# Conversion of Dynamic Social Network Stochastic Differential Equation Model to Fokker-Planck Model

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**Abstract**

Stochastic differential equation (SDE) models offer one formulation for introducing uncertainty in human interactions in a dynamic social network model based on static and/or deterministic ordinary differential equation (ODE) models. A coupled SDE system for agent characteristics and connectivities was developed and investigated in [4]. This SDE model (which tacitly assumed instantaneous influence between agents with connectivity) may be improved by including delays in an SDE model or in an equivalent Fokker-Planck (FP) model if such exists. The coupled model of [4] involved discontinuities and did not yield a Markov diffusion process (for which an equivalent Fokker-Planck formulation is possible). In this project we formulate a new smooth vector SDE system and demonstrate that it generates a Markov diffusion process and provides computational results equivalent to those of the earlier model of [4]. We derive an equivalent Fokker-Planck formulation to this new SDE system. Numerical methods to implement the FP model are formulated. This illustrates the disadvantages of including delays in this already complex framework, suggesting the SDE model may, after all, be more tractable for the consideration of delays.

## 1 Introduction

Social network structures can be represented by graphs in which individuals, or agents, are represented by vertices and the connections between agents are represented by edges. Agents are defined by measurable *characteristics*, and the connections between agents can be defined as either unidirectional, thus creating a directed graph, or bidirectional, thus creating an undirected graph. In social network graphs, the relationship between two agents, or a pairwise relationship, is considered to be the most basic structure used in constructing the graph. More complicated social models, even triads including three agents, are created from using multiple pairwise relationships. Research on social networks may be applied to many areas, including social group structure, information sharing, and disease outbreak [5].

Interactions between agents often have uncertain and unpredictable results, and many static and deterministic social network models fail to consider variability and delay. Complex nonlinear systems are more suited for modeling the uncertainty found in human interactions. A dynamic social network model that includes delay and accounts for variability could allow governmental organizations to predict the propagation of information through a terrorist network [2], allow local officials to examine how a communicable disease would spread through their community [7], or enable corporations to simulate scenarios in which their supply chain operates under strained conditions or with perturbations [1].

One area of current research in social networks is dynamics of connections and agents over a time. The model first proposed by [4] utilizes a coupled system of stochastic differential equations (SDE's). Agents are defined by quantitatively described characteristics and the strength of the pairwise relationship, or *degree of connectivity*, joining two agents is determined by these characteristics. It is assumed that agents participate in *homophilic* relationships and that two agents are more likely to have a positive degree of connectivity if their characteristics are similar. The proposed model accounts for changes in agents and relationships through both observable interactions and unobservable events represented as noise.

Another factor in social network dynamics that can be considered is delay. The effect one agent has on another often is not instantaneous; like diseases in a body, new ideas and opinions are subject to a propagation time in an agent's mind before the agent accepts or rejects the new concept. An agent does not pass on ideas, objects, and diseases immediately after they have come in contact with or possession of them. Consider a conversation between two people of differing religious views. It is unlikely that one will immediately convert to the other's faith; however, after repeated contact and reflection by each individual, the two may become more understanding of each others' religions. Adding constant and stochastic delays to the proposed model will result in a more realistic simulation of agent interactions and internal changes in the agents.

The proposed model is built from the model created by [4]. The model is used with the assumption that time, degrees of connectivity, and agent characteristics are continuous. The coupled SDE model proposed in [4] is recreated and used to reproduce previous results. The SDEs are then changed so that the coefficients used in the model are smooth, allowing the model to satisfy the conditions of a diffusion process, and the results of the model with smooth coefficients are compared to the results of the original model to verify that the change in coefficients does not affect the visible behavior of the model. These equations are then converted to an equivalent Fokker-Planck (FP) model. Numerical implementation of the FP model is outlined. Examination of this reveals the difficulty in adding delays using this formulation, and therefore suggests that it is comparably advantageous to study delays in the equivalent SDE model.

# 2 Mathematical Model

## 2.1 Basic Stochastic Differential Equation

Denote the vector of characteristics of agent $i$, $i \in \mathcal{N} = 1, \ldots, N$ at time $t$ by $\mathbf{C}_i(t) \in K \subset \mathbb{R}^m$, where $K$ is a compact constraint set for the values of characteristics and $m$ is the number of characteristics. We define the degree of connectivity of agent $i$ to agent $i'$ at time $t$ by $e(i, i', t) \in \mathbb{R}$. An agent's connectivity to itself $e(i, i, t) = 0\ \forall i, t$ and an edge between $i$ and $i'$ exists if and only if $e(i, i', t) > 0$. The strength of the link of agent $i$ to agent $i'$ is not necessarily the same as the strength of connection of agent $i'$ to agent $i$, so it is possible that $e(i, i', t) \neq e(i', i, t)$. Define $A_i(t) = \{i' \in \mathcal{N} : e(i, i', t) > 0\}$ to be the set of all agents to which agent $i$ is linked at time $t$, and let $|A_i(t)|$ be the number of elements in $A_i(t)$.

The system of coupled stochastic differential equations as defined by [4] used as the starting point in this model are

$$
\begin{aligned}
d\mathbf{C}_i(t) &= \frac{\beta_i}{\mid A_i(t) \mid} \sum_{i' \in A_i(t)} \Big[\mathbf{C}_{i'}(t) - \mathbf{C}_i(t)\Big] dt + \sigma d\mathbf{W}_i^{\mathbf{C}}(t) && (1) \\
de(i, i', t) &= f\Big(\big\|\mathbf{C}_i - \mathbf{C}_{i'}\big\|^2\Big) dt + \gamma d\mathbf{W}_{i,i'}^e(t). && (2)
\end{aligned}
$$

In (1), the constant $\beta_i$ is an agent-specific value that represents the desire of agent $i$ to be more like agents to which it is linked and $\sigma \mathbf{W}_i^{\mathbf{C}}(\cdot)$ is a Wiener process with variance $\sigma^2$. In (2), $f(\xi) = 2e^{-b\xi} - 1$ where the constant $b > 0$ controls the rate at which edges are formed and $\|\cdot\|^2$ denotes the square of the vector norm in $\mathbb{R}^m$, and $\gamma d\mathbf{W}^e_{i,i'}(\cdot)$ is a Wiener process with variance $\gamma^2$.

## 2.2 Stochastic Differential Equation with Smooth Coefficients

The model described by (1) does not fit the definition of a usual diffusion process because the values of $\mid A_i(t) \mid$ take on integer values, allowing for $d\mathbf{C}_i(t)$ to be discontinuous. An alternative model is proposed by [4]:

$$d\mathbf{C}_i(t) = \frac{\beta_i}{\sum_{i' \neq i} e(i,i',t)} \sum_{i' \neq i} e(i,i',t) \Big[ \mathbf{C}_{i'}(t) - \mathbf{C}_i(t) \Big] dt + \sigma d\mathbf{W}^{\mathbf{C}}_i(t) \qquad (3)$$

The sum $\sum_{i' \neq i} e(i,i',t)$ is continuous for $\gamma = 0$, taking values in $\mathbb{R}$. While this model has smooth coefficients, it fails to follow the assumption of homophily and enables agents who have a non-positive connectivity to a particular agent to influence the characteristics of that particular agent. This in turn causes the degrees of connectivity $e(i,i',t(n)) \to \infty$ as $n \to \infty$, resulting in single cluster scenarios for any value of $b$ in the deterministic case, as illustrated by Figure 1.

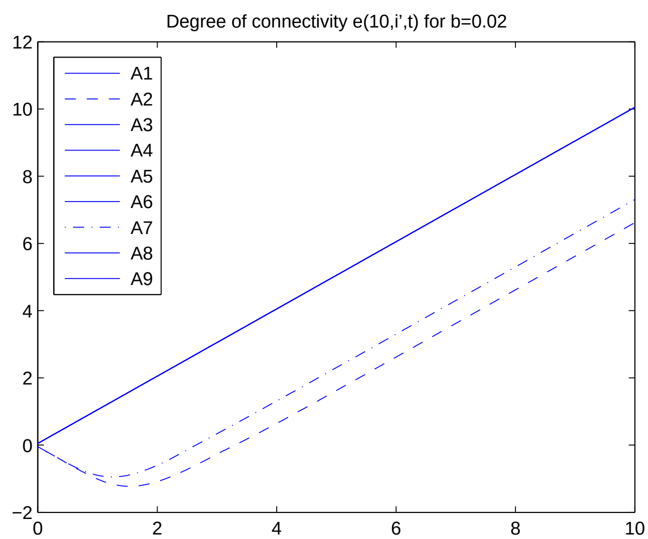


Figure 1: Degree of connectivity $e(10,i',t), i' \neq 10$ of agent 10 with other agents for $b = 0.02$ using smooth model proposed in [4].

Consider the agents $i', i' \neq 2, 7$ and agent 2 in the case study used in [4] in which $\mathbf{C}_{i'}(0) = (10, 10)$, $\mathbf{C}_2(0) = (-10, -10)$ and $\mathbf{C}_7(0) = (-5, -5)$. Assume that these agents' characteristics remain the same for several time steps in the simulation while the degrees of connectivity $e(i', 2, t)$ and $e(2, i', t)$ become negative numbers so that $e(2, i', t_n) = -1.5$ at some time $t_n$. Ignoring agent 7, (3) would reduce in the deterministic case to

$$\begin{aligned}
d\mathbf{C}_2(t_n) &= \frac{\beta}{\sum_{i' \neq 2} e(2, i', t_n)} \sum_{i' \neq 2} e(2, i', t_n) \Big[ \mathbf{C}_{i'}(t_n) - \mathbf{C}_2(t_n) \Big] dt \\
&\quad + \sigma d\mathbf{W}_i^{\mathbf{C}}(t_n) \\
d\mathbf{C}_2(t_n) &= \frac{1}{-12} \sum_{i' \neq 2} -1.5 \left[ \begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} -10 \\ -10 \end{bmatrix} \right] dt + 0 d\mathbf{W}_i^{\mathbf{C}}(t_n) \\
d\mathbf{C}_2(t_n) &= \begin{bmatrix} 20 \\ 20 \end{bmatrix} dt
\end{aligned}$$

As noted before, agents to which agent 2 has a negative degree of connectivity should not alter the characteristic values of agent 2, but this rule is clearly violated by this proposed smooth model.

To avoid agents affecting each other at values of $e(i, i', t) \leq 0$, we excluded these connectivities in the sum in the SDE for $d\mathbf{C}_i(t)$, giving

$$d\mathbf{C}_i(t) = \frac{\beta_i}{\sum_{\substack{i' \neq i \\ i' \in A_i(t)}} e(i, i', t)} \sum_{\substack{i' \neq i \\ i' \in A_i(t)}} e(i, i', t) \Big[ \mathbf{C}_{i'}(t) - \mathbf{C}_i(t) \Big] dt + \sigma d\mathbf{W}_i^{\mathbf{C}}(t). \tag{4}$$

In this form, the model does represent a diffusion process and does not violate the assumptions of the original model in [4]. In addition, this model can be shown to qualitatively reproduce the results from [4] as discussed in 4.

Like in (1), $A_i(t)$ is the set of agents $i' \neq i$ that have connectivity $e(i, i', t) > 0$. Instead of explicitly checking for $i' \in A_i(t)$, which decreases the computational efficiency, an equivalent expression for (4) is

$$d\mathbf{C}_i(t) = \frac{\beta_i}{\sum_{i' \neq i} \varphi} \sum_{i' \neq i} \varphi \Big[ \mathbf{C}_{i'}(t) - \mathbf{C}_i(t) \Big] dt + \sigma d\mathbf{W}_i^{\mathbf{C}}(t) \tag{5}$$

$$de(i, i', t) = f\Big( \big\| \mathbf{C}_i - \mathbf{C}_{i'} \big\|^2 \Big) dt + \gamma d\mathbf{W}^e_{i,i'}(t). \tag{6}$$

where $\varphi = \frac{1}{2}\big(e(i,i',t) + |e(i,i',t)|\big)$. As no agents $i'$ are to affect agent $i$ if $e(i,i',t) \leq 0$, it is assumed that $\frac{\beta_i}{\sum_{i' \neq i} \varphi} = 0$ if $\sum_{i' \neq i} \varphi = 0$. Thus this model is defined if single-agent clusters arise during a simulation. This model maintains the requirement of *homophily* between agents and prevents agents with non-positive degrees of connectivity affecting other agents. When used to run simulations of the test cases first run with (1), similar results are produced.

# 3 Numerical Methods

## 3.1 Numerical Scheme for Stochastic Models

We continue to use the stochastic analogue of the classical fourth-order Runge-Kutta method to solve the system of differential equations as implemented by [4]. Approximating $d\mathbf{C}_i(t)$ remains nearly the same:

$$\mathbf{C}_i(t_n) = \mathbf{C}_i(t_{n-1}) + p_0\mathbf{F}_0 h + p_1\mathbf{F}_1 h + p_2\mathbf{F}_2 h + p_3\mathbf{F}_3 h + \sigma\Delta\mathbf{W}_{in}^{\mathbf{C}} \qquad (7)$$

where

$$\begin{aligned}
\mathbf{F}_0 &= \mathbf{g}\Big(t_{n-1}, \mathbf{C}_i(t_{n-1})\Big) \\
\mathbf{C}_i^{(1)}(t_{n-1}) &= \mathbf{C}_i(t_{n-1}) + \frac{1}{2}\mathbf{F}_0 h + \frac{1}{2}\sigma\Delta\mathbf{W}_{in}^{\mathbf{C}} \\
\mathbf{F}_1 &= \mathbf{g}\Big(t_{n-1} + \frac{1}{2}h, \mathbf{C}_i^{(1)}(t_{n-1})\Big) \\
\mathbf{C}_i^{(2)}(t_{n-1}) &= \mathbf{C}_i(t_{n-1}) + \frac{1}{2}\mathbf{F}_1 h + \frac{1}{2}\sigma\Delta\mathbf{W}_{in}^{\mathbf{C}} \\
\mathbf{F}_2 &= \mathbf{g}\Big(t_{n-1} + \frac{1}{2}h, \mathbf{C}_i^{(2)}(t_{n-1})\Big) \\
\mathbf{C}_i^{(3)}(t_{n-1}) &= \mathbf{C}_i(t_{n-1}) + \mathbf{F}_2 h + \sigma\Delta\mathbf{W}_{in}^{\mathbf{C}} \\
\mathbf{F}_3 &= \mathbf{g}\Big(t_{n-1} + h, \mathbf{C}_i^{(3)}(t_{n-1})\Big),
\end{aligned}$$

$$p_0 = p_3 = \frac{1}{6} \text{ and } p_1 = p_2 = \frac{1}{3}.$$

The value of $\sigma$ was included in the calculation of $\mathbf{C}_i^{(1)}(t_{n-1})$, $\mathbf{C}_i^{(2)}(t_{n-1})$, and $\mathbf{C}_i^{(3)}(t_{n-1})$ to scale the value of $\Delta\mathbf{W}_{in}^{\mathbf{C}}$ so that it reflected the value of $\sigma\Delta\mathbf{W}_{in}^{\mathbf{C}}$ used in (7). The definition of $\mathbf{g}(t, \mathbf{C}_i(t))$ is dependent upon the model. For the original SDE, $\mathbf{g}(t, \mathbf{C}_i(t)) = \frac{\beta_i}{|A_i(t)|} \sum_{i' \in A_i(t)} \Big[\mathbf{C}_{i'}(t) - \mathbf{C}_i(t)\Big]$, and for the modified SDE, $\mathbf{g}(t, \mathbf{C}_i(t)) = \frac{\beta_i}{\sum_{i' \neq i} \varphi} \sum_{i' \neq i} \varphi \Big[\mathbf{C}_{i'}(t) - \mathbf{C}_i(t)\Big]$.

Equation (2) may be solved in a simplified manner since its result is not dependent upon the value of the degree of connectivity $e(i, i', t)$. This can be expressed as

$$\begin{aligned} e(i, i', t_n) &= e(i, i', t_n) + p_0 f\Big(\big\|\mathbf{C}_i(t_{n-1}) - \mathbf{C}_{i'}(t_{n-1})\big\|^2\Big)h \\ &\quad + (p_1 + p_2) f\Big(\big\|\mathbf{C}_i(t_{n-1} + \frac{1}{2}h) - \mathbf{C}_{i'}(t_{n-1} + \frac{1}{2}h)\big\|^2\Big)h \\ &\quad + p_3 f\Big(\big\|\mathbf{C}_i(t_n) - \mathbf{C}_{i'}(t_n)\big\|^2\Big)h + \gamma \Delta \mathbf{W}^e_{in}. \end{aligned} \tag{8}$$

Here $\Delta \mathbf{W}^e_{in} = \mathbf{W}^e_{i,i'}(t_n) - \mathbf{W}^e_{i,i'}(t_{n-1})$ are standard Wiener increments with the distribution $\mathcal{N}(0, h)$ where $h = \Delta t_n$. We assume that the value of $\mathbf{C}_i(t_n + \frac{1}{2}h)$ can be approximated linearly from $\mathbf{C}_i(t_{n-1})$ to $\mathbf{C}_i(t_n)$ when $\Delta t_n$ is small and therefore $\mathbf{C}_i(t_{n-1} + \frac{1}{2}h)$ may be approximated by $\frac{\mathbf{C}_i(t_{n-1}) + \mathbf{C}_i(t_n)}{2}$.

# 4 Numerical Simulations

We use the same case study concerning 10 agents as in [4] over a time interval of $t \in [0, 10]$. To initialize the connectivity values consistent with the characteristics of the agents, we calculate $e(i, i', 0) = f\Big(\big\|\mathbf{C}_i(0) - \mathbf{C}_{i'}(0)\big\|^2\Big)h$. This will cause all connectivities to start with a nonzero value. Consistent with [4], we restrict each characteristic to take values in $[-10, 10]$.

## 4.1 Basic Stochastic Formulation

### 4.1.1 Deterministic Case

The reproduction of the existing model performs similarly to that of [4] in the deterministic case (when $\sigma = 0$ and $\theta = 0$). By properly implementing (2), the values of $b$ that determined the number of clusters created after the same time as used in [4] were found to be different. One cluster formed when $b \in (0, 0.00155)$; two clusters formed when $b \in [0.00155, 0.01390)$; and three clusters formed when $b \in [0.01390, \infty)$. Two agents are members of different clusters if their connectivities $e(i, i', t) \leq 0$ and $e(i', i, t) \leq 0$. The clusters are identical to those described in [4]: the one cluster scenarios had a result of one cluster of all 10 agents; the two cluster scenarios had a result of one cluster of agents 2 and 7 and a second cluster of all other agents; and the three cluster scenario had a result of one cluster of only agent 2, one cluster of only agent 7, and one cluster of all other agents, and the degrees of connectivity between agents changed at a rate identical to the rates found in [4].

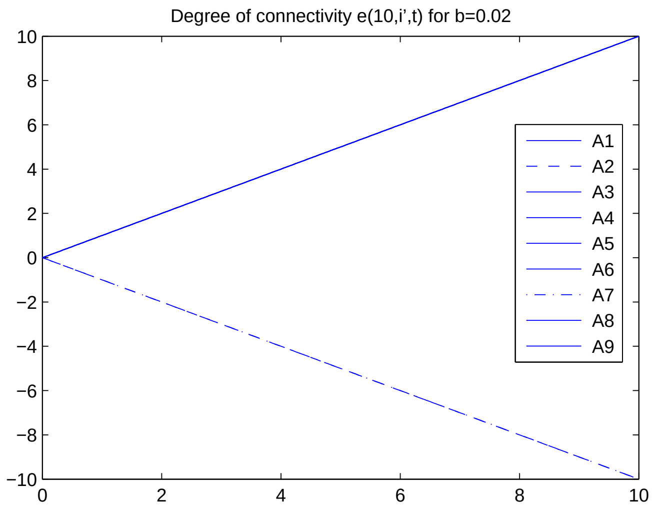


Figure 2: Degrees of connectivity $e(10, i', t), i' \neq 10$ of agent 10 with other agents for $b = 0.02$ (three cluster scenario).

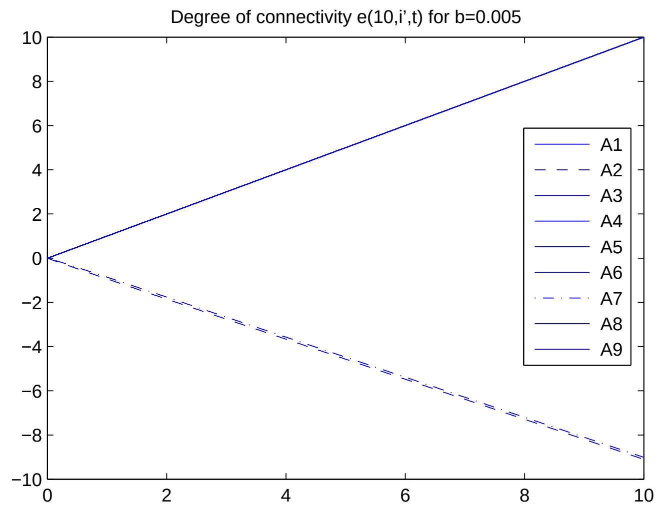


Figure 3: Degree of connectivity $e(10, i', t), i' \neq 10$ of agent 10 with other agents for $b = 0.005$ (two cluster scenario).

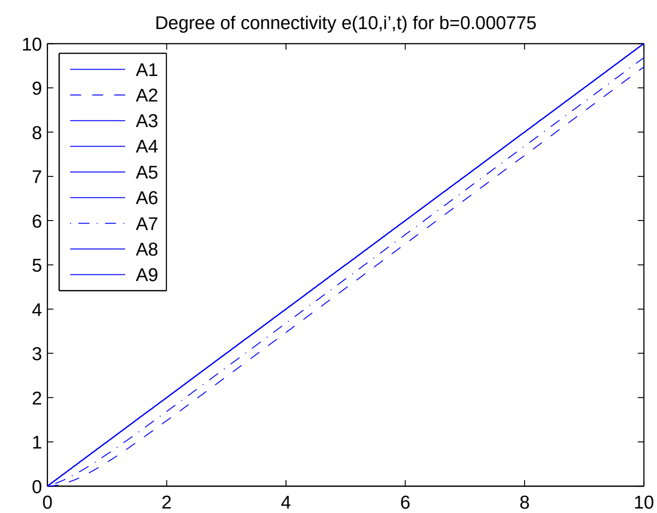


Figure 4: Degree of connectivity $e(10, i', t), i' \neq 10$ of agent 10 with other agents for $b = 0.000775$ (one cluster scenario).

### 4.1.2 Stochastic Case

Four regimes are defined in [4] into which the results of a simulated trial may be organized. These four regimes are labeled essentially deterministic, noise enriched, noise enlarged (with two- and three-cluster possibilities), and noise dominated.

To investigate the occurrence of the regimes defined in [4] using a constant $b$ and variable $\sigma$ and $\gamma$ we ran simulations, with time step $h = \Delta t_n = 0.05$, for selected values of $\sigma \in [0, 40]$ and $\gamma \in [0, 12]$ with fixed $b = 0.000775$. We carried out 100 simulations for each pair $(\sigma_k, \gamma_l)$ in a partition with values $\sigma_k = k = 0, 1, \ldots, 40; \gamma_l = 0.25l, l = 0, 1, \ldots, 48$. To further investigate the relationship between $\sigma$, $\delta$, and the number of trials that produced essentially deterministic results, we also carried out 100 simulations for each pair $(\sigma_k, \gamma_l)$ in a partition with values $\sigma_k = k = 0, 1, \ldots, 40; \gamma_l = 0.0025l, l = 0, 1, \ldots, 48$. The trends seen in Figures 10 - 13 of [4] are reproduced here albeit for different values of $\sigma$ and $\gamma$. Again, the surface of the noise-dominated regime is almost the complement of the surface of the noise-enriched regime with the exception of the occurrence of some essentially deterministic cases at low values of $\sigma$ and $\gamma$. This indicates that it is highly unlikely that the inclusion of noise in the model will cause the simulation to reach a two- or three-clustered scenario if a $b$ in the range of values that will produce a one-clustered scenario is selected.

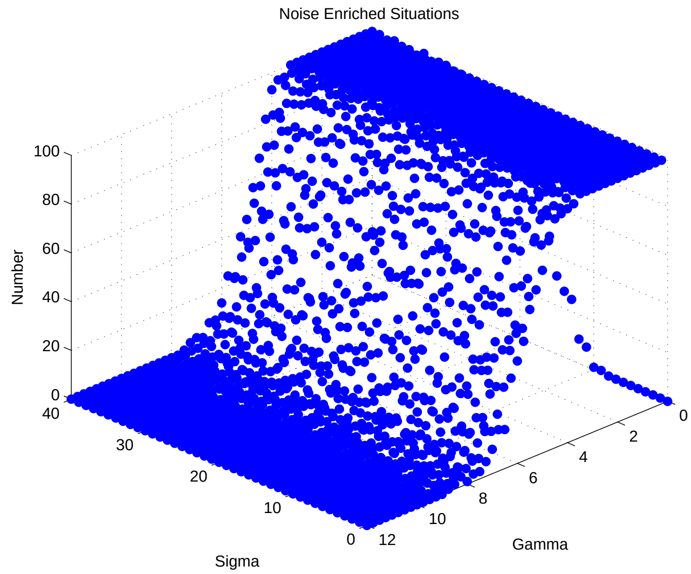


Figure 5: The $(\sigma, \gamma)$ noise-enriched surface along $\sigma \in [0, 40], \gamma \in [0, 12]$ plotted as a number out of 100 simulations.

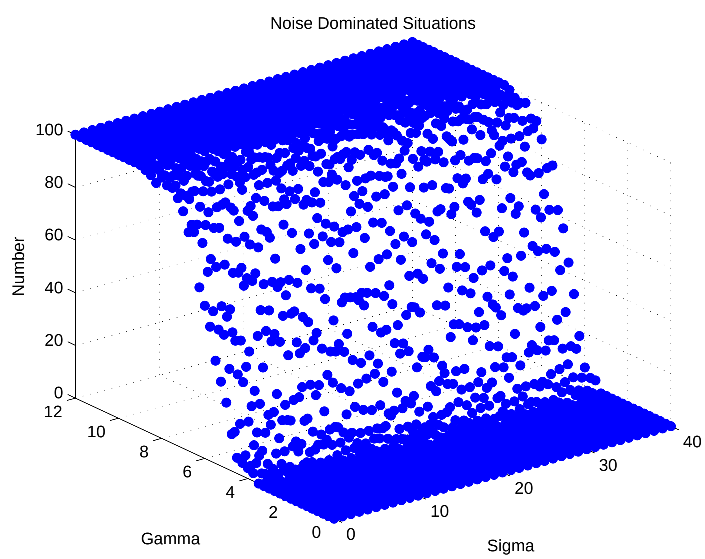


Figure 6: The $(\sigma, \gamma)$ noise-dominated surface along $\sigma \in [0, 40], \gamma \in [0, 12]$ plotted as a number out of 100 simulations.

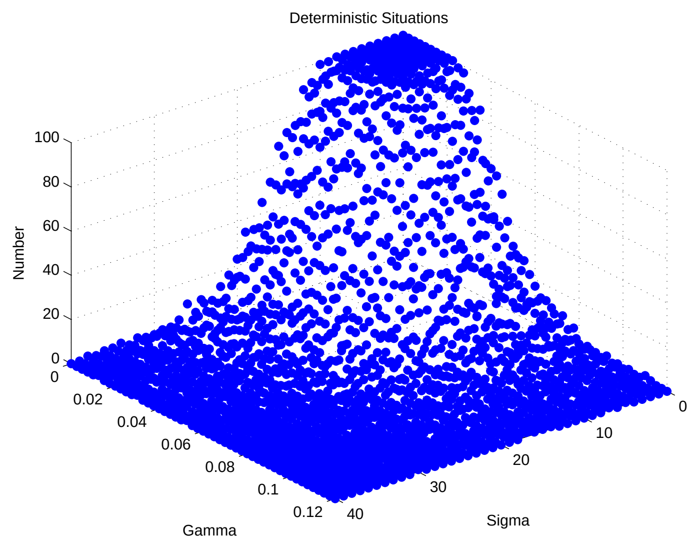


Figure 7: The $(\sigma, \gamma)$ essentially deterministic surface along $\sigma \in [0, 40], \gamma \in [0, 0.12]$ plotted as a number out of 100 simulations.

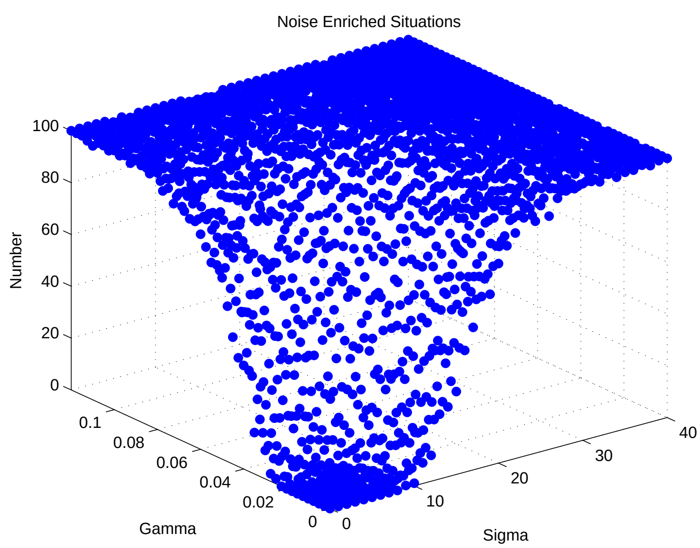


Figure 8: The $(\sigma, \gamma)$ noise enriched surface along $\sigma \in [0, 40], \gamma \in [0, 0.12]$ plotted as a number out of 100 simulations.

### 4.2 Stochastic Model with Smooth Coefficients

The coupled SDE model with smooth coefficients are also used to simulate the case denoted above and produced visually identical results. In the deterministic case, the values of $b$ that control clustering behavior are identical to those found in the deterministic case of the basic SDE system; that is, one cluster formed when $b \in (0, 0.00155)$; two clusters formed when $b \in [0.00155, 0.01390)$; and three clusters formed when $b \in [0.01390, \infty)$. The sample distributions of regimes also appear to be very similar to those established in [4].

To verify that the model exhibited random behavior when $\sigma \neq 0$ and $\gamma \neq 0$, a simulation of the one-cluster scenario was run using the parameters $\sigma = 5$ and $\gamma = 1$. The general behavior exhibited by the deterministic case is also evident in this stochastic case. Note that the trend of increasing connectivities in a one-cluster deterministic case, pictured in Figure 9, is also reflected in the one-cluster stochastic case pictured in Figure 13. Also, the tendency for characteristic values in the one-cluster deterministic case to approach a certain value between all the characteristic values, shown in Figure 10, is also present in the one-cluster stochastic case, as illustrated by Figure 14.

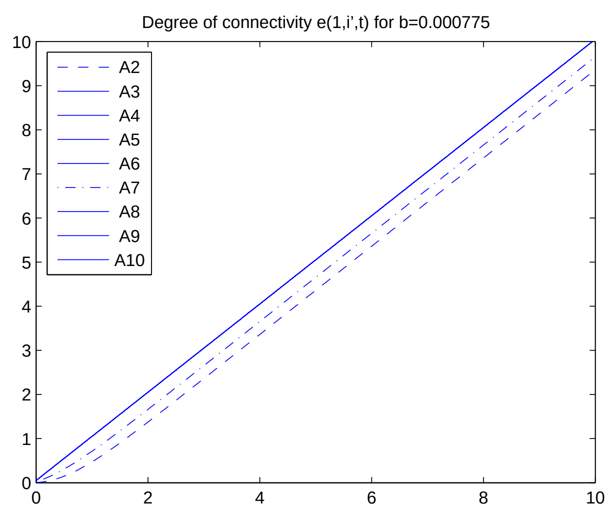


Figure 9: Degree of connectivity $e(1, i', t), i' \neq 1$ of agent 1 with other agents for $b = 0.000775$, $\sigma = 0$, and $\gamma = 0$ using the smooth coefficient model (one cluster scenario, deterministic case).

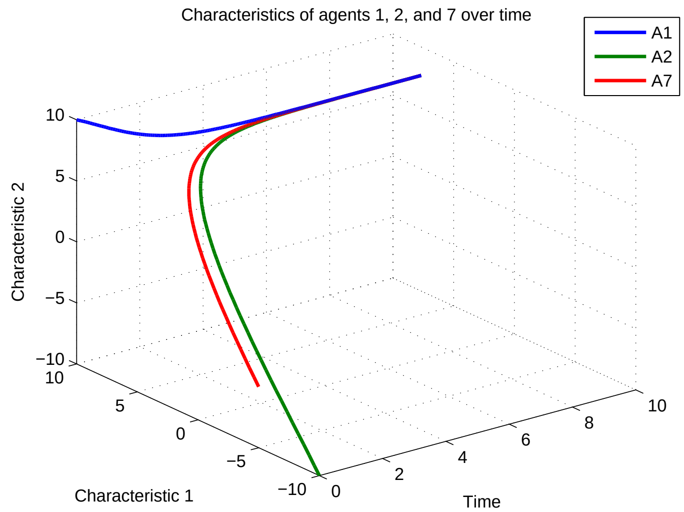


Figure 10: Characteristic values of agents 1, 2, and 7 over $t \in [0, 10]$ for $b = 0.000775$, $\sigma = 0$, and $\gamma = 0$ (one cluster scenario, deterministic case).

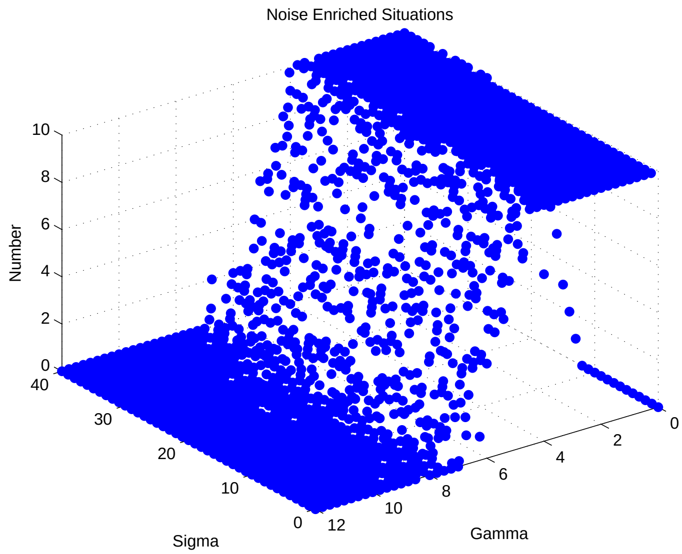


Figure 11: The $(\sigma, \gamma)$ noise enriched surface along $\sigma \in [0, 40], \gamma \in [0, 12]$ plotted as a number out of 10 simulations using the smooth coefficient model.

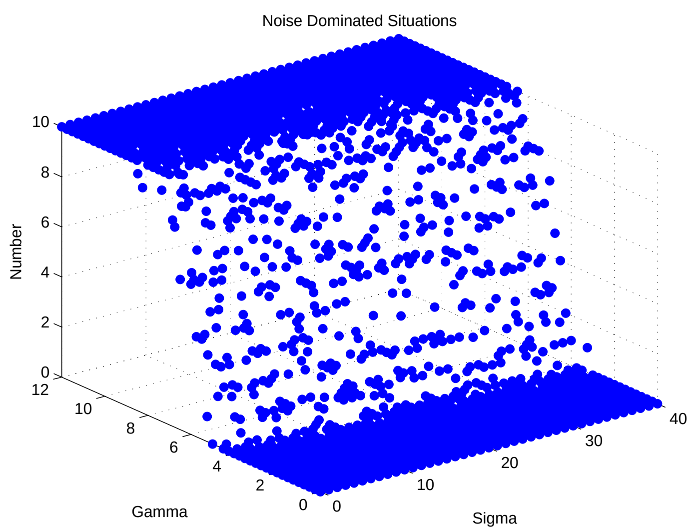


Figure 12: The $(\sigma, \gamma)$ noise dominated surface along $\sigma \in [0, 40], \gamma \in [0, 12]$ plotted as a number out of 10 simulations using the smooth coefficient model.

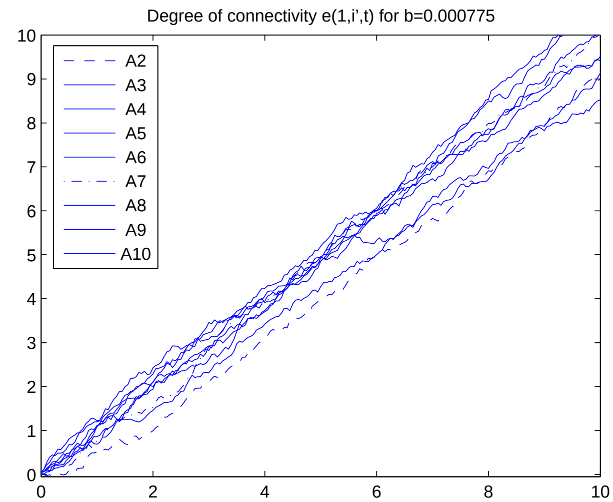


Figure 13: Degree of connectivity $e(1, i', t), i' \neq 1$ of agent 1 with other agents for one trial using the parameters $b = 0.000775$, $\sigma = 5$, and $\gamma = 1$ using the smooth coefficient model (one cluster scenario, stochastic case).

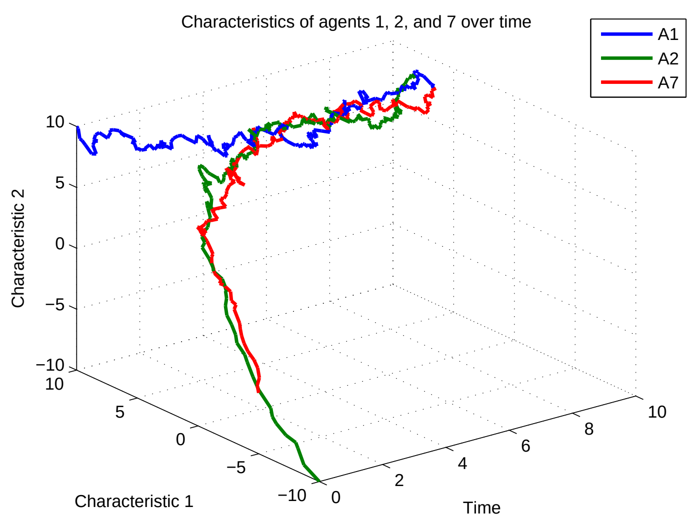


Figure 14: Characteristic values of agents 1, 2, and 7 over $t \in [0, 10]$ for one trial using the parameters $b = 0.000775$, $\sigma = 5$, and $\gamma = 1$ (one cluster scenario, stochastic case).

## 5 Fokker-Planck Equivalent Formulation

With the establishment of the smooth SDE model and its faithfulness to the SDE model first proposed by [4], the development of an equivalent FP model may proceed. Let $\{X(t)\}$ be a stochastic process, with $t \in [t_0, \infty)$, and $X(t)$ a random variable continuous in time and state, $X(t) \in (-\infty, \infty)$. Let $X(t)$ satisfy the *Ito SDE*

$$dX(t) = \alpha(X(t),t)dt + \beta(X(t),t)dW(t), \tag{9}$$

where $dW(t)$ represents an infinitesimal Wiener increment, and $W(t)$ is the standard Wiener process. The mean and variance of $X(t)$ is then $\alpha(X(t),t)$ and $\beta^2(X(t),t)$, respectively. Conditions (10) - (12)

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} |y-x|^{\delta} p(y, t+\Delta t; x, t) dy = 0, \qquad \delta > 2 \tag{10}$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y-x) p(y, t+\Delta t; x, t) dy = \alpha(x,t), \tag{11}$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y-x)^2 p(y, t+\Delta t; x, t) dy = \beta^2(x,t), \tag{12}$$

establish the smoothness of coefficients $\alpha(X(t),t)$ and $\beta(X(t),t)$. Should these conditions be satisfied, it can be shown via the application of the Chapman-Kolmogorov equation and Taylor expansion of a test function, that $X(t)$ is a diffusion process. Equivalently, this means that $X(t)$ satisfies the Kolmogorov equations with drift coefficient $\alpha$ and diffusion coefficient $\beta^2$. The corresponding forward Kolmogorov (or Fokker-Planck) equation is then

$$\frac{\partial p}{\partial t} = -\frac{\partial(\alpha(x,t)p)}{\partial x} + \frac{1}{2}\frac{\partial^2(\beta^2(x,t)p)}{\partial x^2}. \tag{13}$$

In equation (13), $p$ is the pdf of the stochastic process $X(t)$, and the transition from state $x$ at time $t$ to state $y$ at time $s$ is then $p(y,s;x,t)$.

Generalizing (13) to $M$ dimensions as in [8], we find the forward Kol - mogorov/Fokker-Planck is given by

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{M} \frac{\partial}{\partial x_i}\left[S_i p\right] + \frac{1}{2}\sum_{i,j=1}^{M} \frac{\partial^2}{\partial x_i \partial x_j}\left[S_{ij} p\right], \tag{14}$$

where $\mathbf{X} = (X_1, \ldots, X_M)$ is a stochastic process such that

$$S_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\Delta X_i(t) | X_i(t) = x_i],$$

and

$$S_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[\Delta X_i(t) \Delta X_j(t) | X_i(t) = x_i, X_j(t) = x_j].$$

If we let $C_i^k(t)$ represent the $k$th element of $\mathbf{C}_i(t)$, and apply (14), the forward Kolmogorov formulation corresponding to model (5) - (6) is then

$$\begin{aligned}
\frac{\partial p}{\partial t} = &- \sum_{i=1}^{N} \frac{\partial}{\partial C_i^1} \left[ \left( \frac{\beta_i}{\sum_{i' \neq i} \psi} \sum_{i' \neq i} \psi \left[ C_{i'}^1(t) - C_i^1(t) \right] \right) p \right] \\
&- \sum_{i=1}^{N} \frac{\partial}{\partial C_i^2} \left[ \left( \frac{\beta_i}{\sum_{i' \neq i} \psi} \sum_{i' \neq i} \psi \left[ C_{i'}^2(t) - C_i^2(t) \right] \right) p \right] \\
&- \ldots - \sum_{i=1}^{N} \frac{\partial}{\partial C_i^m} \left[ \left( \frac{\beta_i}{\sum_{i' \neq i} \psi} \sum_{i' \neq i} \psi \left[ C_{i'}^m(t) - C_i^m(t) \right] \right) p \right] \\
&+ \sum_{i=1}^{N} \sum_{i'=1}^{N} \frac{\partial}{\partial e_{i,i'}} \left[ f(||\mathbf{C}_{i'}(t) - \mathbf{C}_i(t)||^2) p \right] \\
&+ \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2 (\sigma^2 p)}{\partial (C_i^1)^2} + \sum_{i=1}^{N} \frac{\partial^2 (\sigma^2 p)}{\partial (C_i^2)^2} + \ldots + \sum_{i=1}^{N} \frac{\partial^2 (\sigma^2 p)}{\partial (C_i^m)^2} + \sum_{i=1}^{N} \sum_{i'=1}^{N} \frac{\partial^2 (\gamma^2 p)}{\partial e_{i,i'}^2}.
\end{aligned}$$

Here $p$ is the joint transition probability density function of $p(\mathbf{C}_i, e_{i,i'}, s, \tilde{\mathbf{C}}_i, \tilde{e}_{i,i'}, t, )$. A more concise representation of the above equation is given by

$$\begin{aligned}
\frac{\partial p}{\partial t} = &- \sum_{i=1}^{N} \sum_{k=1}^{m} \frac{\partial}{\partial C_i^k} \left[ \left( \frac{\beta_i}{\sum_{\phi_i} e(i, i', t)} \sum_{\phi_i} e(i, i', t) \left[ C_{i'}^k(t) - C_i^k(t) \right] \right) p \right] \\
&+ \sum_{i=1}^{N} \sum_{i'=1}^{N} \frac{\partial}{\partial e_{i,i'}} \left[ f(||\mathbf{C}_{i'}(t) - \mathbf{C}_i(t)||^2) p \right] \\
&+ \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{m} \frac{\partial^2 (\sigma^2 p)}{\partial (C_i^k)^2} + \sum_{i=1}^{N} \sum_{i'=1}^{N} \frac{\partial^2 (\gamma^2 p)}{\partial e_{i,i'}^2}.
\end{aligned} \tag{15}$$

### 5.1 Finite Difference Scheme for Fokker-Planck Model

The finite difference approximation formulated by [6] preserves the properties of the FP model. This numerical scheme yields a non-negative solution for all values of $\bar{x}(t)$, where we take $x$ to represent a vector of the model state variables $x(t) = (\mathbf{C}_1(t), ..., \mathbf{C}_{10}(t), e(1,1,t), ..., e(10,10,t))^T$ at time $t$. This method conserves the area of the p.d.f. $p$, and is known to be relatively stable. Additionally, this method allows mesh points to be more widely spaced than in other comparable methods.

Let $\Delta\bar{x} = (x_{max} - x_{min})/R$, where $x_{max}$ and $x_{min}$ are the upper and lower boundaries of the mesh $\bar{x}$ and $R$ is the number of intervals into which the domain of $x(t)$ shall be divided. Let $\Delta t = T/S$, where $S$ is the number of time steps. The mesh points shall be given by $x_r = x_{min} + r\Delta x$, $r = 0, 1, 2, \ldots, R$ and $t_s = s\Delta t$, $s = 0, 1, 2, \ldots, S$. The midpoint between two mesh points can be interpolated linearly by $x_{r+\frac{1}{2}} = \frac{x_r + x_{r+1}}{2}$. Define the following functions for any $x \in \bar{x}$

$$\begin{aligned} G(x,t) &= \sum_{i=1}^{N}\sum_{k=1}^{m} \frac{-\beta}{\sum_{i' \neq i}\psi} \sum_{i' \neq i} \psi\left[\mathbf{C}_{i'}^k(t) - \mathbf{C}_i^k(t)\right] \\ &\quad + \sum_{i=1}^{N}\sum_{i'=1}^{N} 2\left[\exp(-b||\mathbf{C}_i(t) - \mathbf{C}_{i'}(t)||^2) - 1\right], \qquad (16) \\ F(x,t) &= Gp + \frac{\sigma^2}{2}\sum_{i=1}^{N}\sum_{k=1}^{m}\frac{\partial p}{\partial C_i^k} + \frac{\gamma^2}{2}\sum_{i=1}^{N}\sum_{i'=1}^{N}\frac{\partial p}{\partial e_{i,i'}}. \qquad (17) \end{aligned}$$

The approximation (17) may be utilized to obtain

$$\frac{p_r^{s+1} - p_r^s}{\Delta t} = \frac{F_{r+\frac{1}{2}}^{s+1} - F_{r-\frac{1}{2}}^{s+1}}{\Delta x} \qquad (18)$$

where $p_r^{s+1} - p_r^s$ denotes the change of the p.d.f. $p$ w.r.t. the change in $x \in \bar{x}$ from one mesh point to the proceeding mesh point. Letting $\lambda = \sigma$ if $x \in \bar{\mathbf{C}}$ and $\lambda = \gamma$ if $x \in \bar{e}$, $F_{r+\frac{1}{2}}^{s+1}$ may be defined as

$$\begin{aligned} F_{r+\frac{1}{2}}^{s+1} &= G_{r+\frac{1}{2}}^{s+1} p_{r+\frac{1}{2}}^{s+1} + \frac{\lambda^2}{2}\frac{p_{r+1}^{s+1} - p_r^{s+1}}{\Delta x} \qquad (19) \\ &= G_{r+\frac{1}{2}}^{s+1}\left[\delta_r^{s+1} p_{r+1}^{s+1} + (1 - \delta_r^{s+1}) p_r^{s+1}\right] + \frac{\lambda^2}{2}\frac{p_{r+1}^{s+1} - p_r^{s+1}}{\Delta x} \\ &= \left[\delta_r^{s+1} G_{r+\frac{1}{2}}^{s+1} + \frac{\lambda^2}{2\Delta x}\right] p_{r+1}^{s+1} + \left[(1 - \delta_r^{s+1}) G_{r+\frac{1}{2}}^{s+1} - \frac{\lambda^2}{2\Delta x}\right] p_r^{s+1} \end{aligned}$$

$$\text{where} \quad \delta_r^{s+1} = \tfrac{1}{\tau_r^{s+1}} - \tfrac{1}{\exp(\tau_r^{s+1})-1}, \quad \tau_r^{s+1} = \frac{G_{r+\frac{1}{2}}^{s+1}\Delta x}{\lambda^2}$$

To preserve the zero-flux boundary conditions of $F(\bar{x}|x_{min} \in \bar{x}, t_{s+1}) = 0$ and $F(\bar{x}|x_{max} \in \bar{x}, t_{s+1}) = 0$, let $F_{\frac{-1}{2}}^{s+1} = 0$ and $F_{r+\frac{1}{2}}^{s+1} = 0$. Equation (18) can be solved using the system

$$-\phi_{1,r}^{s+1} p_{r+1}^{s+1} + \phi_{0,r}^{s+1} p_r^{s+1} - \phi_{-1,r}^{s+1} p_{r-1}^{s+1} = p_r^s \tag{20}$$

where

$$\begin{aligned}
\phi_{1,r}^{s+1} &= \frac{\Delta t}{\Delta x}\left[\frac{\lambda^2}{4\Delta x} - \delta_r^{s+1} G_{r+\frac{1}{2}}^{s+1}\right] \\
\phi_{0,r}^{s+1} &= 1 + \frac{\Delta t}{\Delta x}\left[(1-\delta_r^{s+1}) G_{r+\frac{1}{2}}^{s+1} - \delta_{r+1}^{s+1} G_{r-\frac{1}{2}}^{s+1}\right] + \frac{\lambda^2 \Delta t}{2\Delta x^2} \\
\phi_{-1,r}^{s+1} &= \frac{\Delta t}{\Delta x}\left[(1-\delta_r^{s+1}) G_{r+\frac{1}{2}}^{s+1} + \frac{\lambda^2}{4\Delta x}\right]
\end{aligned}$$

While it is not conceptually daunting to implement this numerical method there are some difficulties. Note the multiple summations in the expressions for $F(x,t)$ and $G(x,t)$, some of which involve the calculation of partial derivatives. These summations involve many terms and therefore, operations, which are due to the high dimension of our SDE model. Other lower dimension SDEs have an equivalent FP formulation which do not require as many operations to compute these quantities. In those cases, it may be advantageous to use the FP formulation to study the behavior of the model. However, we opt to use the SDE framework to study additional complexities such as delays in the context of network models, as networks inherently are of a large dimension.

## 6 Concluding Remarks

The coupled SDE model proposed in [4] was converted to an SDE model with smooth coefficients. The results found in [4] were duplicated using this modified version of the model. Equation (2) orginally possessed smooth coefficients, and to change (1) to be composed of only smooth coefficients involved scaling the difference in two agents' characteristic values by the connectivity between those two agents if the connectivity is positive. Other methods of smoothing the coefficients were considered but did not fulfill the requirement of homophily between agents, thus violating the original assumptions of the model. Changing the system (1)-(2) to (5)-(6) we found no discernable change in model behavior in both deterministic and stochastic cases and no change in the effect of parameters $\beta$ and $b$ or variables $\sigma$ and $\gamma$ on the system.

With smooth coefficients, the SDE fulfills the definition of a diffusion process, thus allowing an equivalent multidimensional Fokker-Planck (FP) model to be formulated from the coupled SDE system (5) - (6). Use of the FP model is nearly ubiquitous in many scientific fields, and current research is being performed to further develop and understand the FP equation [3], [5]. The FP formulation possesses drift coefficients as described in (16) and diffusion coefficients equal to $\sigma$ in $\mathcal{N}m$ dimensions of the process and $\gamma$ in an additional $\mathcal{N}(\mathcal{N}-1)$ dimensions. It is possible to approximate the FP model by a Finite Difference Scheme; however, it requires significantly more operations to implement this method when the dimension of the model is high. The modeling of networks inherently involves a large number of state variables, and therefore it appears to be more covenient to use the SDE framework for computations. The implementation of the SDE is better suited to incorporate further complexity such as delays, which add to the physical relevance of network modeling efforts.

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