A Fast Algorithm for Determining Optimal Feedback Gain

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July 18, 2008

Abstract

A fast algorithm to compute the optimal feedback gain for the linear quadratic regulator problem is developed and analyzed. The algorithm utilizes the relation between an invariant subspace of the corresponding Hamiltonian operator and the solution to the Riccati equation and the reduced order methods. It is based on the inverse of the Hamiltonian operator on the reduced order subspace. Large scale control systems that arise from a discretization of a class of control problems governed by partial differential equations is used to demonstrate the feasibility and applicability of Algorithm. A sparsity and structural property of system matrices are incorporated in Algorithm and it enables us to compute a stabilizing feedback law for a large class of distributed control systems.

1 Introduction

Consider the Linear Quadratic Regulator (LQR) problem;

$$\min \quad \int_0^\infty ((x(t), Qx(t))_X + |u(t)|^2)\, dt \quad \text{over} \quad u \in L^2(0, \infty, U), \qquad (1.1)$$

*Center for Research in Scientific Computation, Department of Mathematics, North Carolina State University; research partially supported by the Army Research Office under DAAD19-02-1-039 and the AFOSR under FA9550-04-1-0220.

subject to the linear control dynamics

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X. \tag{1.2}$$

Here, $x(t) \in X$ and $u(t) \in U$ is the state and control functions, respectively. The optimal control to (1.1)–(1.2) is given in the feedback form $u(t) = -B^*\Pi x(t)$ where the bounded, self-adjoint (operator) Π on X is the unique nonnegative solution to the algebraic Riccati equation (ARE)

$$A^*\Pi x + \Pi A x - \Pi B B^*\Pi x + Qx = 0 \tag{1.3}$$

for every $x \in dom\,(A)$. Here A^*, B^* are the Hilbert space adjoints of A and B, respectively and $\Pi x \in dom\,(A^*)$ for $x \in dom\,(A)$, and we assume that (A, B) is (exponentially) stabilizable and (A, Q) is (exponentially) detectable. This result on the LQR problem is stated for the case when X and U are Hilbert spaces and A is an infinitesimal generator of C_0 semigroup on X [18] and is standard, we refer e.g. to, [5, 8].

In order to construct the optimal feedback gain $K^{op} = B^*\Pi$ via ARE (1.3) it is normally done that we consider a sequence of LQR problems (A^N, B^N, Q^N) on R^N based on approximation methods. It follows from the Gibson's approximation theory for ARE that if a sequence of approximating finite dimensional control problem (A^N, B^N, Q^N) satisfies the stability and consistency condition, e.g., see [6, 8], then the feedback law by $K^N = (B^N)^*\Pi^N$ for (A^N, B^N, Q^N) converges to K^{op} in norm. Certain class of control systems, including the fluid control and vibration control problems results in a large scale control system (A^N, B^N, Q^N) via discretizations. So, it is essential to have an efficient method for the Riccati equation that overcomes problems associated with large dimensionality N.

In this paper we develop and analyze a fast method to compute the optimal feedback gain K^{op} based on the corresponding Hamiltonian, i.e., if we let $p(t) = \Pi x(t)$, then we have

$$\frac{d}{dt}(x(t), p(t))^T = \mathcal{H}(x(t), p(t))^T,$$

where the Hamiltonian operator $\mathcal{H}$ in $X \times X$ is defined by

$$\mathcal{H} = \begin{pmatrix} A & -BB^* \\ -Q & -A^* \end{pmatrix}.$$

If λ is a closed loop eigenvalue of $A - BB^*\Pi$, then λ is an eigenvalue of $\mathcal{H}$. In fact, if $x \in dom\,(A)$ satisfies $(A - BB^*\Pi)x = \lambda\, x$, then $\Pi x \in dom\,(A^*)$ and

$$\begin{aligned} (A - BB^*\Pi)x &= \lambda\, x \\ -Qx - A^*\Pi x &= \Pi(A - BB^*\Pi)x = \lambda\, \Pi x. \end{aligned} \tag{1.4}$$

Thus, λ is an eigenvalue of $\mathcal{H}$ and the corresponding eigenfunction is given by $(x, \Pi x)$. That is, if $X = R^N$ is finite dimensional, then $\Pi = YV^{-1}$ where (V, Y) is eigenvectors (in general Schur vectors) corresponding to eigenvalues $\lambda(\mathcal{H})$ with $Re\lambda < 0$, e.g., [19, 15, 20]. While these "eigenvector" methods can be used satisfactorily (for a discussion of real advantages offered by the Laub-Schur approach over Potter method, see [15]) for systems with N relatively small, the computational effort (and time) grows like N^3 and the storage requirement is of order N^2 and they become prohibitive for large systems.

Our proposed algorithms overcome this difficulty by the reduced order approach in [9]. Algorithm I computes a sub-invariant subspace of $\mathcal{H}$ that corresponds to eigenvalues $\lambda \in \sigma(\mathcal{H})$ whose real part $Re\,\lambda < 0$ is large and then forms a feedback gain on the invariant subspace. Algorithm II uses the inverse of $\mathcal{H}$ on the reduced subspace and approximates the invariant subspace of $\mathcal{H}$. Both Algorithms employ an implicitly Re-Started Arnold iteration algorithm [21, 22] for finding sub-invariant subspace $\mathcal{H}$ (which is a generalization of the inverse power iterate for $\mathcal{H}$) and use the subspace method (2.3)–(2.4) for computation of $\mathcal{H}^{-1}(f, g)$ which requires the basic operations $A^{-1}f$ and $A^{-*}g$. Algorithm I computes the exact eigenvalues and associated invariant subspace X_1 of $A - BB^*\Pi$ and thus $K^{op}x = K^L x$ for $x \in X_1$. Algorithm II yields approximations to such eigenvalues and invariant subspaces. However, for Algorithm II these basic operations are restricted on the subspace spanned by the reduced order basis of dimension

L. The operation count and storage requirements are of $O(L^2)$. As a result it does not have a storage limitation as Algorithm I may have (see Section 2) and it is very efficient and accurate as well. We strongly believe that by Algorithms the myth of "solving Riccati equation to determine the optimal feedback gain is very impractical" is no longer true. We will demonstrate it through some specific control problems in Section 2. It is of our plans to apply Algorithms to concrete control problems including 3D Navier Stokes control flow problems.

In [2] a hybrid method combining the Chandrasekhar algorithm [3], Newton-Kleiman method [13] along with an innovative use of the Smith algorithm [23] for solution of Lyapunov equations is developed. It requires the basic operation $(I - r\,A)^{-1}x,\ x \in X$ and a very efficient method when the rank of Q is small. In [17] an improvement of the hybrid method using ADI method [16] for Lyapunov equations is reported.

The outline of the paper is as follows. Algorithms I and II are introduced and explained and the computational issues are addressed in Section 2. Also, numerical tests for control problems of the heat equation are presented. The corresponding formula to (2.3)–(2.4) for the generalized control system of the form (2.5) and the corresponding second order control system is developed in Section 3. Section 4 presents a convergence analysis of Algorithm I. In Section 5 the reduced order approaches to address the storage problem of Algorithm I are described. Section 6 validates Algorithm II.

2 Algorithms

Algorithm I Find the (generalized) eigenpairs $(\lambda_i, (x_i, p_i))$ with $Re\,\lambda_i < 0$ of $\mathcal{H}$, where the eigenvalues λ_i are ordered with respect to their real part. Form the matrices V and Y consisting of the first L vectors of x_i and p_i, respectively. Define $K^L\phi = B^*Y(V^*V)^{-1}(V,\phi)_X,\ \phi \in X$, where $\phi \to (V^*V)^{-1}(V,\phi)_X$ is the orthogonal projection of X onto $X_1 = range(V)$. Moreover, the span $X_1 = \{x_i,\ 1 \le i \le L\}$ forms an invariant subspace of the

closed loop system $A - BK^{op}$ and $K^{op}x = K^L x$ for $x \in X_1$.

If we use the approximation system (A^N, B^N, Q^N) and approximate $\mathcal{H}$ by $\mathcal{H}^N$;

$$\mathcal{H}^N = \begin{pmatrix} A^N & -B^N(B^N)^T \\ -Q^N & -(A^N)^T \end{pmatrix}.$$

in Algorithm I, then L can be much smaller than N and thus Algorithm offers a reduced order method for construction of the optimal feedback gain [9]. If $X = R^N$ is equipped with the norm $\sqrt{x^T M x}$, then K^L is the matrix representation

$$K^L = B^* Y (V^* M V)^{-1} V^* \tag{2.1}$$

where (V, Y) is consisting of the first L eigen (Schur)-vectors of $\mathcal{H}^N$.

Algorithm I requires to find sub-invariant subspaces of the Hamiltonian $\mathcal{H}$. We employ an implicitly Re-Started Arnold iteration algorithm [21, 22] for finding the sub-invariant subspace X_1 for a linear system

$$\mathcal{H}(x, p) = (f, g) \quad \text{in} \quad X \times X. \tag{2.2}$$

Thus, the efficiency of the combined algorithm depends on an efficient method to solve (2.2). To this end, we apply the subspace method [12] (see, Remark 1.1) for solving (2.2); i.e.,

$$(x, p) = \mathcal{H}^{-1}(f, g) = (A^{-1}(Bu + f), -A^{-*}(g + QA^{-1}(Bu + f)). \tag{2.3}$$

where $u = B^* p \in R^m$ satisfies

$$u + B^* A^{-*} Q A^{-1} B u = -B^* A^{-*}(g + QA^{-1} f) \quad \text{on} \quad R^m. \tag{2.4}$$

In fact, if $u = B^* p$, then

$$0 = u - B^* p = u + B^* A^{-*} Q A^{-1} B u + B^* A^{-*}(g + QA^{-1} f).$$

Equation (2.4) is on R^m for u with symmetric positive matrix $(I + B^* A^{-*} Q A^{-1} B)$. Here, (2.3)–(2.4) proceeds with

- evaluation of $A^{-1}f,\ r = -B^*A^{-*}(g + QA^{-1}f)$,
- solution u to (2.4) with right hand side r
- $x = A^{-1}f + A^{-1}Bu$ and $p = A^{-*}(QA^{-1}(g + Qx))$ to complete the solution to (2.3).

Thus, the subspace method for solving (2.2) needs only the one solution for $Ax = f$, the one solution $A^*p = \tilde{g}$ and the solution to (2.4) in R^m. In summary it reduces dramatically storage and computational requirements for solving (2.2). The proposed combined algorithm is very efficient and enables us to compute the Riccati-based feedback gain for a (super) large system. If m is very large, then we use the conjugate gradient method for solving (2.4) which only needs the vector operation $B^*A^{-*}QA^{-1}Bu,\ u \in R^m$. Otherwise, we form $A^{-1}B$ and solve (2.4) using the Cholesky factorization on R^m.

An implementation of the proposed algorithm using *matlab* is as follows.

$[u, e] = eigs(@ricc, 2 * N^2, 2 * L, 0, a, b, q);\ e = diag(e);$
$j = find(real(e) < 0);\ u = v(:, j);\ [t, i] = sort(-real(e(j)));$
$Y = u([N + 1 : 2 * N], i(1 : L));\ V = u(1 : N, i(1 : L));$
$K = V * (V' * M * V) \backslash (Y' * b);$

$function\ [y] = ricc(xx)$
$n = size(xx, 1)/2;\ x = reshape(xx, n, 2);\ f = x(:, 1);\ g = x(:, 2);$
$f0 = a \backslash f;\ b0 = a \backslash b;\ u = -(eye(m) + b0' * q * b0) \backslash (b0' * (g + q * f0));$
$x = f0 + b0 * u;\ p = -(a') \backslash (g + q * x);\ y = [x; p];$

where **eigs** is an *matlab* routine and **ricc.m** is an *matlab* m-file. A user can provide the routine to evaluate $a \backslash f$ and $(a') \backslash g$.

Remark 1.1

- Formula (2.3)–(2.4) directly apply to $\mathcal{H}$ for the original control problem (1.1)–(1.2). In practice we use an approximating system (A^N, B^N, Q^N) to approximate $\mathcal{H}$ by $\mathcal{H}^N$ and then a matlab routine **eigs**, which implements the Arnold method among with (2.3)–(2.4) is used to find invariant subspaces of $\mathcal{H}^N$.

- Accuracy of sub-invariant subspaces based on $\mathcal{H}^N$ depends on those of (A^N, B^N, Q^N).

- In general, an approximating system has the form

$$M\frac{d}{dt}x(t) = Hx(t) + B_0u(t) \tag{2.5}$$

where M is the mass (symmetric, positive) matrix on R^N, H is the stiffness operator on R^N and B_0 is the input matrix. Thus,

$$A = M^{-1}H, \quad B = M^{-1}B_0$$

and $Q^N = M$ if Q =the identity operator in (1.1). It will be shown in Section 2 that for system of the form (2.5) we have an equivalent formulation of (2.3)–(2.4) in which one can avoid forming $A = M^{-1}H$ and $B = M^{-1}B_0$ and performing $M^{-1}f$ completely. This is especially efficient for the second order control (3.3) as described in Section 2.

- Formula (2.3)–(2.4) is the right precondition system [12] of (2.2);

$$\mathcal{H}\mathcal{R}^{-1}z = (f, g), \quad (x, p) = \mathcal{R}^{-1}z$$

with preconditioner

$$\mathcal{R} = \begin{pmatrix} A & 0 \\ -Q & -A^* \end{pmatrix}.$$

In fact, note that if $\hat{z} = z - (f, g)$, then

$$\mathcal{H}\mathcal{R}^{-1}\hat{z} = (f, g) - \mathcal{R}^{-1}(f, g) = -(\mathcal{H} - \mathcal{R})\mathcal{R}^{-1}(f, g) = \hat{r} \in Y$$

where $\mathcal{H}\mathcal{R}^{-1} = (\mathcal{H} - \mathcal{R})\mathcal{R}^{-1} + I$ and $Y = range(\mathcal{H} - \mathcal{R})$. Thus $\hat{z} \in Y$ satisfies

$$\hat{z} + (\mathcal{H} - \mathcal{R})\mathcal{R}^{-1}\hat{z} = \hat{r} \text{ in } Y \tag{2.6}$$

This is the reduced equation in the sparse subspace $Y = R^m$ with

$$\mathcal{H} - \mathcal{R} = \begin{pmatrix} 0 & BB^* \\ 0 & 0 \end{pmatrix} \quad \text{and } Y = range(B) \times \{0\}$$

It is easy to see that (2.6) is equivalent (2.4).

In order to demonstrate an applicability of Algorithm I we consider a boundary control of 2 and 3 D heat equations on domain $\Omega = (0,1)^d,\ d = 2, 3$;

$$\frac{\partial}{\partial t} y(t,x) = \nu\, \Delta y(t,x), \quad x \in \Omega$$

$$\frac{\partial}{\partial n} y(t,x) = \sum_{\ell=1}^{m} u_\ell(t)\, b_\ell(x), \quad x \in \partial\Omega$$

where $\nu > 0$ is a diffusion constant and $b_i(x)$ are the distribution functions on the boundary $\partial\Omega$. Suppose we use the central difference approximation based on the uniform Cartesian gridpoints with mesh size $h = \dfrac{1}{n}$ in each direction. Then we have $N = (n+1)^d$ and for the two dimensional case (2.5) is with

$$H = \nu\,(H_1 \otimes Q_1 + Q_1 \otimes H_1), \quad M = h\,(Q_1 \otimes Q_1)$$

$$(B_0)_{ijk} = \nu\, b(x_i, y_j, z_k) \in \partial\Omega,$$

where $\otimes$ denotes the Kronecker product and H_1 is the tridiagonal matrix of the form

$$H_1 = \frac{\nu}{h} \begin{pmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1. \end{pmatrix}$$

and Q_1 is the diagonal matrix of the form $Q_1 = diag([.5, 1, \cdots, 1, .5])$. In this case $H^{-1}x$ can carried out by FFT with order $N\, log(N)$ operations and $O(N)$ storage. In the followings we summarize the results for the case $Q = I$ and $\nu = .1$. In Figure 1 the resulting feedback gains for $L = 30, 50, 100, N$ are shown for the two dimensional case with the control at the $x = 0$ side with $b(0,y) = exp(-50(y-.5)^2)$ and $N = 21^2$. Table 1 shows Error defined by

$$Error = |K^L - K^N|_2 / |K^N|_2$$

and CPU times for the above matlab implementation and for the case using FFT solver. The full dimension K^N is computed by a matlab routine **lqr**

with CPU 24.39 (sec). With our algorithm it takes 27.53 (sec) and 9.40 (sec) using FFT, i.e., even for full dimension case $N = L$ Algorithm I can be more efficient due to Formula (2.3)–(2.4).

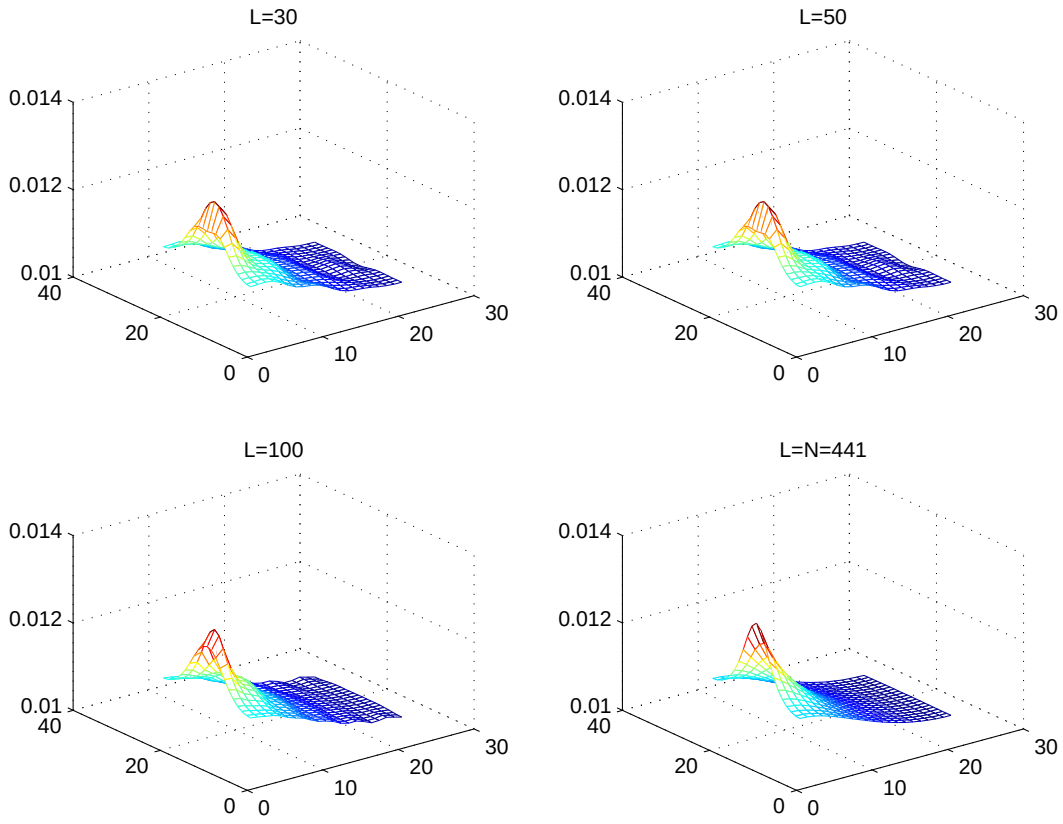

Figure 1: Feedback Gains based on eign-subspace with dimension L

For the three dimensional case in Table 2 we summarize our results.

With $2N = 2\,81^3 = 1,062,882$ it was not possible to use Algorithm I with $L = 20$ due to lack of memory space ($> 1GB$). CPU time glows linearly in L for this case. The storage becomes a problem with increasing L before CPU time does for a large scale problem.

This storage problem is resolved by the reduced order approach in Section 5 and by the inverse Hamiltonian based reduced order method in Section 6.

L	Error	CPU (sec)	CPU(FFT)
30	0.0124	4.00	0.50
50	0.0086	5.97	0.91
100	0.0053	14.58	4.45

Table 1: Feedback Gains based on eign-subspace with dimension L

L	N	CPU(sec)	iter.
50	21^3	33.82	7
100	21^3	69.68	4
200	21^3	206.15	2
50	41^3	234.75	7
100	41^3	601.96	4
10	81^3	421.36	7

Table 2: 3D boundary control for heat equation with Dimension L of eigen-subspace, iteration counts, and CPU times

It is based on the fact that invariant subspaces of $\mathcal{H}^{-1}$ are same as those of $\mathcal{H}$ and that $\mathcal{H}^{-1}$ is Hamiltonian. It uses the restriction of $\mathcal{H}^{-1}$ on $\hat{X} \times \hat{X}$ by

$$\hat{\mathcal{H}}^{-1} = \begin{pmatrix} W^* & 0 \\ 0 & W^* \end{pmatrix} \mathcal{H}^{-1} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

where $\hat{X} = W^* X$ and W is the reduced order orthonormal basis of X. Such a basis can be generated by applying the proper orthogonal decomposition approach [1, 14] and the reduced-basis method [10, 11]. Let $L = dim(\hat{X})$, $\hat{\mathcal{H}}^{-1}$ has the $(2L) \times (2L)$ matrix representation. Thus, we just need to store $(2L) \times (2L)$ Schur basis (compared to $(2N) \times (2L)$ Schur basis for Algorithm I).

Algorithm II Find the (generalized) eigenpairs $(\lambda_i, (\hat{x}_i, \hat{p}_i))$ with $Re\, \lambda_i < 0$ of $\hat{\mathcal{H}}^{-1}$. Form the matrices $\hat{V}$ and $\hat{Y}$ consisting of vectors of x_i and p_i, respectively. Define $\hat{K}^L \phi = B^* W^* \hat{Y} (\hat{V}^* \hat{V})^{-1} \hat{V} W^* \phi,\ \phi \in X$.

The convergence of Algorithm II follows from the completeness of the reduced

order basis and further we refer it to the analysis in Section 6. The computation of $\hat{\mathcal{H}}^{-1}(\hat{f},\hat{g})$ can be carried out exactly as for $\mathcal{H}^{-1}$ (see, (6.2)–(6.3)) and is restricted to the reduced basis. The other advantage of Algorithm II is that *matlab* routine **eigs** is applied for $(2L)\times(2L)$ matrix (compared to $(2N)\times(2N)$ for Algorithm I) and results in a significant reduction of storage requirement and computational complexity. On the other hand, Algorithm I leads to the exact eigenvalues and associated invariant subspace X_1 of $A-BB^*\Pi$ and thus $K^{op}x = K^L x$ for $x\in X_1$. Algorithm II yields approximations to such eigenvalues and invariant subspaces.

Our numerical tests of Algorithm II for our heat equation examples show that it performs as well as Algorithm I. We are able to solve 3D problem with $L=10^3=1000$ and the resulting feedback is very accurate.

3 Second order System

For system of the form (2.5) $A=M^{-1}H,\ B=M^{-1}B_0$ and $A^*=H^*M^{-1}$ and (2.3)–(2.4) can be equivalently written as

$$(x,p)=\mathcal{H}^{-1}(f,g)=(H^{-1}(B_0u+Mf),-MH^{-*}(g+QH^{-1}(B_0u+Mf)). \tag{3.1}$$

where $u=B_0^*M^{-1}p\in R^m$ satisfies

$$u+B_0^*H^{-*}QH^{-1}B_0u=-B_0^*H^{-*}(g+QH^{-1}Mf)\quad \text{on}\ \ R^m. \tag{3.2}$$

In fact,

$$0=u-B_0^*(M^{-1}p)=u+B_0^*H^{-*}QH^{-1}B_0u+B_0^*H^{-*}(g+QH^{-1}Mf).$$

Thus, we can avoid forming $A,\ B$ and performing $M^{-1}f$ completely for (2.3)–(2.4).

Furthermore, for the second order control system;

$$M_0\frac{d}{dt}y(t)+D_0\frac{d}{dt}y(t)+H_0y(t)=B_0u(t) \tag{3.3}$$

where $M_0,\ H_0,\ D_0$ are mass, stiffness and damping operators, respectively, we can take an advantage of the structure property as follows. If we let $x(t) = (y(t), \frac{d}{dt}y(t)) \in X$, then

$$\begin{bmatrix} H_0 & 0 \\ 0 & M_0 \end{bmatrix} \frac{d}{dt}x(t) = \begin{bmatrix} 0 & H_0 \\ -H_0 & -D_0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_0 \end{bmatrix} u(t) \tag{3.4}$$

and we equip X by norm $|(u,v)|_X^2 = (M_0 v, v) + (H_0 u, u)$. Note that for (3.4)

$$A^{-1}f = H^{-1}M(f_1, f_2) = (-H_0^{-1}(M_0 f_2 + D_0 f_1), f_1)$$

$$A^{-*}g = MH^{-*}(g_1, g_2) = (g_2 - D_0^*(H_0^{-1}g_1), -M_0 H_0^{-1} g_1)$$

and

$$H^{-1}B_0 u = (-H_0^{-1}B_0 u, 0), \quad B_0^* H^{-*}(g_1, g_2) = -B_0 H_0^{-1} g_1$$

Thus, it follows from (3.1)–(3.2) we have for (3.3)–(3.4)

$$(x,p) = \mathcal{H}^{-1}(f,g) \quad \text{with}$$

$$x = (-H_0^{-1}(B_0 u + M_0 f_2 + D_0 f_1), f_1) \tag{3.5}$$

$$p = ((g + Qx)_2 - D_0^* H_0^{-1}(g + Qx)_1, M_0 H_0^{-1}(g + Qx)_1),$$

where $u \in R^m$ satisfies

$$u + B_0^* H_0^{-*} Q_{11} H_0^{-1} B_0 u = B_0^* H_0^{-*}(g + QH^{-1}Mf)_1 \quad \text{on} \ \ R^m. \tag{3.6}$$

All required operations are carried out with the original system matrices (M_0, H_0, D_0) of (3.3).

4 Convergence Analysis of Algorithm I

Let us assume that A has a compact resolvent. Let P be the projection operator defined by

$$P = \int_\Gamma (\lambda I - \tilde{A})^{-1}\, d\lambda$$

where Γ is a closed Jordan curve isolating the first L eigenvalues of the closed-loop operator $\tilde{A} = A - BK^{op}$. We let $X_1 = PX$ and $X_2 = (I-P)X$. Then we have the decomposition on $X_1 \times X_2$

$$A - BK^L = A - BK^{op} + \begin{bmatrix} 0 & PB(K^{op} - K^L) \\ 0 & (I-P)B(K^{op} - K^L) \end{bmatrix}$$

Note that $K^L = K^{op}\tilde{P}$ where $\tilde{P}$ is the orthogonal projection of X onto X_1. Thus,

- If the eigenspace spanned by $\{x_i\}$ of $A - BK^{op}$ is complete, then $\tilde{P} \to I$ and thus
$$|K^{op} - K^L| = |K_{op}(I - \tilde{P})| \to 0 \text{ as } L \to \infty.$$
- If $|(I-P)BK^{op}(I-\tilde{P})|$ is sufficiently small, then $A - BK^L$ generates an exponentially stable semigroup on X.

It follows from (1.4) that for $\phi \in X_1$

$$\Pi(A - BB^*\Pi)\phi + A^*\Pi\phi + Q\phi = 0,$$

or equivalently

$$(\Pi\psi, (A - BB^*\Pi)\phi) + ((A - BB^*\Pi)\psi, \Pi\phi) + ((Q + (K^{op})^*K^{op})\psi, \phi)) \quad (4.1)$$

for all $\phi \in X_1$ and $\psi \in X$. Let $\hat{\Pi} = \tilde{P}\Pi\tilde{P}$. Then, $\hat{\Pi}$ has a matrix representation on X_1 as

$$MV(V^*MV)^{-1}Y(V^*MV)^{-1}V^*M.$$

It follows from (4.1) that

$$(\hat{\Pi}\psi, (A - BB^*\Pi)\phi) + ((A - BB^*\Pi)\psi, \hat{\Pi}\phi) + ((Q + (K^{op})^*K^{op})\psi, \phi))$$

for all $\phi,\ \psi \in X_1$. That is,

$$\hat{\Pi} = \int_0^\infty \tilde{P}S^*(t)\tilde{P}((Q + (K^{op})^*K^{op})\tilde{P}S(t)\tilde{P}\,dt$$

where $S(t)$ is the semigroup on X generated by $A - BK^{op}$.

5 Proper Orthogonal Decomposition and Reduced-order Control

In this section we describe the reduced order approaches. As pointed out in Introduction the storage requirement for Algorithm I becomes a problem with increasing L. It is necessary to reduce the dimension of X for large scale systems.

Let $W \in R^{N\times\hat{N}}$ be the reduced order orthonormal basis of X. Such a basis can be generated by applying the proper orthogonal decomposition approach [1, 14] and the reduced-basis method [10, 11]. If Q has a small rank, the basis can be generated by the Krylov subspace

$$span\,\{[B,Q], A^{-1}[B,Q], \ldots, A^{n-1}[B,Q]\}. \tag{5.1}$$

This subspace is based on the fact that the solution Π to the Riccati equation satisfies a Liyapunov equation;

$$(A - BK^{op})^*\Pi + \Pi(A - BK^{op}) + (K^{op})^*K^{op} + Q = 0.$$

The reduced order control system on $\hat{X} = W^*X$ is

$$\min \quad \int_0^\infty ((QW\hat{x}, W\hat{x})_X + |u(t)|^2)\,dt$$

subject to

$$W^*MW\frac{d}{dt}\hat{x}(t) = W^*HW\hat{x}(t) + W^*B_0u(t), \quad \hat{x}(t) \in \hat{X}.$$

Thus, we can apply our algorithm (3.1)–(3.2) for the reduced order system with $(W^*MW, W^*HW, W^*B_0, W^*QW)$.

6 Algorithm II and Inverse Hamiltonian

Let $\hat{X} = M^{\frac{1}{2}}W^*X$ with $W^*MW = I$ be the reduced order subspace and $\hat{\mathcal{H}}^{-1}$ be the reduced restriction of $\mathcal{H}^{-1}$ on $\hat{X}\times\hat{X}$ defined by

$$\hat{\mathcal{H}}^{-1} = \begin{pmatrix} W^*M^{\frac{1}{2}} & 0 \\ 0 & W^*M^{\frac{1}{2}} \end{pmatrix} \mathcal{H}^{-1} \begin{pmatrix} M^{\frac{1}{2}}W & 0 \\ 0 & M^{\frac{1}{2}}W \end{pmatrix}. \tag{6.1}$$

Then, Algorithm II uses

$$(x,p) = \hat{\mathcal{H}}^{-1}(f,g)$$

$$= (W^*MH^{-1}(B_0u + MWf), -W^*MH^{-*}(MWg + QH^{-1}(B_0u + MWf)). \tag{6.2}$$

where $u = B_0^*Wp \in R^m$ satisfies

$$u + B_0^*H^{-*}QH^{-1}B_0u = -B_0^*H^{-*}(Wg + QH^{-1}MWf) \quad \text{on } R^m. \tag{6.3}$$

In fact let $\xi = M^{\frac{1}{2}}x(t)$, then (2.5) is equivalently written as

$$\frac{d}{dt}\xi(t) = A\xi(t) + Bu(t)$$

with

$$A = M^{-\frac{1}{2}}HM^{-\frac{1}{2}}, \quad B = M^{-\frac{1}{2}}B_0, \quad \tilde{Q} = M^{-\frac{1}{2}}QM^{-\frac{1}{2}}.$$

Then, the corresponding Hamiltonian $\mathcal{H}$ satisfies

$$\mathcal{H}^{-1} = \begin{pmatrix} M^{\frac{1}{2}} & 0 \\ 0 & M^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} H & -B_0B_0^* \\ -Q & -H^* \end{pmatrix}^{-1} \begin{pmatrix} M^{\frac{1}{2}} & 0 \\ 0 & M^{\frac{1}{2}} \end{pmatrix}$$

and thus from (6.1)

$$\hat{\mathcal{H}}^{-1} = \begin{pmatrix} W^*M & 0 \\ 0 & W^*M \end{pmatrix} \begin{pmatrix} H & -B_0B_0^* \\ -Q & -H^* \end{pmatrix}^{-1} \begin{pmatrix} MW & 0 \\ 0 & MW \end{pmatrix}$$

which results in (6.2)–(6.3). Note that Formula (6.2)–(6.3) uses the same basic (required) operations as (3.1)–(3.2).

Next, we claim that $\hat{\mathcal{H}}^{-1}$ is Hamiltonian. In fact,

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \hat{\mathcal{H}}^{-1} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} W^* M^{\frac{1}{2}} & 0 \\ 0 & W^* M^{\frac{1}{2}} \end{pmatrix} \mathcal{H}^{-1} \begin{pmatrix} M^{\frac{1}{2}} W & 0 \\ 0 & M^{\frac{1}{2}} W \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$= \begin{pmatrix} W^* M^{\frac{1}{2}} & 0 \\ 0 & W^* M^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{H}^{-1} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$$

$$= - \begin{pmatrix} W^* M^{\frac{1}{2}} & 0 \\ 0 & W^* M^{\frac{1}{2}} \end{pmatrix} (\mathcal{H}^{-1})^* \begin{pmatrix} M^{\frac{1}{2}} W & 0 \\ 0 & M^{\frac{1}{2}} W \end{pmatrix} = -(\hat{\mathcal{H}}^{-1})^*$$

Thus, the Potter theory [19, 20] validates Algorithm II.

6.1 Spectral Approximations

Let λ be an eigenvalue of $\mathcal{H}^{-1}$ and $\Gamma = \{z \in C : |z - \lambda| = \delta\}$ is a counterclockwise oriented curve in $\rho(\mathcal{H}^{-1})$ and isolates λ and define the spectral projection P:

$$P = \frac{1}{2\pi i} \int_{\Gamma} (z\, I - \mathcal{H}^{-1})^{-1}\, dz \tag{6.4}$$

Let $\mathcal{H}_n^{-1}$ be a (finite rank) approximating sequence of $\mathcal{H}^{-1}$. For example,

$$\mathcal{H}_n^{-1} = \tilde{P}_n (\mathcal{H}^{N_n})^{-1} \tilde{P}_n,$$

where $\tilde{P}_n$ is the orthogonal projection on to $X_n \times X_n$ and $\mathcal{H}^{N_n} : (X_{N_n} \times X_{N_n}) \to (X_{N_n} \times X_{N_n})$ approximates $\mathcal{H}$. Let λ be an eigenvalue of $\mathcal{H}^{-1}$ and Δ is the domain enclosed by Γ. Assume the strong stability: for each $z \in \Delta \setminus \{\lambda\}$

$$\mathcal{H}_n^{-1} \phi \to \mathcal{H}^{-1} \phi \text{ for all } \phi \in X \times X$$

and there exists $M = M(z)$ such that

$$|(z\, I - \mathcal{H}_n^{-1})^{-1}| \le M$$

for sufficiently large n. Then, the corresponding spectral projection P_n to $\mathcal{H}_n^{-1}$ is defined for sufficiently large n and

$$|(P - P_n)\phi| \leq \delta M(\Gamma) \max_{z \in \Gamma} |(\mathcal{H}^{-1} - \mathcal{H}_n^{-1})(z\, I - \mathcal{H}^{-1})^{-1}\phi| \tag{6.5}$$

for $\phi \in X \times X$. It follows from [4] that

$$\begin{aligned} &\lim_n [\sigma(\mathcal{H}_n^{-1}) \cap \Delta] = \{\lambda\} \\ &\lim_n \, dim\, P_n(X \times X) \geq dim\, P(X \times X). \end{aligned} \tag{6.6}$$

Moreover, assume the uniform radial convergence: for all $\epsilon > 0$ and every compact subset K of $\rho(\mathcal{H}^{-1})$ there exists an N such that for $n \geq N$

$$\sup_{z \in K} \, r_\sigma([(\mathcal{H}^{-1} - \mathcal{H}_n^{-1})(z\, I - \mathcal{H}^{-1})^{-1}]) \leq \epsilon$$

where $r_\sigma(S)$ denotes the spectral radius of a bounded operator S. It then follows from [4]

$$\lim_n \, dim\, P_n(X \times X) = dim\, P(X \times X). \tag{6.7}$$

For example, if $\mathcal{H}_n^{-1}$ is defined by (6.1) with $W = W_n$ increasing family of the reduced order basis, then it follows from [7] that the strong stability and uniform radial convergence hold and the accuracy estimate of Algorithm II is reduced from (6.5).

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