hyperbolic model with time dependent stiffness and damping given by

$$\langle \ddot{u}(t), \psi \rangle_{V^*,V} + d(t; \dot{u}(t), \psi) + a(t; u(t), \psi) = \langle f(t), \psi \rangle_{V^*,V}$$

where $V \subset V_D \subset H \subset V_D^* \subset V^*$ are Hilbert spaces with continuous and dense injections, where H is identified with its dual and $\langle \cdot, \cdot \rangle$ denotes the associated duality product. We show under reasonable assumptions on the time-dependent sesquilinear forms $a(t; \cdot, \cdot) : V \times V \to C$ and $d(t; \cdot, \cdot) : V_D \times V_D \to C$ that this model allows a unique solution and that the solution depends continuously on the data of the problem. We also consider well-posedness as well as finite element type approximations in associated inverse problems. The problem above is a weak formulation that includes models in abstract differential operator form that include plate, beam and shell equations with several important kinds of damping.

1 Introduction

In this paper we consider a general abstract operator formulation for damped hyperbolic partial differential equations with time dependent parameters. Our motivation is to extend the general theory for inverse problems for such abstract systems as developed in [5] to systems with time dependent damping and stiffness operators $A(t)$ and $D(t)$, respectively. Applications for such systems are abundant and range from systems with periodic or structured pattern time dependence that occurs for example in thermally dependent systems orbiting in space (periodic exposure to sunlight) to earth bound structures (bridges, buildings) and aircraft/space structure components with extreme temperature exposures (winter vs. summer). On a longer time scale, such systems are important in time dependent health of elastic structures where long term (slowly varying) time dependence of parameters may be used in detecting aging/fatiguing as represented by changes in stiffness and/or damping. Moreover, applications may be found in modern "smart material structures" [5, 18] where one "controls" damping and/or elasticity via piezoceramic patches, electrically active polymers, etc.

Let V and H be complex Hilbert spaces forming a "Gelfand triple" $V \subset H = H^* \subset V^*$, where we write V^* for the dual of V, with duality product $\langle \cdot, \cdot \rangle_{V^*,V}$. The injections are assumed to be dense and continuous and the spaces are assumed to be separable. Moreover, we assume that there exists a Hilbert space V_D (the damping space), such that $V \subset V_D \subset H = H^* \subset V_D^* \subset V^*$, allowing for a wide class of damping models. Thus the duality products $\langle \cdot, \cdot \rangle_{V^*,V}$ and $\langle \cdot, \cdot \rangle_{V_D^*,V_D}$ are the natural extensions by continuity of the inner product $(\cdot, \cdot)$ in H to $V^* \times V$ and $V_D^* \times V_D$, respectively. The H-norm is denoted by $|\cdot|$ or $|\cdot|_H$ for clarity if needed and the V and V_D-norms are denoted $|\cdot|_V$ and $|\cdot|_{V_D}$, respectively. We denote $\partial_t u$ by $\dot{u}$, where $u = u(t)$ is considered as a function of time taking values in one of the occurring Hilbert spaces, that is, $u(t,x) = u(t)(x)$ where $u(t) \in H$, for example.

Our goal is to investigate the problem

$$\ddot{u}(t) + D(t)\dot{u}(t) + A(t)u(t) = f(t), \quad \text{in } V^*, \quad t \in (0,T); \tag{1}$$

with $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$ which is equivalent to

$$\langle \ddot{u}(t), \psi \rangle_{V^*,V} + \langle D(t)\dot{u}(t), \psi \rangle_{V_D^*,V_D} + \langle A(t)u(t), \psi \rangle_{V^*,V} = \langle f(t), \psi \rangle_{V^*,V}, \tag{2}$$

for all $\psi \in V$ and $t \in (0,T)$, with $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$.

We will call a function $u \in L^2(0,T;V)$, with $\dot{u} \in L^2(0,T;H)$ and $\ddot{u} \in L^2(0,T;V^*)$ a *weak solution* of the initial-value problem (1) if it solves the equation (2) or, equivalently, solves the equation (11) below, with $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$ given.

We will assume that the operators A and D arise (as defined precisely in (8)-(9) below) from time-dependent sesquilinear forms a and d satisfying the following natural ellipticity, coercivity and differentiability conditions.

First we assume hermitian symmetry, that is

(H1) $a(t,\phi,\psi) = \overline{a(t,\psi,\phi)}$ for all $\phi,\psi \in V$.

(H2) $|a(t;\phi,\psi)| \leq c_1|\phi|_V|\psi|_V, \quad \phi,\psi \in V$ where c_1 is independent of t.

We assume, further, that

(H3) $a(t;\phi,\psi)$ for $\phi,\psi \in V$ fixed is *continuously differentiable* with respect to t for $t \in [0,T]$ (T finite) and

$$|\dot{a}(t;\phi,\psi)| \leq c_2|\phi|_V|\psi|_V, \quad \forall t \in [0,T], \tag{3}$$

c_2 once again independent of t.

We also assume that the sesquilinear form $a(t;\phi,\psi)$ is V-*elliptic*, so that

(H4) There exist a constant $\alpha > 0$ such that

$$|a(t;\phi,\phi)| \geq \alpha|\phi|_V^2 \text{ for all } t \in [0,T], \text{ and for all } u \in V. \tag{4}$$

For the sesquilinear form d we assume similarly

(H5) $|d(t;\phi,\psi)| \leq c_3|\phi|_{V_D}|\psi|_{V_D}, \quad \phi,\psi \in V_D$ where c_3 is independent of t.

We assume, further, that

(H6) $d(t;\phi,\psi)$ for $\phi,\psi \in V_D$ fixed is *continuously differentiable* with respect to t for $t \in [0,T]$ (T finite) and

$$|\dot{d}(t;\phi,\psi)| \leq c_4|\phi|_{V_D}|\psi|_{V_D}, \quad \forall t \in [0,T], \tag{5}$$

c_4 once again independent of t.

Then $t \to d(t;\phi,\psi)$ and $t \to a(t;\phi,\psi)$ are $C^1[0,T]$ for all $\phi,\psi \in V_D$, and $\phi,\psi \in V$, respectively, which implies that $d(t;\phi,\psi)$ and $a(t;\phi,\psi)$ are sufficiently well-behaved in order to have existence for (1) or (2). We also assume that the sesquilinear form $d(t;\phi,\psi)$ is V_D-*coercive*. That is,

(H7) There exist constants λ_d and $\alpha_d > 0$, such that

$$\operatorname{Re} d(t;\phi,\phi) + \lambda_d|\phi|^2 \geq \alpha_d|\phi|_{V_D}^2 \text{ for all } t \in [0,T], \text{ and for all } \phi \in V_D. \tag{6}$$

We know then from [11, 12] that there exist representation operators $A(t)$ and $D(t)$

$$A(t) : V \to V^*, \quad D(t) : V_D \to V_D^*, \tag{7}$$

which for each fixed t are continuous and linear, with

$$a(t; \phi, \psi) = \langle A(t)\phi, \psi \rangle_{V^*,V}, \quad \text{for all} \quad \phi, \psi \in V, \tag{8}$$

and

$$d(t; \phi, \psi) = \langle D(t)\phi, \psi \rangle_{V_D^*,V_D}, \quad \text{for all} \quad \phi, \psi \in V_D. \tag{9}$$

We will now consider the following problem: Given finite T and $f \in L^2(0,T;V_D^*)$ along with initial conditions

$$u^0 \in V, \quad u^1 \in H,$$

we wish to find a function $u \in L^2(0,T;V), \dot{u} \in L^2(0,T;V_D)$ such that in V^* we have

$$\begin{cases} \ddot{u}(t) + D(t)\dot{u}(t) + A(t)u(t) = f(t), \quad t \in (0,T), \\ u(0) = u^0, \ \dot{u}(0) = u^1. \end{cases} \tag{10}$$

That is, for $f \in L^2(0,T;V_D^*)$

$$\langle \ddot{u}(t), \psi \rangle_{V^*,V} + d(t; \dot{u}(t), \psi) + a(t; u(t), \psi) = \langle f(t), \psi \rangle_{V^*,V} \quad \text{for all } \psi \in V. \tag{11}$$

(Observe that (11) makes sense since $f(t) \in V_D^* \subset V^*$.)

This formulation covers linear beam, plate and shell models with numerous damping models (Kelvin-Voigt, viscous, square-root, structural and spatial hysteresis) frequently studied in the literature. The formulation above is non-standard in the sense that the damping sesquilinear form is incorporated in the variational model *and* is time-dependent. The problem without damping ($d = 0$) and $f \in L^2(0,T;H)$ was already treated by Lions in [11], and subsequently in [20]. The less general case without damping and $V = H_0^1(\Omega)$ is treated for example in [8]. The model above, but with $d : V_D \to C$ independent of time appears in the literature for the first time (to our knowledge) in [5]. The following theorem is the time-dependent extension of the previous results.

Theorem 1 *Assume that $(f, u^0, u^1) \in L^2(0,T;V_D^*) \times V \times H$. Then there exists a unique solution u to (11) with $(u, \dot{u}) \in L^2(0,T;V) \times L^2(0,T;V_D)$, and the mapping*

$$(f, u^0, u^1) \to (u, \dot{u}), \tag{12}$$

is continuous and linear on

$$L^2(0,T;V_D^*) \times V \times H \to L^2(0,T;V) \times L^2(0,T;V_D). \tag{13}$$

As we will see, Theorem 1 can then be to extended to the following:

Theorem 2 *Assume that* $(f, u^0, u^1) \in L^2(0,T;V_D^*) \times V \times H$. *Then there exists (perhaps after modifications on a set of measure zero) a unique solution* u *to (11) with* $(u, \dot{u}) \in C(0,T;V) \times (C(0,T;H) \cap L^2(0,T;V_D))$, *and the mapping*

$$(f, u^0, u^1) \to (u, \dot{u}), \tag{14}$$

is continuous and linear on

$$L^2(0,T;V_D^*) \times V \times H \to C(0,T;V) \times (C(0,T;H) \cap L^2(0,T;V_D)). \tag{15}$$

Remark 1 *If we only have that the inequality (6) for the damping sesquilinear form* d *is satisfied with* $\alpha_d = 0$, *the results are still true with modifications. Then it will be necessary that* $f \in L^2(0,T;H)$ *and one obtains only that* $\dot{u} \in L^2(0,T;H)$; *that is, we have the same results as if there was no damping. (See e.g., [10].)*

2 Discussion of the Model

It is well known from the literature that the strong form of the operator formulation (1) of the problem in general causes computational problems due to irregularities stemming from non-smooth terms - typically in the force/moment terms in for example elasticity problems. The weak formulation has proven advantageous both for theoretical and practical purposes, specifically in the effort to estimate parameters or for control purposes [5]. To give a particular example illustrating and motivating our discussions here, we consider the basic structural models for a fixed ends Euler-Bernoulli beam of length l, width b, thickness h and linear density ρ, where the parameters b, h, ρ may be functions that depend on time and/or spatial position x along the beam. The Euler-Bernoulli equation for transverse displacements $w = w(t,x)$ (in strong form [5]) is given by

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left\{ \widetilde{C_D I} \frac{\partial^3 w}{\partial x^2 \partial t} + \widetilde{EI} \frac{\partial^2 w}{\partial x^2} \right\} = f \quad 0 < x < l, \tag{16}$$

with fixed end boundary conditions

$$w(t,0) = \frac{\partial w}{\partial x}(t,0) = w(t,l) = \frac{\partial w}{\partial x}(t,l), \tag{17}$$

where we have assumed Kevin-Voigt structural damping with damping coefficient $\widetilde{C_D I} = \widetilde{C_D I}(t,x)$ and the possibly time and spatially dependent stiffness coefficient given by $\widetilde{EI} = \widetilde{EI}(t,x)$. For simplicity we assume ρ is constant and scale the system by taking $\rho = 1$. One can readily compute that $\widetilde{EI} = Eh^3b/12, \widetilde{C_D I} = C_D h^3 b/12$, where the Young's modulus E, the damping coefficient C_D and the geometric parameters h, b may in general all be time and/or spatially dependent. In weak form this can be written

$$\langle \ddot{w}(t), \psi \rangle_{V^*,V} + \langle \widetilde{C_D I} \frac{\partial \dot{w}^2(t)}{\partial x^2} + \widetilde{EI} \frac{\partial^2 w(t)}{\partial x^2}, \frac{\partial^2 \psi}{\partial x^2} \rangle_H = \langle f(t), \psi \rangle_{V^*,V} \quad \text{for all } \psi \in V, \tag{18}$$

where $H = L^2(\Omega) \equiv L^2(0, l)$ and $V = V_D = H_0^2(\Omega) \equiv H_0^2(0, l)$ with

$$H_0^2(0, l) \equiv \{\psi \in H^2(0, l) | \psi(0) = \psi'(0) = \psi(l) = \psi'(l) = 0\}. \tag{19}$$

Here we have adopted the usual notation $\psi' = \frac{\partial \psi}{\partial x}$.

This has the form (2) or (11) with

$$a(t; \phi, \psi) = \langle A(t)\phi, \psi \rangle_{V^*,V} = \int_0^l \widetilde{EI}(t, x)\phi''(x)\psi''(x)dx \tag{20}$$

$$d(t; \phi, \psi) = \langle D(t)\phi, \psi \rangle_{V^*,V} = \int_0^l \widetilde{C_D I}(t, x)\phi''(x)\psi''(x)dx. \tag{21}$$

Models such as this can be generalized to higher dimensions (in R^n for $n = 2, 3$) to treat more general beams, plates, shells, and solid bodies [5, 18]. Of great interest are a number of useful damping models that can be readily used with these equations and treated using the abstract formulation developed here. We discuss briefly some of these damping models without going into much detail, as the purpose of this paper is to establish well-posedness and approximation properties of the abstract model. First, we note that in general, as shown in [13, 14, 15, 16], the damping term in the abstract PDE-model can also arise from a damping term on the boundary of the spatial domain. But is this case the theory here must be modified because the sesquilinear form d usually does not satisfy the V_D coercive condition of $(H6)$ if $V_D \subset H$. We turn briefly to several damping models of interest in practice.

Time-dependent Kelvin-Voigt damping: Let $\Omega \subset R^n$ be a bounded domain with density $\rho(x)$ and let $\omega \subset \Omega$, with 1_ω denoting the characteristic function of ω. Let $\gamma, \delta > 0$ denote material parameters and let $t \to k(t)$ denote a sufficiently smooth function. Then a $time-dependent$ damping sesquilinear form is given by

$$d(t; \phi, \psi) = \int_\Omega (\gamma + \delta k(t) 1_\omega(x)) \Delta\phi(x) \Delta\psi(x) \rho(x) dx, \tag{22}$$

for $\phi, \psi \in V_D = V$, where V can be taken for example as an appropriate subspace of $H^2(\Omega)$ which incorporates essential boundary conditions. This gives a model for a mechanical structure damped by a time-varying actuator, localized somewhere inside the structure. This could be piezoceramic actuators or other "smart" devices, with the possibility of them varying in time.

Time-dependent viscous damping:

This is a velocity-proportional damping, given (with the notation from above) by the sesquilinear form

$$d(t; \phi, \psi) = \int_\Omega k(t,x)u(t,x)v(t,x)\rho(x)dx, \tag{23}$$

with $k \in C^1(0,T; L^\infty(\Omega))$ denoting the damping coefficient. One can take $V_D = H$ here.

Time-dependent spatial hysteresis damping:

This model, without time-dependence, is discussed by Russell in [17], and, as mentioned in [5], it has been shown to be appropriate for models where graphite fibers are embedded in an epoxy matrix. The time-dependent sesquilinear form that we consider here can now be constructed with the following compact operator $K(t)$ on $L^2(\Omega)$:

$$(K(t)\varphi)(x) = \int_\Omega k(t,x,y)\varphi(y)dy, \tag{24}$$

where the nonnegative integral kernel k belongs to $C^1(0,T; L^\infty(\Omega) \times L^\infty(\Omega))$. Letting

$$\nu(x) = \int_\Omega \kappa(x,y)dy \tag{25}$$

denote some material property, we can define

$$d(t; \phi, \psi) = \int_\Omega ((\nu(x) - K(t))\nabla_x u(t,x)) \cdot (\nabla_x v(t,x))\mu(x)dx, \tag{26}$$

taking $V_D = H^1(\Omega)$.

Time-dependent "square-root" damping:

This model (without the time-dependence) has been used frequently in the literature as a device to obtain exponentially decaying solutions of "wave-like" partial differential equations, the model we study in this paper is actually a generalized, weak formulation of such an equation. So taking $V_D = D(A^{\frac{1}{2}})$, we could define the damping sesquilinear form by

$$d(t; \phi, \psi) = \int_\Omega c(t,x)(A^{\frac{1}{2}}\phi)(x)(A^{\frac{1}{2}}\psi)(x)\rho(x)dx \tag{27}$$

for some nonnegative function $c \in C^1(0,T;L^\infty(\Omega))$.

3 Proof of Theorems 1 and 2

We will follow a Galerkin approximation method (see for example [11, 20, 5]) with necessary, non-trivial modifications due to the presence of the time-dependent form(s). So, let $\{w_j\}_1^\infty$ denote an orthonormal basis in H that is also an orthogonal basis in V. This is possible since V is dense in H. For a fixed m we denote by V_m the finite dimensional subspace spanned by $\{w_j\}_1^m$, and we let u_m^0 and u_m^1 be chosen in V_m such that

$$u_m^0 \to u^0 \quad \text{in } V, \quad u_m^1 \to u^1 \quad \text{in } H, \text{ for } m \to \infty. \tag{28}$$

We now define the approximate solution $u_m(t)$ of order m of our problem in the following way:

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j, \tag{29}$$

where the $g_{jm}(t)$ are determined uniquely from the m-dimensional linear system:

$$(\ddot{u}_m(t), w_j) + d(t; \dot{u}_m(t), w_j) + a(t; u_m(t), w_j) = \langle f(t), w_j \rangle_{V^*,V}, \quad j = 1, 2, ..., m; \tag{30}$$

with $u_m(0) = u_m^0$ and $\dot{u}_m(0) = u_m^1$. Multiplying (30) with $\dot{\bar{g}}_{jm}(t)$ and summing over j yeilds

$$(\ddot{u}_m(t), \dot{u}_m(t)) + d(t; \dot{u}_m(t), \dot{u}_m(t)) + a(t; u_m(t), \dot{u}_m(t)) = \langle f(t), \dot{u}_m(t) \rangle_{V^*,V}. \tag{31}$$

Now, since

$$\frac{d}{dt} a(t; u_m(t), u_m(t)) = 2 \operatorname{Re} a(t; u_m(t), \dot{u}_m(t)) + \dot{a}(t; u_m(t), u_m(t)), \tag{32}$$

we see that

$$\frac{d}{dt}\{|\dot{u}_m(t)|^2 + a(t; u_m(t), u_m(t))\} + 2 \operatorname{Re} d(t; \dot{u}_m(t), \dot{u}_m(t)) = \dot{a}(t; u_m(t), u_m(t)) + 2 \operatorname{Re} \langle f(t), \dot{u}_m(t) \rangle_{V^*,V}$$

and by integrating this equality we find

$$|\dot{u}_m(t)|^2 + a(t; u_m(t), u_m(t)) + \int_0^t 2 \operatorname{Re} d(t; \dot{u}_m(s), \dot{u}_m(s)) ds = |\dot{u}_m(0)|^2 + a(0; u_m^0, u_m^0) + \int_0^t \dot{a}(s; u_m(s), u_m(s)) ds + \int_0^t 2 \operatorname{Re} \langle f(s), \dot{u}_m(s) \rangle_{V^*,V} ds.$$

Using the coercivity conditions for a and d, together with the inequality (recall that $f(s) \in V_D^*$)

$$|\langle f(s), \dot{u}_m(s)\rangle_{V^*,V}| \leq \frac{1}{4\epsilon}|f(s)|^2_{V_D^*} + \epsilon|\dot{u}_m(s)|^2_{V_D} \tag{33}$$

we obtain, for all $\epsilon > 0$

$$|\dot{u}_m(t)|^2 + \alpha|u_m(t)|^2_V + \int_0^t 2(\alpha_d - \epsilon)|\dot{u}_m(s)|^2_{V_D} ds \leq |u_m^1|^2 + c_1|u_m^0|^2_V + \\ c_2\int_0^t |u_m(s)|^2_V ds + 2\lambda_d \int_0^t |\dot{u}_m(s)|^2 ds + \int_0^t \frac{1}{2\epsilon}|f(s)|^2_{V_D^*} ds. \tag{34}$$

Since $u_m^0 \to u^0$ in V, $u_m^1 \to u^1$ in H and $f \in L^2(0,T;V_D^*)$, we have that, for $\epsilon > 0$ fixed and m large, there exist a constant $C > 0$, such that

$$|u_m^1|^2 + c_1|u_m^0|^2_V + \int_0^t \frac{1}{2\epsilon}|f(s)|^2_{V_D^*} ds \leq C, \tag{35}$$

hence

$$|\dot{u}_m(t)|^2 + \alpha|u_m(t)|^2_V + \int_0^t 2(\alpha_d - \epsilon)|\dot{u}_m(s)|^2_{V_D} ds \leq \\ C + c_2\int_0^t |u_m(s)|^2_V ds + 2\lambda_d \int_0^t |\dot{u}_m(s)|^2 ds. \tag{36}$$

Then, in particular

$$|\dot{u}_m(t)|^2 + \alpha|u_m(t)|^2_V \leq \\ C + c_2\int_0^t |u_m(s)|^2_V ds + 2\lambda_d \int_0^t |\dot{u}_m(s)|^2 ds. \tag{37}$$

By Gronwall's inequality we then see that the sequence $\{\dot{u}_m\}$ is bounded in $C(0,T;H)$ and that the sequence $\{u_m\}$ is bounded in $C(0,T;V)$. From this fact together with the inequality (36) we conclude that $\{\dot{u}_m\}$ is also bounded in $L^2(0,T;V_D)$. Then it is possible to extract a subsequence $\{u_{m_k}\} \subset \{u_m\}$ and functions $u \in L^2(0,T;V)$ and $\tilde{u} \in L^2(0,T;V_D)$, such that $u_{m_k} \rightharpoonup u$, weakly in $L^2(0,T;V)$ and $\dot{u}_{m_k} \rightharpoonup \tilde{u}$, weakly in $L^2(0,T;V_D)$. But for $0 \leq t < T$ we have in V, hence in V_D and H, that

$$u_{m_k}(t) = u_{m_k}(0) + \int_0^t \dot{u}_{m_k}(s) ds. \tag{38}$$

But $u_{m_k}(0) \to u^0$ in V and hence in V_D, while, for t fixed, $\int_0^t \dot{u}_{m_k}(s)ds \rightharpoonup \int_0^t \tilde{u}(s)ds$, weakly in V_D. So, by taking the weak limit in V_D in (38), we obtain in V_D the equality

$$u(t) = u^0 + \int_0^t \tilde{u}(s)ds, \tag{39}$$

from which we conclude that $\dot{u}(t)$ is in V_D *a.e.*, with $\dot{u} = \tilde{u}$ and $u(0) = u^0$.

We need now to show that u is actually a solution to the problem (11), with $\dot{u}(0) = u^1$. To see this, take a function $\varphi \in C^1([0,T])$, satisfying $\varphi(T) = 0$, and define, for $j < m$, the function φ_j by $\varphi_j(t) = \varphi(t)w_j$, where $\{w_j\}_1^m$ was the basis spanning V_m. Now, for a fixed $j < m$, we multiply (30) with $\overline{\varphi}(t)$ and integrate to obtain

$$\int_0^T ((\ddot{u}_m(s), \varphi_j(s)) + d(s; \dot{u}_m(s), \varphi_j(s)) + a(s; u_m(s), \varphi_j(s)))ds = \int_0^T \langle f(s), \varphi_j(s)\rangle_{V_D^*, V_D} ds.$$

Noticing that, for each t, we have that $d(t; \cdot, \varphi_j(t)) \in V_D^*$ and $a(t; \cdot, \varphi_j(t)) \in V^*$, we find, using the weak convergence above that for $m = m_k \to \infty$ and integration by parts in the first term, that

$$\int_0^T (-(\dot{u}(s), \dot{\varphi}_j(s)) + d(s; \dot{u}(s), \varphi_j(s)) + a(s; u(s), \varphi_j(s)))ds = \int_0^T \langle f(s), \varphi_j(s)\rangle_{V_D^*, V_D} ds + (u^1, \varphi_j(0)), \tag{40}$$

for every j. Now further restrict φ to also satisfy $\varphi \in C_0^\infty(0,T)$ and write (40) as

$$\int_0^T \dot{\overline{\varphi}}(s)(-\dot{u}(s), w_j) + \int_0^T \overline{\varphi}(s)(d(s; \dot{u}(s), w_j) + a(s; u(s), w_j) - \langle f(s), w_j\rangle_{V_D^*, V_D})ds = 0, \tag{41}$$

for each j. But by (41), we have

$$\frac{d}{dt}(\dot{u}(t), w_j) + d(t; \dot{u}(t), w_j) + a(t; u(t), w_j) = \langle f(t), w_j\rangle_{V_D^*, V_D} \tag{42}$$

for all w_j. By density of V_m in V we conclude that $\ddot{u} \in L^2(0,T; V^*)$ and that for all $\psi \in V$

$$(\ddot{u}(t), \psi) + d(t; \dot{u}(t), \psi) + a(t; u(t), \psi) = \langle f(t), \psi\rangle_{V_D^*, V_D}, \tag{43}$$

which was (11). Hence the u we have constructed is indeed a solution to the equation and by (39) we have that $u(0) = u^0$. In order to verify that $\dot{u}(0) = u^1$ we integrate by parts in (40), and by application of (42) we find that, for all j:

$$-(\dot{u}(s), \varphi_j(s))|_{s=0}^{s=T} = (u^1, \varphi_j(0)), \tag{44}$$

or, equivalently

$$(\dot{u}(0), w_j)\overline{\varphi}(0) = (u^1, w_j)\overline{\varphi}(0). \tag{45}$$

Hence $\dot{u}(0) = u^1$.

In order to prove uniqueness, let u be a solution of our problem (11) corresponding to $(u^0, u^1, f) = (0, 0, 0)$, and define for a fixed $t_1 \in (0, T)$ (arbitrarily chosen) the function ψ by

$$\psi(t) = \begin{cases} -\int_t^{t_1} u(s)ds & \text{for } t < t_1, \\ 0 & \text{for } t \geq t_1, \end{cases} \tag{46}$$

so $\psi(T) = 0$. Obviously $\psi(t) \in V$ for all t, so we can take $\psi(t) = \psi$ in (11) which yeilds

$$\langle \ddot{u}(t), \psi(t) \rangle_{V^*,V} + d(t; \dot{u}(t), \psi(t)) + a(t; u(t), \psi(t)) = \langle f(t), \psi(t) \rangle_{V^*,V}. \tag{47}$$

Because $\dot{\psi}(t) = u(t)$ for $t < t_1$ (a.e), we have that

$$\int_0^{t_1} (\langle \ddot{u}(t), \psi(t) \rangle_{V^*,V} + \langle \dot{u}(t), u(t) \rangle_{V^*,V})dt = \int_0^{t_1} \frac{d}{dt}(\langle \dot{u}(t), \psi(t) \rangle_{V^*,V})dt = 0, \tag{48}$$

due to $\psi(t_1) = 0$ and the initial conditions. Using this and by integration of (47) we find

$$\int_0^{t_1} (\langle \dot{u}(t), u(t) \rangle_{V^*,V} - d(t; \dot{u}(t), \psi(t)) - a(t; u(t), \psi(t)))dt = 0; \tag{49}$$

hence

$$\int_0^{t_1} \frac{d}{dt}(|u(t)|^2 - a(t; \psi(t), \psi(t)))dt = 2\int_0^{t_1} (\dot{a}(t; \psi(t), \psi(t)) + \operatorname{Re} d(t; \dot{u}(t), \psi(t)))dt. \tag{50}$$

Because $\psi(t_1) = 0$ and $u(0) = u^0 = 0$ this yields

$$|u(t_1)|^2 + a(0; \psi(0), \psi(0)) = 2\int_0^{t_1} (\dot{a}(t; \psi(t), \psi(t)) + \operatorname{Re} d(t; \dot{u}(t), \psi(t)))dt. \tag{51}$$

From the assumptions on a and $\dot{a}$ we arrive at

$$|u(t_1)|^2 + \alpha|\psi(0)|_V^2 \leq 2\int_0^{t_1} (c_2|\psi(t)|_V^2 + \operatorname{Re} d(t;\dot{u}(t),\psi(t)))dt. \tag{52}$$

Now notice that

$$d(t;\dot{u}(t),\psi(t)) = \frac{d}{dt}(d(t;u(t),\psi(t))) - \dot{d}(t;u(t),\psi(t)) - d(t;u(t),u(t)), \tag{53}$$

so (from the initial conditions)

$$\int_0^{t_1} d(t;\dot{u}(t),\psi(t))dt = \int_0^{t_1} (-\dot{d}(t;u(t),\psi(t)) - d(t;u(t),u(t)))dt. \tag{54}$$

Because

$$-\operatorname{Re} d(t;u(t),u(t)) \leq \lambda_D|u(t)|^2 - \alpha_d|u(t)|_{V_D}^2 \tag{55}$$

we have that

$$|u(t_1)|^2 + \alpha|\psi(0)|_V^2 \leq$$
$$2\int_0^{t_1} (c_2|\psi(t)|_V^2 + \lambda_D|u(t)|^2 - \alpha_d|u(t)|_{V_D}^2 + \operatorname{Re}\dot{d}(t;u(t),\psi(t)))dt. \tag{56}$$

Now we introduce the function $\omega(t) = \int_0^t u(s)ds$ and use

$$|\psi(t)|_V^2 = |\omega(t) - \omega(t_1)|_V^2 \leq 2|\omega(t)|_V^2 + 2|\omega(t_1)|_V^2 \tag{57}$$

to obtain

$$|u(t_1)|^2 + (\alpha - 4c_2t_1)|\omega(t_1)|_V^2 \leq$$
$$2\int_0^{t_1} (2c_2|\omega(t)|_V^2 + \lambda_D|u(t)|^2 - \alpha_d|u(t)|_{V_D}^2 + \operatorname{Re}\dot{d}(t;u(t),\psi(t)))dt. \tag{58}$$

Finally we use the differentiability of the damping sesquilinear form d:

$$\begin{aligned} |\dot{d}(t;u(t),\psi(t))| &\leq c_4|u(t)|_{V_D}|\psi(t)|_{V_D} \\ &\leq \frac{c_4}{2}(\epsilon|u(t)|_{V_D}^2 + \frac{1}{\epsilon}|\psi(t)|_{V_D}^2) \\ &\leq \frac{c_4}{2}(\epsilon|u(t)|_{V_D}^2 + \frac{2}{\epsilon}|\omega(t)|_{V_D}^2 + \frac{2}{\epsilon}|\omega(t_1)|_{V_D}^2) \end{aligned}$$

for all $\epsilon > 0$. We now have an inequality

$$|u(t_1)|^2 + (\alpha - 4c_2t_1)|\omega(t_1)|_V^2 - \frac{2c_4t_1}{\epsilon}|\omega(t_1)|_{V_D}^2 \leq$$
$$2\int_0^{t_1} (2c_2|\omega(t)|_V^2 + \lambda_D|u(t)|^2 - \alpha_d|u(t)|_{V_D}^2 + \frac{c_4}{2}(\epsilon|u(t)|_{V_D}^2 + \frac{2}{\epsilon}|\omega(t)|_{V_D}^2))dt \tag{59}$$

and using here that

$$(\alpha - 4c_2t_1)|\omega(t_1)|_V^2 \geq (\frac{\alpha}{2} - 2c_2t_1)(|\omega(t_1)|_V^2 + |\omega(t_1)|_{V_D}^2), \tag{60}$$

we finally arrive at the formidable inequality

$$|u(t_1)|^2 + (\frac{\alpha}{2} - 2c_2t_1)|\omega(t_1)|_V^2 + (\frac{\alpha}{2} - 2c_2t_1 - \frac{2c_4t_1}{\epsilon})|\omega(t_1)|_{V_D}^2 \leq$$
$$2\int_0^{t_1}(2c_2|\omega(t)|_V^2 + \lambda_D|u(t)|^2 + (\frac{c_4\epsilon}{2} - \alpha_d)|u(t)|_{V_D}^2 + \frac{c_4}{\epsilon}|\omega(t)|_{V_D}^2)dt.$$

Now fix $\epsilon > 0$ such that $(\frac{c_4\epsilon}{2} - \alpha_d) < 0$ and fix $t_1 = \frac{\alpha}{8(c_2+\frac{c_4}{\epsilon})}$ such that $(\frac{\alpha}{2} - 2c_2t_1 - \frac{2c_4t_1}{\epsilon}) = \frac{\alpha}{4}$. Then also $(\frac{\alpha}{2} - 2c_2t_1) > 0$ and the inequality above implies that for some constant $M > 0$:

$$|u(t_1)|^2 + |\omega(t_1)|_V^2 + |\omega(t_1)|_{V_D}^2 \leq M\int_0^{t_1}(|u(t)|^2 + |\omega(t)|_V^2 + |\omega(t)|_{V_D}^2)dt, \tag{61}$$

and from Gronwall's inequality we see that $u = 0$ in the interval $[0, t_1]$. Since the length of t_1 is independent of the choice of origin, we conclude that $u = 0$ on $[t_1, 2t_1]$, etc. Hence $u = 0$ and uniqueness is proved. That the solution depends continously on the data is obvious from the inequalities used to show existence; indeed, from (34) and (37) and the weak lower semicontinuity of norms we conclude that the constructed solution satisfies

$$|u(t)|_V^2 + |\dot{u}(t)|^2 + \delta\int_0^t |\dot{u}(t)|_{V_D}^2 \leq$$
$$K(|u^0|_V^2 + |u^1|^2 + \int_0^t |f(s)|_{V_D^*}^2 ds) \tag{62}$$

for some positive constants δ and K. Integrating from 0 to T yields the desired result (since $(u^0, u^1, f) \to (u, \dot{u})$ is linear). This completes the proof of Theorem 1.

The proof of Theorem 2 follows from the inequality (61) and the original proof in the case $d = 0$ from p. 275–279 of [11], because we do not gain any additional regularity from the form d in this case.

4 Formulation of the Parameter Estimation Problem

In the generic abstract parameter estimation problem, we consider a dynamic model of the form (1) where the operators A and D and possibly the input f depend on some unknown (i.e., to be estimated) functional parameters q in an admissible family $\mathcal{Q} \subset C^1(0,T;L^\infty(\Omega;Q))$ of parameters. We assume that the time dependence of the operators A and D are through the time dependence of the parameters $q(t) \in L^\infty(\Omega;Q)$ where $Q \subset R^p$ is a given constraint set for the values of the parameters. That is, $A(t) = A_1(q(t)), D(t) = A_2(q(t))$ so that we have

$$\begin{aligned} &\ddot{u}(t) + A_2(q(t))\dot{u}(t) + A_1(q(t))u(t) = f(t,q) \quad \text{in } V^* \\ &u(0) = u_0, \quad \dot{u}(0) = u_1. \end{aligned} \tag{63}$$

Thus we introduce the sesquilinear forms σ_1, σ_2 by

$$a(t;\phi,\psi) \equiv \sigma_1(q(t))(\phi,\psi) = \langle A_1(q(t))\phi,\psi\rangle_{V^*,V}, \tag{64}$$

$$d(t;\phi,\psi) \equiv \sigma_2(q(t))(\phi,\psi) = \langle A_2(q(t))\phi,\psi\rangle_{V_D^*,V_D}. \tag{65}$$

It will be convenient in subsequent arguments to use the notation

$$\dot{\sigma}_i(q)(\phi,\psi) \equiv \frac{d}{dt}\sigma_i(q)(\phi,\psi) \tag{66}$$

which in the event that σ_i is linear in q becomes $\dot{\sigma}_i(q)(\phi,\psi) = \sigma_i(\dot{q})(\phi,\psi)$. For example, in the Euler Bernoulli example of (18), we would have

$$\sigma_1(q(t))(\phi,\psi) = \int_0^l \widetilde{EI}(t,x)\phi''(x)\psi''(x)dx \tag{67}$$

$$\sigma_2(q(t))(\phi,\psi) = \int_0^l \widetilde{C_D I}(t,x)\phi''(x)\psi''(x)dx, \tag{68}$$

where $q = (\widetilde{EI}, \widetilde{C_D I}) \in C^1(0,T;L^\infty(0,l;R_+^2))$. Note that in this case we do have $\dot{\sigma}_i(q)(\phi,\psi) = \sigma_i(\dot{q})(\phi,\psi)$.

In terms of parameter dependent sesquilinear forms we thus will write (63) as

$$\begin{aligned} &\langle \ddot{u}(t),\phi\rangle + \sigma_2(q(t))(\dot{u}(t),\phi) + \sigma_1(q(t))(u(t),\phi) = \langle f(t,q),\phi\rangle \\ &u(0) = u_0, \quad \dot{u}(0) = u_1 \end{aligned} \tag{69}$$

for all $\phi \in V$. As in (2), $\langle\cdot,\cdot\rangle$ denotes the duality product $\langle\cdot,\cdot\rangle_{V^*,V}$. In some problems the initial data u_0, u_1 may also depend on parameters $\tilde{q}$ to be estimated, i.e.,

$u_0 = u_0(\tilde{q}), u_1 = u_1(\tilde{q})$. We shall not discuss such problems here, although the ideas we present can be used to effectively treat such problems. We instead refer readers to [4] for discussions of general estimation problems where not only the initial data but even the underlying spaces V and H themselves may depend on unknown parameters.

It is assumed that the parameter-dependent sesquilinear forms $\sigma_1(q), \sigma_2(q)$ of (69) satisfy the continuity and ellipticity conditions (H2)-(H7) of Section 1 uniformly in $q \in \mathcal{Q}$; that is, the constants $c_1, c_2, \alpha, c_3, c_4, \lambda_d, \alpha_d$ of (H2)-(H7) can be found independently of $q \in \mathcal{Q}$.

In general inverse problems, one must estimate the functional parameters q from dynamic observations of the system (63) or (69). A fundamental consideration in problem formulation involves what will be measured in the dynamic experiments producing the observations. To discuss these measurements in a specific setting, we consider the transverse vibrations (for example, equation (18)) of our beam example of Section 2 where $u(t) = w(t, \cdot)$. Measurements, of course, depend on the sensors available. If one considers a truly smart material structure as in [5, 18], it contains both sensors and actuators which may or may not rely on the same physical device or material. In usual mechanical experiments, there are several popular measurement devices [5, 6], some of which could possibly be used in a smart material configuration. If one uses an accelerometer placed at the point $\bar{x} \in (0, \ell)$ along the beam, then one obtains observations $\ddot{w}(t, \bar{x})$ of beam *acceleration.* A laser vibrometer will yield data of *velocity* $\dot{w}(t, \bar{x})$ while proximity probes including displacement solenoids produce measurements of *displacement* $w(t, \bar{x})$. In the case of a beam (or structure) with piezoceramic patches, the patches may be used as sensors as well as actuators. In this case one obtains observations of voltages which are proportional to the *accumulated strain*; this is discussed fully in [5, 18].

Whatever the measuring devices, the resulting observations can be used in a maximum likelihood or one of several least squares formulations of the parameter estimation problem, depending on assumptions about the statistical model [1, 6] for errors in the observation process. In the least squares formulations, the problems are stated in terms of finding parameters which give the best fit of the parameter-dependent solutions of the partial differential equation to dynamic system response data collected after various excitations, while the maximum likelihood estimator results from maximizing a given (assumed) likelihood function for the parameters, given the data.

In the beam example, the parameters to be estimated include the stiffness coefficient $\widetilde{EI}(t, x)$, the Kelvin-Voigt damping parameter $\widetilde{c_D I}(t, x)$, and any control related parameters that arise in the actuator input f. Details regarding the estimation of these parameters in the time independent case for an experimental beam are given in Section 5.4 of [5], while similar experimental results for a plate are summarized in Section 5.5 of the same reference.

The general ordinary least squares parameter estimation problem can be formulated as follows. For a given discrete set of measured observations $z = \{z_i\}_{i=1}^{N_t}$ corresponding to model observations $z_{ob}(t_i)$ at times t_i as obtained in most practical cases, we consider the problem of minimizing over $q \in \mathcal{Q}$ the least squares output functional

$$J(q, z) = \left| \tilde{C}_2 \left\{ \tilde{C}_1 \{w(t_i, \cdot; q)\} - \{z_i\} \right\} \right|^2, \tag{70}$$

where $\{w(t_i, \cdot; q)\}$ are the parameter dependent solutions of (63) or (69) evaluated at each time $t_i, i = 1, 2, \ldots, N_t$ and $|\cdot|$ is an appropriately chosen Euclidean norm. Here the operators $\tilde{C}_1$ and $\tilde{C}_2$ are observation operators that depend on the type of observed or measured data available. The operator $\tilde{C}_1$ may have several forms depending on the type of sensors being used. When the collected data z_i consists of time domain displacement, velocity, or acceleration values at a point $\bar{x}$ on the beam as discussed above, the functional takes the form

$$J_\nu(q, z) = \sum_{i=1}^{N_t} \left| \frac{\partial^\nu w}{\partial t^\nu}(t_i, \bar{x}; q) - z_i \right|^2, \tag{71}$$

for $\nu = 0, 1, 2$, respectively. In this case the operator $\tilde{C}_1$ involves differentiation (either $\nu = 0, 1$ or 2 times, respectively) with respect to time followed by pointwise evaluation in t and $\bar{x}$. We shall in our presentation of the next section adopt the ordinary least squares functional (70) to formulate and develop our results.

5 Approximation and Convergence

In this section we present a *corrected version* (Theorem 5.2 of [5] contains slight error in statement and proof) and extension (to treat time dependent coefficients) of arguments for approximation and convergence in inverse problems found in Section 5.2 of [5]. For more details on general inverse problem methodology in the context of abstract structural systems, the reader may consult [5]. The Banks-Kunisch book [4] contains a general treatment of inverse problems for partial differential equations in a functional analytic setting.

The minimization in our general abstract parameter estimation problems for (70) involves an infinite dimensional state space H and an infinite dimensional admissible parameter set $\mathcal{Q}$ (of functions). To obtain computationally tractable methods, we thus consider Galerkin type approximations in the context of the variational formulation (69). Let H^N be a sequence of finite dimensional subspaces of H, and $\mathcal{Q}^M$ be a sequence of finite dimensional sets approximating the parameter set $\mathcal{Q}$. We denote by P^N the orthogonal projections of H onto H^N. Then a family of approximating estimation problems with finite dimensional state spaces and parameter sets can be formulated by seeking $q \in \mathcal{Q}^M$ which minimizes

$$J^N(q, z) = \left| \tilde{C}_2 \left\{ \tilde{C}_1 \{u^N(t_i, \cdot; q)\} - \{z_i\} \right\} \right|^2, \tag{72}$$

where $u^N(t; q) \in H^N$ is the solution to the finite dimensional approximation of (69)) given by

$$\begin{aligned} &\langle \ddot{u}^N(t), \phi \rangle + \sigma_2(q(t))(\dot{u}^N(t), \phi) + \sigma_1(q(t))(u^N(t), \phi) = \langle f(t, q), \phi \rangle \\ &u^N(0) = P^N u_0, \quad \dot{u}^N(0) = P^N u_1, \end{aligned} \tag{73}$$

for $\phi \in H^N$. For the parameter sets $\mathcal{Q}$ and $\mathcal{Q}^M$, and state spaces H^N, we make the following hypotheses.

(A1M) The sets $\mathcal{Q}$ and $\mathcal{Q}^M$ lie in a metric space $\tilde{\mathcal{Q}}$ with metric d. It is assumed that $\mathcal{Q}$ and $\mathcal{Q}^M$ are compact in this metric and there is a mapping $i^M : \mathcal{Q} \to \mathcal{Q}^M$ so that $\mathcal{Q}^M = i^M(\mathcal{Q})$. Furthermore, for each $q \in \mathcal{Q}$, $i^M(q) \to q$ in $\tilde{\mathcal{Q}}$ with the convergence uniform in $q \in \mathcal{Q}$.

(A2N) The finite dimensional subspaces H^N satisfy $H^N \subset V$ as well as the approximation properties of the next two statements.

(A3N) For each $\psi \in V, |\psi - P^N \psi|_V \to 0$ as $N \to \infty$.

(A4N) For each $\psi \in V_D, |\psi - P^N \psi|_{V_D} \to 0$ as $N \to \infty$.

The reader is referred to Chapter 4 of [5] for a complete discussion motivating the spaces H^N and V_D.

We also need some regularity with respect to the parameters q in the parameter dependent sesquilinear forms σ_1, σ_2. In addition to (uniform in $\mathcal{Q}$) ellipticity/coercivity and continuity conditions (H2)-(H7), the sesquilinear forms $\sigma_1 = \sigma_1(q)$, $\sigma_2 = \sigma_2(q)$ and $\dot{\sigma}_1 = \dot{\sigma}_1(q)$ are assumed to be defined on $\mathcal{Q}$ and satisfy the continuity-with-respect-to-parameter conditions

(H8) $|\sigma_1(q)(\phi, \psi) - \sigma_1(\tilde{q})(\phi, \psi)| \leq \gamma_1 d(q, \tilde{q}) |\phi|_V |\psi|_V$, for $\phi, \psi \in V$

(H9) $|\dot{\sigma}_1(q)(\phi, \psi) - \dot{\sigma}_1(\tilde{q})(\phi, \psi)| \leq \gamma_3 d(q, \tilde{q}) |\phi|_V |\psi|_V$, for $\phi, \psi \in V$

(H10) $|\sigma_2(q)(\xi, \eta) - \sigma_2(\tilde{q})(\xi, \eta) \leq \gamma_2 d(q, \tilde{q}) |\xi|_{V_D} |\eta|_{V_D}$, for $\xi, \eta \in V_D$

for $q, \tilde{q} \in \mathcal{Q}$ where the constants $\gamma_1, \gamma_2, \gamma_3$ depend only on $\mathcal{Q}$.

Solving the approximate estimation problems involving (72),(73), we obtain a sequence of parameter estimates $\{\bar{q}^{N,M}\}$. It is of paramount importance to establish conditions under which $\{\bar{q}^{N,M}\}$ (or some subsequence) converges to a solution for the original infinite dimensional estimation problem involving (69),(70). Toward this goal we have the following results.

Theorem 3 *To obtain convergence of at least a subsequence of* $\{\bar{q}^{N,M}\}$ *to a solution* $\bar{q}$ *of minimizing* (70) *subject to* (69)*, it suffices, under assumption (A1M), to argue that for arbitrary sequences* $\{q^{N,M}\}$ *in* $\mathcal{Q}^M$ *with* $q^{N,M} \to q$ *in* $\mathcal{Q}$*, we have*

$$\tilde{C}_2 \tilde{C}_1 u^N(t; q^{N,M}) \to \tilde{C}_2 \tilde{C}_1 u(t; q). \tag{74}$$

Proof: Under the assumptions (A1M), let $\{\bar{q}^{N,M}\}$ be solutions minimizing (72) subject to the finite dimensional system (73) and let $\hat{q}^{N,M} \in \mathcal{Q}$ be such that $i^M(\hat{q}^{N,M}) = \bar{q}^{N,M}$. From the compactness of $\mathcal{Q}$, we may select subsequences, again denoted by $\{\hat{q}^{N,M}\}$ and $\{\bar{q}^{N,M}\}$, so that $\hat{q}^{N,M} \to \bar{q} \in \mathcal{Q}$ and $\bar{q}^{N,M} \to \bar{q}$ (the latter follows the last statement of (A1M)). The optimality of $\{\bar{q}^{N,M}\}$ guarantees that for every $q \in \mathcal{Q}$

$$J^N(\bar{q}^{N,M}, z) \leq J^N(i^M(q), z). \tag{75}$$

Using (74), the last statement of (A1M) and taking the limit as $N, M \to \infty$ in the inequality (75), we obtain $J(\bar{q}, z) \leq J(q, z)$ for every $q \in \mathcal{Q}$, or that $\bar{q}$ is a solution of the problem for (69),(70). We note that under uniqueness assumptions on the problems (a situation that we hasten to add is not often realized in practice), one can actually guarantee convergence of the entire sequence $\{\bar{q}^{N,M}\}$ in place of subsequential convergence to solutions.

We note that the essential aspects in the arguments given above involve compactness assumptions on the sets $\mathcal{Q}^M$ and $\mathcal{Q}$. Such compactness ideas play a fundamental role in other theoretical and computational aspects of these problems. For example, one can formulate distinct concepts of *problem stability* and *method stability* as in [4] involving some type of continuous dependence of solutions on the observations z, and use conditions similar to those of (74) and (A1M), with compactness again playing a critical role, to guarantee stability. We illustrate with a simple form of *method stability* (other stronger forms are also amenable to this approach–see [4]).

We might say that an *approximation method,* such as that formulated above involving $\mathcal{Q}^M, H^N$ and (72), is *stable* if

$$dist(\tilde{q}^{N,M}(z^k), \tilde{q}(z^*)) \to 0$$

as $N, M, k \to \infty$ for any $z^k \to z^*$ (in this case in the appropriate Euclidean space), where $\tilde{q}(z)$ denotes the set of all solutions of the problem for (70) and $\tilde{q}^{N,M}(z)$

denotes the set of all solutions of the problem for (72). Here "dist" represents the usual distance set function. Under (74) and (A1M), one can use arguments very similar to those sketched above to establish that one has this method stability. If the sets $\mathcal{Q}^M$ are not defined through a mapping i^M as supposed above, one can still obtain this method stability if one replaces the last statement of (A1M) by the assumptions:

(i) If $\{q^M\}$ is *any* sequence with $q^M \in \mathcal{Q}^M$, then there exist q^* in $\mathcal{Q}$ and subsequence $\{q^{M_k}\}$ with $q^{M_k} \to q^*$ in the $\tilde{\mathcal{Q}}$ topology.

(ii) For *any* $q \in \mathcal{Q}$, there exists a sequence $\{q^M\}$ with $q^M \in \mathcal{Q}^M$ such that $q^M \to q$ in $\tilde{\mathcal{Q}}$.

Similar ideas may be employed to discuss the question of *problem stability* for the problem of minimizing (70) over $\mathcal{Q}$ (i.e., the original problem) and again compactness of the admissible parameter set plays a critical role.

Compactness of parameter sets also plays an important role in computational considerations. In certain problems, the formulation outlined above (involving $\mathcal{Q}^M = i^M(\mathcal{Q})$) results in a computational framework wherein the $\mathcal{Q}^M$ and $\mathcal{Q}$ all lie in some uniform set possessing compactness properties. The compactness criteria can then be reduced to uniform constraints on the derivatives of the admissible parameter functions. There are numerical examples (for example, see [2]) which demonstrate that imposition of these constraints is necessary (and sufficient) for convergence of the resulting algorithms. (This offers a possible explanation for some of the numerical failures [21] of such methods reported in the engineering literature.)

Thus we have that compactness of admissible parameter sets play a fundamental role in a number of aspects, both theoretical and computational, in parameter estimation problems. This compactness may be assumed (and imposed) explicitly as we have outlined here, or it may be included implicitly in the problem formulation through *Tikhonov regularization* as discussed for example by Kravaris and Seinfeld [9], Vogel [19] and widely by many others. In the regularization approach one restricts consideration to a subset $\mathcal{Q}_1$ of parameters which has compact embedding in $\mathcal{Q}$ and modifies the least-squares criterion to include a term which insures that minimizing sequences will be $\mathcal{Q}_1$ bounded and hence compact in the original parameter set $\mathcal{Q}$.

After this short digression on general inverse problem concepts, we return to the condition (74). To demonstrate that this condition can be readily established in many problems of interest to us here, we give the following general convergence results.

Theorem 4 *Suppose that H^N satisfies (A2N),(A3N),(A4N) and assume that the sesquilinear forms $\sigma_1(q), \dot{\sigma}_1(q)$ and $\sigma_2(q)$ satisfy (H8),(H9),(H10), respectively, as well as (H1)-(H7) of Section 1 (uniformly in $q \in \mathcal{Q}$). Furthermore, assume that*

$$q \to f(\cdot; q) \text{ is continuous from } \mathcal{Q} \text{ to } L^2(0,T;V_D^*). \tag{76}$$

Let q^N be arbitrary in $\mathcal{Q}$ such that $q^N \to q$ in $\mathcal{Q}$. Then if in addition $\dot{u} \in L^2(0,T;V)$, we have as $N \to \infty$,

$$\begin{aligned} u^N(t;q^N) &\to u(t;q) \quad &&\text{in } V \text{ norm for each } t > 0 \\ \dot{u}^N(t;q^N) &\to \dot{u}(t;q) \quad &&\text{in } L^2(0,T;V_D) \cap C(0,T;H), \end{aligned}$$

where $(u^N, \dot{u}^N)$ are the solutions to (73) and $(u, \dot{u})$ are the solutions to (69).

Proof: From Theorem 2 of Section 1 we find that the solution of (69) satisfies $u(t) \in V$ for each t, $\dot{u}(t) \in V_D$ for almost every $t > 0$. Because

$$|u^N(t;q^N) - u(t;q)|_V \leq |u^N(t;q^N) - P^N u(t;q)|_V + |P^N u(t;q) - u(t;q)|_V,$$

and (A3N) implies that second term on the right side converges to 0 as $N \to \infty$, it suffices for the first convergence statement to show that

$$|u^N(t;q^N) - P^N u(t;q)|_V \to 0 \quad \text{as } N \to \infty.$$

Similarly, we note that this same inequality with u^N, u replaced by $\dot{u}^N, \dot{u}$ and the V-norm replaced by the V_D-norm along with (A4N) permits us to claim that the convergence

$$|\dot{u}^N(t;q^N) - P^N \dot{u}(t;q)|_{V_D} \to 0 \quad \text{as } N \to \infty,$$

is sufficient to establish the second convergence statement of the theorem. We shall, in fact, establish the convergence of $\dot{u}^N - P^N \dot{u}$ in the stronger V norm.

Let $u^N = u^N(t;q^N), u = u(t;q)$, and $\Delta^N = \Delta^N(t) \equiv u^N(t;q^N) - P^N u(t;q)$. Then

$$\dot{\Delta}^N = \dot{u}^N - \frac{d}{dt} P^N u = \dot{u}^N - P^N \dot{u}$$

and

$$\ddot{\Delta}^N = \ddot{u}^N - \frac{d^2}{dt^2} P^N u$$

because $\dot{u} \in L^2((0,T),V_D), \ddot{u} \in L^2((0,T),V^*)$. We suppress the dependence on t in the arguments below when no confusion will result. From (69) and (73), we have for $\psi \in H^N$

$$\begin{aligned}\langle \ddot{\Delta}^N, \psi\rangle_{V^*,V} &= \langle \ddot{u}^N - \ddot{u} + \ddot{u} - \frac{d^2}{dt^2}P^N u, \psi\rangle_{V^*,V} \\ &= \langle f(q^N), \psi\rangle_{V_D^*,V_D} - \sigma_2(q^N)(\dot{u}^N, \psi) - \sigma_1(q^N)(u^N, \psi) \\ &\quad - \langle f(q), \psi\rangle_{V_D^*,V_D} + \sigma_2(q)(\dot{u}, \psi) + \sigma_1(q)(u, \psi) \\ &\quad + \langle \ddot{u} - \frac{d^2}{dt^2}P^N u, \psi\rangle_{V^*,V}.\end{aligned}$$

This can be written as

$$\begin{aligned}&\langle \ddot{\Delta}^N, \psi\rangle_{V^*,V} + \sigma_1(q^N)(\Delta^N, \psi) \\ = \;& \langle \ddot{u} - \frac{d^2}{dt^2}P^N u, \psi\rangle_{V^*,V} - \langle f(q) - f(q^N), \psi\rangle_{V_D^*,V_D} \\ & + \sigma_2(q^N)(\dot{u} - P^N\dot{u}, \psi) + \sigma_2(q)(\dot{u}, \psi) - \sigma_2(q^N)(\dot{u}, \psi) \\ & + \sigma_1(q^N)(u - P^N u, \psi) + \sigma_1(q)(u, \psi) - \sigma_1(q^N)(u, \psi) \\ & - \sigma_2(q^N)(\dot{\Delta}^N, \psi).\end{aligned} \tag{77}$$

Choosing $\dot{\Delta}^N$ as the test function ψ in (77) and employing the equality $\langle \ddot{\Delta}^N, \dot{\Delta}^N\rangle_{V^*,V} = \frac{1}{2}\frac{d}{dt}|\dot{\Delta}^N|_H^2$ (this follows using definitions of the duality mapping - see [3] and the hypothesis (A2N)), we have using the symmetry of σ_1

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{|\dot{\Delta}^N|_H^2 + \sigma_1(q^N)(\Delta^N, \Delta^N)\Big\} \\ &= Re\Big\{\langle \ddot{u} - \frac{d^2}{dt^2}P^N u, \dot{\Delta}^N\rangle_{V^*,V} - \langle f(q) - f(q^N), \dot{\Delta}^N\rangle_{V_D^*,V_D} \\ &\quad + \sigma_2(q^N)(\dot{u} - P^N\dot{u}, \dot{\Delta}^N) + \sigma_2(q)(\dot{u}, \dot{\Delta}^N) \\ &\quad - \sigma_2(q^N)(\dot{u}, \dot{\Delta}^N) + \sigma_1(q^N)(u - P^N u, \dot{\Delta}^N) \\ &\quad + \sigma_1(q)(u, \dot{\Delta}^N) - \sigma_1(q^N)(u, \dot{\Delta}^N) - \sigma_2(q^N)(\dot{\Delta}^N, \dot{\Delta}^N) \\ &\quad + \dot{\sigma}_1(q^N)(\Delta^N, \Delta^N)\Big\}.\end{aligned} \tag{78}$$

We observe that $\langle \ddot{u} - \frac{d^2}{dt^2}P^N u, \dot{\Delta}^N\rangle_{V^*,V} \equiv 0$ because P^N is an orthogonal projection. Thus, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big\{|\dot{\Delta}^N|_H^2 + \sigma_1(q^N)(\Delta^N, \Delta^N)\Big\} \\ = \;& Re\Big\{-T_3^N + \sigma_2(q^N)(\dot{u} - P^N\dot{u}, \dot{\Delta}^N) + T_2^N + \frac{d}{dt}(\sigma_1(q^N)(u - P^N u, \Delta^N)) \\ & - \sigma_1(q^N)(\frac{d}{dt}(u - P^N u), \Delta^N) - \dot{\sigma}_1(q^N)(u - P^N u, \Delta^N) \\ & + T_1^N + \sigma_2(q^N)(\dot{\Delta}^N, \dot{\Delta}^N) + \dot{\sigma}_1(q^N)(\Delta^N, \Delta^N)\Big\},\end{aligned} \tag{79}$$

where

$$
\begin{aligned}
T_1^N &= \Delta\sigma_1^N(u, \dot{\Delta}^N) \equiv \sigma_1(q)(u, \dot{\Delta}^N) - \sigma_1(q^N)(u, \dot{\Delta}^N) \\
T_2^N &= \Delta\sigma_2^N(\dot{u}, \dot{\Delta}^N) \equiv \sigma_2(q)(\dot{u}, \dot{\Delta}^N) - \sigma_2(q^N)(\dot{u}, \dot{\Delta}^N) \\
T_3^N &= \langle \Delta f^N, \dot{\Delta}^N \rangle_{V_D^*, V_D} \equiv \langle f(q) - f(q^N), \dot{\Delta}^N \rangle_{V_D^*, V_D}.
\end{aligned} \tag{80}
$$

Note that here we have used that $\dot{u} \in L^2(0, T; V)$. Integrating the terms in (79) from 0 to t and using the initial conditions

$$
\begin{aligned}
\Delta^N(0) = u^N(0) - P^N u(0) = u^N(0) - P^N u_0 = 0 \\
\dot{\Delta}^N(0) = \dot{u}^N(0) - P^N \dot{u}(0) = \dot{u}^N(0) - P^N u_1 = 0,
\end{aligned}
$$

we obtain (here we do include arguments (t) and (s) in our calculations and estimates to avoid confusion)

$$
\begin{aligned}
&\frac{1}{2}|\dot{\Delta}^N(t)|_H^2 + \sigma_1(q^N(t))(\Delta^N(t), \Delta^N(t)) \\
&\qquad = \int_0^t \Big\{ Re\{\sigma_2(q^N(s))(\dot{u}(s) - P^N \dot{u}(s), \dot{\Delta}^N(s)) \\
&\quad - \sigma_1(q^N(s))(\dot{u}(s) - P^N \dot{u}(s), \Delta^N(s)) - \dot{\sigma}_1(q^N(s))(u(s) - P^N u(s), \Delta^N(s)) \\
&\qquad - \sigma_2(q^N(s))(\dot{\Delta}^N(s), \dot{\Delta}^N(s)) + T_1^N(s) \\
&\qquad + T_2^N(s) + T_3^N(s) + \dot{\sigma}_1(q^N(s))(\Delta^N(s), \Delta^N(s))\} \Big\} ds \\
&\qquad\qquad + Re\Big\{ \sigma_1(q(t))(u(t) - P^N u(t), \Delta^N(t)) \Big\}.
\end{aligned} \tag{81}
$$

We consider the T_i^N terms in this equation. We have

$$
T_1^N = \frac{d}{dt} \Delta\sigma_1^N(u, \Delta^N) - \Delta\sigma_1^N(\dot{u}, \Delta^N) - \Delta\dot{\sigma}_1^N(u, \Delta^N)
$$

so that

$$
\begin{aligned}
\int_0^t T_1^N(s) ds &= \Delta\sigma_1^N(u(t), \Delta^N(t)) \\
&\quad - \int_0^t \Big\{ \Delta\sigma_1^N(\dot{u}(s), \Delta^N(s)) - \Delta\dot{\sigma}_1^N(u(s), \Delta^N(s)) \Big\} ds.
\end{aligned}
$$

Using (H8),(H9) we thus obtain

$$Re\int_0^t T_1^N(s)ds \le \frac{\gamma_1^2}{4\epsilon}d(q^N,q)^2|u(t)|_V^2 + \epsilon|\Delta^N(t)|_V^2 + \int_0^t \Big\{\frac{\gamma_1^2}{4\epsilon}d(q^N,q)^2|\dot{u}(s)|_V^2 + \epsilon|\Delta^N(s)|_V^2 + \frac{\gamma_3^2}{4\epsilon}d(q^N,q)^2|u(s)|_V^2 + \epsilon|\Delta^N(s)|_V^2\Big\}ds. \quad (82)$$

Similarly, using (H10) we find

$$Re\int_0^t (T_2^N(s)+T_3^N(s))ds \le \int_0^t \Big\{\frac{\gamma_2^2}{4\epsilon}d(q^N,q)^2|\dot{u}(s)|_V^2 + 2\epsilon|\dot{\Delta}^N(s)|_{V_D}^2 + \frac{1}{4\epsilon}|\Delta f^N(s)|_V^2\Big\}ds. \quad (83)$$

These can then be used in (81) to obtain the estimate

$$\frac{1}{2}|\dot{\Delta}^N(t)|_H^2 + \alpha|\Delta^N(t)|_V^2 + \int_0^t \alpha_d|\dot{\Delta}^N(s)|_{V_D}^2 ds \le \int_0^t \Big\{\lambda_d|\dot{\Delta}^N(s)|_H^2 + \frac{c_3^2}{4\epsilon}|\dot{u}(s) - P^N\dot{u}(s)|_{V_D}^2 + 3\epsilon|\dot{\Delta}^N(s)|_{V_D}^2 + \frac{c_1^2}{2}|\dot{u}(s) - P^N\dot{u}(s)|_V^2 + \frac{c_2^2}{2}|u(s) - P^N u(s)|_V^2 + (1+\epsilon+c_2)|\Delta^N(s)|_V^2 + \frac{\gamma_1^2}{4\epsilon}d(q^N,q)^2|\dot{u}(s)|_V^2 + \frac{\gamma_3^2}{4\epsilon}d(q^N,q)^2|u(s)|_V^2 + \frac{\gamma_2^2}{4\epsilon}d(q^N,q)^2|\dot{u}(s)|_V^2 + \frac{1}{4\epsilon}|\Delta f^N(s)|_V^2\Big\}ds + \frac{c_1^2}{4\epsilon}|u(t) - P^N u(t)|_V^2 + 2\epsilon|\Delta^N(t)|_V^2 + \frac{\gamma_1^2}{4\epsilon}d(q^N,q)^2|u(t)|_V^2.$$

This finally reduces to

$$\frac{1}{2}|\dot{\Delta}^N(t)|_H^2 + (\alpha - 2\epsilon)|\Delta^N(t)|_V^2 + \int_0^t (\alpha_d - 3\epsilon)|\dot{\Delta}^N(s)|_{V_D}^2 ds$$

$$\leq \int_0^t \Big\{ \lambda_d |\dot{\Delta}^N(s)|_H^2 + (1 + \epsilon + c_2)|\Delta^N(s)|_V^2 \Big\} ds$$

$$+ \underbrace{\frac{c_1^2}{4\epsilon}|u(t) - P^N u(t)|_V^2 + \int_0^t \Big\{ \frac{c_3^2}{4\epsilon}|\dot{u}(s) - P^N \dot{u}(s)|_{V_D}^2 + \frac{c_1^2}{2}|\dot{u}(s) - P^N \dot{u}(s)|_V^2 \Big\} ds}_{\delta_1^N(t)}$$

$$+ \underbrace{\frac{\gamma_1^2}{4\epsilon} d(q^N, q)^2 |u(t)|_V^2 + \int_0^t \Big\{ \frac{1}{4\epsilon}|\Delta f^N(s)|_V^2 + \frac{c_2^2}{2}|u(s) - P^N u(s)|_V^2 \Big\} ds}_{\delta_2^N(t)}$$

$$\underbrace{+ \int_0^t \Big\{ \frac{\gamma_1^2}{4\epsilon} d(q^N, q)^2 |\dot{u}(s)|_V^2 + \frac{\gamma_3^2}{4\epsilon} d(q^N, q)^2 |u(s)|_V^2 + \frac{\gamma_2^2}{4\epsilon} d(q^N, q)^2 |\dot{u}(s)|_V^2 + \Big\} ds}_{\delta_3^N(t)} .$$

Therefore, under the assumptions we have as $N \to \infty$ and $q^N \to q$

$$\sup_{t \in (0,T)} [\delta_1^N(t) + \delta_2^N(t) + \delta_3^N(t)] \to 0.$$

Applying Gronwall's inequality, we find that

$$\dot{\Delta}^N \to 0 \text{ in } C(0,T;H)$$
$$\Delta^N \to 0 \text{ in } C(0,T;V)$$
$$\dot{\Delta}^N \to 0 \text{ in } L^2(0,T;V_D),$$

and hence the convergence statement of the theorem holds.

Remark: The condition "$\dot{u} \in L^2(0,T;V)$" routinely occurs if the structure under investigation has sufficiently strong damping so that V_D is equivalent to V.

6 Concluding Remarks

We have provided sufficient conditions and detailed arguments for existence and uniqueness of solutions to abstract second order non-autonomous hyperbolic systems such as those with time dependent "stiffness", "damping" and input parameters. In addition, we have considered a class of corresponding inverse problems for these systems and argued convergence results for approximating problems that yield a type of method stability as well as a framework for finite element type computational techniques. The efficacy of such methods have been demonstrated for autonomous systems in earlier efforts [5]. The approaches developed here can readily be extended (albeit with considerable tedium) for higher dimensional spatial systems such as plates, shells, etc.

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