Mathematical Model and Analysis of a Laminated Curved Beam with Shear*

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Abstract

We present a mathematical framework to provide general well-posedness and approximation results for a laminated curved beam. The layered beam consists of an elastic core layer, symmentric viscoelastic damping layers, symmetric constraining layers and bonded piezoceramic patch pairs on the outer surfaces. The model includes unbounded inputs from the piezoceramic patches, hysteresis and shear in the viscoelastic layers. The viscoelastic layers are held in place by sandwiching them between the elastic core and elastic constraining layers. Families of linear and cubic splines are used to illustrate the approximation ideas.

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1 Introduction

There is a widespread interest in the engineering and industrial community over the past several decades in the use of composite (elastic/viscoelastic smart material) structures such as beams, plates and complex articulated structures. There is a large body of literature dealing with the design of viscoelastic material (VEM) layers to enhance passive damping ([MM], [YD], [R], [JK], [Ma], and references therein). Recently, however, there is a growing interest in active constrained layer (ACL) structures ([ATW], [HWT], [RWTW], [RWTW1], [LW], [KKS], [BGM], [BMZ], [BR], [VKI] and references therein). In such structures sensor/actuator devices such as piezoceramic patches or layers with appropriate circuitry and power supply are bonded to a parent material that is elastic. Between the parent material and the piezoceramic patches another material, VEM is often embedded to provide passive damping.

There is very little literature on rigorous mathematical formulations for the ACL devices mentioned above. The present paper is a continuation of the simpler model in [BMZ]. Here we have added constraining layers to hold the viscoelastic layers in place, and further we account for the shear in the viscoelastic layers. As in [BMZ] we establish well-posedness and numerical approximation framework. The multilayer curved beam configuration presented produces fully coupled transverse and longitudinal vibrations. In addition to the shear we have included "integral hysteresis" in our VEM layers. Such curved configurations with VEM hysteresis are important in automobile and aircraft industrial applications.

Our approach, which is functional analytic in nature, combines the ideas on treatment of time hysteresis and approximation in partial differential equation models given in [BFW1] and [BFW2] with general well-posedness (in terms of semigroup) arguments given in [BIW] and [BKW]. These ideas and results can be generalized and extended to treat many of the more realistic (linear) physical models found in the literature. The resulting theoretical and computational framework can be used in control, estimation and simulation applications.

In the next section we present our example model along with a brief outline of the elements of its derivatives. We recast this model in abstract Cauchy system form in Section 3 and show that it can be associated with a C_0-semigroup in appropriately chosen spaces. In Section 4, we give convergence results for a typical family (linear splines for longitudinal vibrations, cubic splines for transverse vibrations) of approximations. We have successfully used this family in related computational studies, the results of which are detailed elsewhere (see [BZ]).

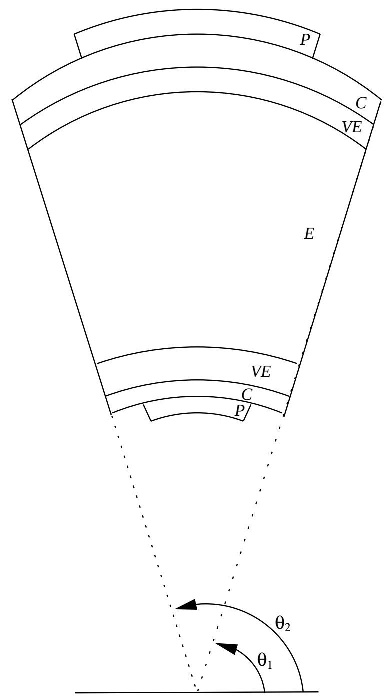
2 Model Development

2.1 Physical Model and Assumptions

We study a laminated beam that contains an elastic core with two viscoelastic layers, two constraining layers, and two piezoceramic patches. The viscoelastic layers are bonded to the elastic core, the constraining layers to the viscoelastic layers, and the piezoceramic patches to the constraining layers. Pairs of the viscoelastic layers, constraining layers, and piezoceramic patches are used to produce geometrical symmetry. Each component has a finite radius of curvature. A view of a middle section of the composite curved beam from the side is given in Figure 1.

The transverse vibrations of the composite beam are our primary consideration. However, due to the beam curvature there are two displacements involved: $v(t, \theta)$ in the tangential ($\hat{\theta}$) direction and $w(t, \theta)$ in the radial ($\hat{n}$) direction. Any motion in the direction of the width of the beam, $\hat{x}$, is assumed negligible. We assume that the radial displacement $w(t, \theta)$ is the same for all layers and we derive tangential displacements for each layer. Moreover, we make the following assumptions for our model:

- The viscoelastic material under consideration is linear.
- Every material layer is bonded perfectly to others in contact, i.e., no slip, and no disbonds are allowed.
- Every material layer deforms uniformly with respect to thickness; that is, displacements are only functions of t and θ.
- The thickness of the composite beam is very small in comparison with other dimensions such as radius of curvature and length.
- Composite beam deformations are sufficiently small so as to allow higher order motions to be neglected, i.e., we use first order models.
- Transverse normal stresses (i.e., the stresses in the direction, $\hat{n}$, normal to the thin dimension) are taken to be negligible.
- A cross section which is originally normal to the reference surfaces of the elastic core, the constraining layers, and the piezoceramic patches will remain normal to the deformed reference surfaces and will remain unstrained.

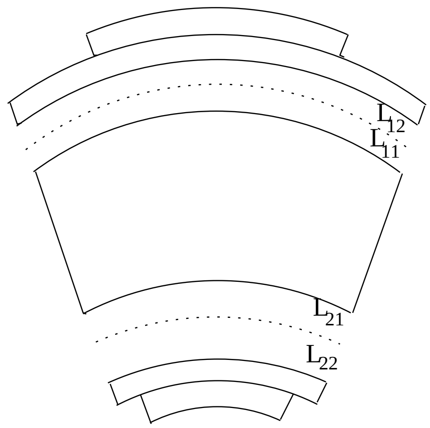


E:Elastic Core thickness h
VE: Viscoelastic layer, thickness Δ
C: Constraining layer, thickness τ
P: Piezo patch, thickness T.

Figure 1. Middle Section of the Curved Composite Beam

2.2 Mathematical Development of Model

To develop the mathematical model we treat each of the layers as a shell. Further, we introduce four sublayers, L_{ij}, $i = 1, 2$, $j = 1, 2$, with nonzero thickness Δ_{ij}, respectively in the viscoelastic layers (see Figure 2). We would like the viscoelastic layers to have the same circumferential displacements as the elastic core at the interfaces between the two materials. We would like a similar equality at the interfaces of the viscoelastic layers with the constraining layers. However, unlike the elastic and constraining layers, the viscoelastic layers undergo shear distortion. This can be efficiently captured by considering sublayers of the viscoelastic layer divided by a neutral line of zero shear which is no longer necessarily the geometrical center line. Moreover, its location, which changes with time during deformations, is in general unknown. By introducing equations for two sublayers in each viscoelastic layer, we can account for this neutral shear axis without actually knowing its location. We emphasize that this is simply a device to develop a model which allows shear in the viscoelastic layers as well as different materials in the constraining layers and elastic core.



The viscoelastic layers divided into sublayers L_{11}, L_{12}, L_{21} and L_{22}.

Figure 2. Viscoelastic Layer Subdivisions

In what follows we write ∂_t for $\partial/\partial t$ and ∂ for $\partial/\partial\theta$. In the elastic core, referring the motion to its neutral line we write

$$v(t,\theta,z) = v(t,\theta) + z\left(\frac{v(t,\theta)}{R} - \frac{\partial w(t,\theta)}{R}\right), \qquad -\frac{h}{2} \le z \le \frac{h}{2}, \tag{2.1}$$

where R is the radius of curvature, h the thickness. The formula (2.1) comes from standard shell theory (see [BSW]). The strain at any point on the elastic core is given by

$$e_\theta(t) = \frac{1}{R}\partial v(t,\theta) + \frac{1}{R}w(t,\theta) - \frac{z}{R^2}\partial^2 w(t,\theta) \tag{2.2}$$

and the corresponding stress is given by

$$\sigma_\theta(t) = \frac{E^e}{1-\nu_e^2}e_\theta(t), \tag{2.3}$$

where E^e is the Young's modulus, and ν_e the Poisson ratio. In (2.1) and (2.2) the variable z is the normal distance of the point (θ, z) from the neutral line.

Moving up to the viscoelastic layers L_{11}, and L_{12} we treat each sublayer as a shell and refer its motion to its geometrical center line. Since we include shear deformation in these layers (L_{ij}), Kirchhoff's fourth hypothesis, which implies $\gamma_{xz} = \gamma_{\theta z} = e_z = 0$ is no longer valid. Denoting the circumferential displacements by $V_{11}^{ve}(t,\theta,\tilde z)$, and $V_{12}^{ve}(t,\theta,\tilde z)$ we write

$$V_{11}^{ve}(t,\theta,\tilde z) = V_{11}^{ve}(t,\theta,0) + \tilde z S_{11}, \qquad -\frac{\Delta_{11}}{2} \le \tilde z \le \frac{\Delta_{11}}{2} \tag{2.4}$$

$$V_{12}^{ve}(t,\theta,\tilde{z}) = V_{12}^{ve}(t,\theta,0) - \tilde{z}S_{12}, \qquad -\frac{\Delta_{12}}{2} \le \tilde{z} \le \frac{\Delta_{12}}{2}. \tag{2.5}$$

In (2.4) and (2.5), $\tilde{z}$ denotes the normal distance of a point $(\theta,\tilde{z})$ from the geometrical centerline. The fact that Kirchhoff's fourth hypothesis is not valid in the sublayers L_{ij} prohibits us from expressing S_{ij} in terms of $V_{ij}^{ve}(t,\theta,0)$, w, and R_{ij}, the radius of curvature of the geometrical center line of the sublayer L_{ij}, in a simple relationship of the form (2.1). Thus, we have to seek an alternative way of determining the S_{ij}'s. This will be done later.

To express the strain $(e_{\theta,ij}^{ve}(t,\theta))$ and the stress $(\sigma_{\theta,ij}^{ve}(t,\theta))$ relationship in the sublayer L_{ij} we use Boltzmann superposition model. Thus,

$$\sigma_{\theta,ij}^{ve}(t) = \frac{E^{ve}}{1-\nu_{ve}^2}\left(e_{\theta,ij}^{ve}(t) - \int_{-r}^{0} g(s)e_{\theta,ij}^{ve}(t+s)ds\right), \tag{2.6}$$

where

$$e_{\theta,ij}^{ve}(t) \approx \frac{1}{R_{ij}}\left(\partial V_{ij}^{ve}(t,\theta,\tilde{z}) + w(t,\theta)\right), \tag{2.7}$$

and E^{ve} is a modulus of elasticity, ν_{ve} the Poisson ratio of the material.

To express the shear strain $(\gamma_{\theta z,ij}^{ve}(t,\theta))$ and shear stress $\sigma_{\theta z,ij}^{ve}(t,\theta)$ relationship we again use the Boltzmann superposition model. Thus,

$$\sigma_{\theta z,ij}^{ve}(t) = \frac{E^{ve}}{2(1+\nu_{ve})}\left[\gamma_{\theta z,ij}^{ve}(t) - \int_{-r}^{0} g(s)\gamma_{\theta z,ij}^{ve}(t+s)ds\right], \tag{2.8}$$

where

$$\gamma_{\theta z,ij}^{ve}(t) = \frac{1}{R_{ij}+\tilde{z}}\left(\partial w(t,\theta) - V_{ij}^{ve}(t,\theta,\tilde{z})\right) + (-1)^{i+j}S_{ij}. \tag{2.9}$$

Next, we move up to the constraining layer. Let $V_1^c(t,\theta,\tilde{z})$ denote the circumferential displacement, and R_1^c the radius of curvature of the neutral line. Then, we have

$$V_1^c(t,\theta,\tilde{z}) = V_1^c(t,\theta,0) + \tilde{z}\left(\frac{V_1^c(t,\theta,0)}{R_1^c} - \frac{\partial w(t,\theta)}{R_1^c}\right), \qquad -\frac{\tau}{2} \le \tilde{z} \le \frac{\tau}{2}, \tag{2.10}$$

where τ is the thickness of the constraining layer and $\tilde{z}$ is the local coordinate for the distance from its neutral line. Using a form similar to (2.2) for the elastic core layer, the strain is given by

$$e_{\theta,1}^c(t,\theta) = \frac{1}{R_1^c}\left(\partial V_1^c(t,\theta,0) + w(t,\theta)\right) - \frac{\tilde{z}}{(R_1^c)^2}\partial^2 w(t,\theta). \tag{2.11}$$

The strain $(e_{\theta,1}^c(t,\theta))$ and stress $(\sigma_{\theta,1}^c(t,\theta))$ relationship is given by

$$\sigma_{\theta,1}^c(t,\theta) = \frac{E^c}{1-\nu_c^2}e_{\theta,1}^c(t,\theta), \tag{2.12}$$

where E^c is the Young's modulus, ν_c the Poisson ratio.

Finally, we move up to the piezoceramic patch. Let $V_1^p(t,\theta,\tilde{z})$ be the circumferential displacement, R_1^p, the radius of curvature of the neutral line, T the thickness. Then, we write

$$V_1^p(t,\theta,\tilde{z}) = V_1^p(t,\theta,0) + \tilde{z}\left(\frac{V_1^p(t,\theta,0)}{R_1^p} - \frac{\partial w(t,\theta)}{R_1^p}\right), \qquad -\frac{T}{2} \le \tilde{z} \le \frac{T}{2}. \tag{2.13}$$

Once again the strain is given by

$$e_{\theta,1}^p(t) = \frac{1}{R_1^p}\partial V_1^p(t,\theta,0) + \frac{w(t,\theta)}{R_1^p} - \frac{\tilde{z}}{(R_1^p)^2}\partial^2 w. \tag{2.14}$$

The strain $(e_{\theta,1}^p(t))$ and stress $(\sigma_{\theta,1}^p(t))$ relationship is given by

$$\sigma_{\theta,1}^p(t) = \frac{E^p}{1-\nu_p^2} e_{\theta,1}^p(t)\chi_1^p(\theta), \tag{2.15}$$

where E^p is the Young's modulus, ν_p the Poisson ratio, and $\chi_1^p(\theta) = 1$ where the piezo patch is present and zero elsewhere.

We repeat the above steps for the layers below the elastic core. Starting from the elastic core we move down to encounter the sublayer L_{21}, then L_{22}. Let V_{2i}^{ve}, $i = 1, 2$ be the circumferential displacements of these sublayers relative to the geometrical center lines. Then, we write

$$V_{21}^{ve}(t,\theta,\tilde{z}) = V_{21}^{ve}(t,\theta,0) - \tilde{z}S_{21}, \qquad -\frac{\Delta_{21}}{2} \le \tilde{z} \le \frac{\Delta_{21}}{2}, \tag{2.16}$$

$$V_{22}^{ve}(t,\theta,\tilde{z}) = V_{22}^{ve}(t,\theta,0) + \tilde{z}S_{22}, \qquad -\frac{\Delta_{22}}{2} \le \tilde{z} \le \frac{\Delta_{22}}{2}. \tag{2.17}$$

Moving down to the constraining layer, we let $V_2^c(t,\theta,\tilde{z})$ denote the circumferential displacement, and R_2^c the radius of curvature of the neutral line. Thus, we write

$$V_2^c(t,\theta,\tilde{z}) = V_2^c(t,\theta,0) + \tilde{z}\left(\frac{V_2^c(t,\theta,0)}{R_2^c} - \frac{\partial w(t,\theta)}{R_2^c}\right), \qquad -\frac{\tau}{2} \le \tilde{z} \le \frac{\tau}{2}, \tag{2.18}$$

where τ is the thickness of the constraining layer. This constraining layer interfaces the viscoelastic layer above it when $\tilde{z} = \tau/2$ and the piezo patch below it when $\tilde{z} = -\tau/2$.

Finally, we move down to the piezo patch. Let $V_2^p(t,\theta,z)$ be the circumferential displacement, R_2^p the radius of curvature of the neutral line, T the thickness. Then, we write

$$V_2^p(t,\theta,\tilde{z}) = V_2^p(t,\theta,0) + \tilde{z}\left(\frac{V_2^p(t,\theta,0)}{R_2^p} - \frac{\partial w(t,\theta)}{R_2^p}\right), \qquad -\frac{T}{2} \le \tilde{z} \le \frac{T}{2}. \tag{2.19}$$

When $\tilde{z} = T/2$ the piezo patch interfaces the constraining layer above it.

Again, as we did for the layers above the elastic core, we write the stress-strain relationships. In the viscoelastic layers we have shear displacements and have ignored compression.

Finally the no-slippage and no-disbond conditions allow us to give the circumferential displacements and the S_{ij}'s (see (2.4), (2.5), (2.16), (2.17)) in terms of v, w, and $V_{ij}^{ve}(t, \theta, 0)$. Thus, we take v, w, and $V_{ij}^{ve}(t, \theta, 0)$ as our dynamic state variables.

2.2.1 Internal Forces, Moment Resultants and Equations of Motions

In the elastic layer, constraining layers, and piezo patches we denote the total internal force by N_θ^e, $N_{\theta,i}^c$, $i = 1, 2$, and $N_{\theta,i}^p$, $i = 1, 2$ respectively. Here $N_{\theta,1}^c$ and $N_{\theta,1}^p$ are the total internal force resultants along the neutral surfaces in the upper constraining layer and upper piezo patch. Similarly $N_{\theta,2}^c$ and $N_{\theta,2}^p$ denote the total internal force resultants for the lower ones. Next, the total internal force resultant along the geometrical mid surface in the viscoelastic layer L_{ij} is denoted by $N_{\theta,ij}^{ve}$, $i = 1, 2$, $j = 1, 2$. The corresponding resultant shear forces in the viscoelastic layers are denoted by $Q_{\theta,ij}^{ve}$, $i = 1, 2$; $j = 1, 2$.

We have (see [BSW],[BSW2])

$$N_\theta^e = \int_{-h/2}^{h/2} \sigma_\theta^e dz = \frac{E^e h}{R(1 - \nu_e^2)} (\partial v + w) \tag{2.20}$$

$$N_{\theta,ij}^{ve} = \int_{-\Delta_{ij}/2}^{\Delta_{ij}/2} \sigma_{\theta,ij}^{ve}(t) dz, \tag{2.21}$$

where

$$\sigma_{\theta,ij}^{ve}(t) = \frac{E^{ve}}{1 - \nu_{ve}^2} \left[e_{\theta,ij}^{ve}(t) - \int_{-r}^{0} g(s) e_{\theta,ij}^{ve}(t + s) ds \right], \tag{2.22}$$

and

$$e_{\theta,ij}^{ve}(t) \approx \frac{1}{R_{ij}} (\partial V_{ij}^{ve} + w). \tag{2.23}$$

Continuing, we have

$$\begin{aligned} N_{\theta,i}^c &= \int_{-\tau/2}^{\tau/2} \sigma_{\theta,i}^c d\tilde{z} \\ &= \int_{-\tau/2}^{\tau/2} \frac{E^c}{1 - \nu_c^2} \left(\frac{1}{R_i^c} \partial V_i^c(t, \theta, 0) + \frac{1}{R_i^c} w - \frac{\tilde{z}}{(R_i^c)^2} \partial^2 w \right) d\tilde{z}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} N_{\theta,i}^p &= \int_{-T/2}^{T/2} \sigma_{\theta,i}^p d\tilde{z} \\ &= \int_{-T/2}^{T/2} \frac{E^p}{1 - \nu_p^2} \left(\frac{1}{R_i^p} \partial V_i^p(t, \theta, 0) + \frac{1}{R_i^p} w - \frac{\tilde{z}}{(R_i^p)^2} \partial^2 w \right) \chi_i^p(\theta) d\tilde{z}. \end{aligned} \tag{2.25}$$

The shear in the viscoelastic layer is given by

$$Q_{\theta,ij}^{ve} = \int_{-\Delta_{ij}/2}^{\Delta_{ij}/2} \sigma_{\theta\tilde{z},ij}^{ve}(t) d\tilde{z}, \tag{2.26}$$

where

$$\sigma^{ve}_{\theta\tilde{z},ij}(t) = \frac{E^{ve}}{2(1+\nu_{ve})}\left[\gamma^{ve}_{\theta\tilde{z},ij}(t) - \int_{-r}^{0} g(s)\gamma^{ve}_{\theta\tilde{z},ij}(t+s)ds\right], \tag{2.27}$$

$$\gamma^{ve}_{\theta\tilde{z},ij}(t) = \frac{1}{R_{ij}+\tilde{z}}\partial w - \frac{1}{R_{ij}+\tilde{z}}V^{ve}_{ij} + (-1)^{i+j}S_{ij}. \tag{2.28}$$

Next we present the internal moments

$$\begin{aligned} M^e_\theta &= \int_{-h/2}^{h/2} \sigma^e_\theta \tilde{z} d\tilde{z} \qquad (2.29)\\ &= \frac{E^e}{1-\nu_e^2}\int_{-h/2}^{h/2}\left(\frac{\partial v}{R} + \frac{w}{R} - \frac{\tilde{z}}{R^2}\partial^2 w\right)\tilde{z}\, d\tilde{z} = \frac{-E^e h^3}{12R^2(1-\nu_e^2)}\partial^2 w. \end{aligned}$$

The internal moments for the constraining layers about the neutral line of the elastic core are given by

$$\begin{aligned} M^c_{\theta,i} &= \frac{E^c}{1-\nu_c^2}\int_{-\tau/2}^{\tau/2}\left(\frac{1}{R^c_i}\partial V^c_i(t,\theta,0) + \frac{w}{R^c_i} - \frac{\tilde{z}}{(R^c_i)^2}\partial^2 w\right)(R^c_i + \tilde{z} - R)d\tilde{z} \\ &= \frac{\tau E^c}{1-\nu_c^2}\frac{R^c_i - R}{R^c_i}(\partial V^c_i(t,\theta,0) + w) - \frac{\tau^3 E^c}{12(1-\nu_c^2)}\frac{\partial^2 w}{(R^c_i)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} M^c_{\theta,1} + M^c_{\theta,2} &= \left[\sum_{i=1}^{2}\frac{\tau E^c}{1-\nu_c^2}\frac{R^c_i - R}{R^c_i}(\partial V^c_i(t,\theta,0) + w)\right] \\ &- \frac{\tau^3 E^c}{12(1-\nu_c^2)}\partial^2 w\left(\frac{1}{(R^c_1)^2} + \frac{1}{(R^c_2)^2}\right). \end{aligned} \tag{2.30}$$

Note that

$$\frac{R^c_1 - R}{R^c_1} \approx \frac{h/2 + \Delta_{11} + \Delta_{12} + \tau/2}{R},$$

$$\frac{R^c_2 - R}{R^c_2} \approx \frac{-(h/2 + \Delta_{21} + \Delta_{22} + \tau/2)}{R}.$$

The internal moments for the piezo layers about the neutral line of the elastic core are given by (see [BSW],[BSW2])

$$M^p_{\theta,i} = \frac{\tau E^p}{1-\nu_p^2}\frac{R^p_i - R}{R^p_i}(\partial V^p_i(t,\theta,0) + w) - \frac{2E^p T^3}{3(1-\nu_p^2)}\frac{\chi^p_i(\theta)}{(R^p_i)^2}\partial^2 w.$$

Thus,

$$\begin{aligned} M^p_{\theta,1} + M^p_{\theta,2} &= \left[\sum_{i=1}^{2}\frac{\tau E^p}{1-\nu_p^2}\frac{R^p_i - R}{R^p_i}(\partial V^p_i(t,\theta,0) + w)\right] \\ &- \frac{2E^p T^3}{3(1-\nu_p^2)}\partial^2 w\left(\frac{\chi^p_1(\theta)}{(R^p_1)^2} + \frac{\chi^p_2(\theta)}{(R^p_2)^2}\right). \end{aligned} \tag{2.31}$$

To simplify our presentation we define the following entities:

$$a_1 = \frac{E^{ve}}{1-\nu_{ve}^2}\sum_{i,j}\frac{\Delta_{ij}}{R_{ij}} \tag{2.32}$$

$$a_2 = \frac{E^{ve}}{2(1+\nu_{ve})} \sum_{i,j} \frac{\Delta_{ij}}{R_{ij}} \tag{2.33}$$

$$b_1 = \frac{hE^{ve}}{2R(1-\nu_{ve}^2)} \sum_{ij} \frac{(-1)^i \Delta_{ij}}{R_{ij}} \tag{2.34}$$

$$b_2 = \frac{E^c \tau}{1-\nu_c^2} \left(\frac{1}{R_2^c} - \frac{1}{R_1^c} \right) \left(\frac{\tau}{2} + \frac{h}{2} \right) \tag{2.35}$$

$$b_3(\theta) = \frac{2E^p T}{1-\nu_p^2} \left(\frac{1}{R_2^p} \chi_2^p - \frac{1}{R_1^p} \chi_1^p \right) \left(\frac{\tau}{2} + \frac{h}{2R} \right) \tag{2.36}$$

$$\begin{aligned} b_4(\theta) &= \frac{2E^p T}{1-\nu_p^2} \left(\frac{1}{2R_2^p R_2^c} \chi_2^p - \frac{1}{2R_1^p R_1^c} \chi_1^p \right) \tau \\ &+ \frac{2E^p T}{1-\nu_p^2} \left(\frac{1}{2(R_2^p)^2} \chi_2^p - \frac{1}{2(R_1^p)^2} \chi_1^p \right) \end{aligned} \tag{2.37}$$

$$D_1 = \frac{E^{ve}}{2(1+\nu_{ve})} (\Delta_{11}, -\Delta_{12}, -\Delta_{21}, \Delta_{22})^T \tag{2.38}$$

$$J^{ve} = \frac{E^{ve}}{1-\nu_{ve}^2} \left(\frac{\Delta_{11}}{R_{11}}, \frac{\Delta_{12}}{R_{12}}, \frac{\Delta_{21}}{R_{21}}, \frac{\Delta_{22}}{R_{22}} \right)^T \tag{2.39}$$

$$J^c = \frac{E^c \tau}{1-\nu_c^2} \left(\frac{1}{R_1^c}, \frac{1}{R_1^c}, \frac{1}{R_2^c}, \frac{1}{R_2^c} \right)^T \tag{2.40}$$

$$J^p = \frac{2TE^p}{1-\nu_p^2} \left(\frac{\chi_1^p}{R_1^p}, \frac{\chi_1^p}{R_1^p}, \frac{\chi_2^p}{R_2^p}, \frac{\chi_2^p}{R_2^p} \right)^T \tag{2.41}$$

$$T_1 = \begin{pmatrix} \frac{1}{2}\Delta_{11} & 0 & 0 & 0 \\ \Delta_{11} & -\frac{1}{2}\Delta_{12} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\Delta_{21} & 0 \\ 0 & 0 & -\Delta_{21} & \frac{1}{2}\Delta_{22} \end{pmatrix} \tag{2.42}$$

$$T_2 = \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & -\Delta_{12} & 0 & 0 \\ 0 & 0 & -\Delta_{21} & 0 \\ 0 & 0 & 0 & \Delta_{22} \end{pmatrix} \tag{2.43}$$

$$D(\theta) = T_1^t J^{ve} + T_2^t (J^c + J^p) \tag{2.44}$$

$$\xi_1(\theta) = \frac{E^e h}{R(1-\nu_e^2)} + b_1 + \frac{E^c \tau}{1-\nu_c^2} \left(\frac{1}{R_1^c} + \frac{1}{R_2^c} \right) \tag{2.45}$$

$$\begin{aligned} \xi_2(\theta) &= \frac{E^e h^3}{12R^2(1-\nu_e^2)} + \frac{E^c \tau^3}{12(1-\nu_c^2)} \left(\frac{1}{(R_1^c)^2} + \frac{1}{(R_2^c)^2} \right) \\ &+ \frac{2E^p T^3}{3(1-\nu_p^2)} \left(\frac{\chi_1^p}{(R_1^p)^2} + \frac{\chi_2^p}{(R_2^p)^2} \right) \end{aligned} \tag{2.46}$$

$$S = (S_{11}, S_{12}, S_{21}, S_{22})^T. \tag{2.47}$$

Furthermore, define the resultant moment for the nonviscoelastic materials

$$M_\theta = M_\theta^e + \sum_i M_{\theta,i}^c + \sum_i M_{\theta,i}^p. \tag{2.48}$$

Kirchhoff's fourth hypothesis implies $\gamma_{xz} = \gamma_{\theta z} = e_z = 0$ and hence we cannot use $\gamma_{\theta z}$ to compute shear stress or the associated moment. However, from moment balance one can relate internal moment to shear stress. Let $\tilde{Q}_\theta$ be the sum of the shear stresses in the elastic, constraining, and piezo layers. Then, we have the relationship

$$\tilde{Q}_\theta \approx \frac{1}{R}\partial M_\theta. \tag{2.49}$$

Let

$$Q^{ve} = \sum_{ij} Q_{\theta,ij}^{ve} \approx a_2(\partial w - v) + \langle D_1, S \rangle. \tag{2.50}$$

Thus, the sum Q_θ of all the shear stresses is given by

$$Q_\theta = \tilde{Q}_\theta + Q_\theta^{ve}.$$

Next, we assemble the internal forces of the beam. Thus, the sum of the internal force resultants N_θ is given by

$$N_\theta = \sum_{ij} N_{\theta,ij}^{ve} + \sum_i N_{\theta,i}^c + \sum_i N_{\theta,i}^p + N_\theta^e.$$

Here we are treating the entire curved laminated beam as a shell and N_θ as the total internal resultant force along the middle surface of the elastic core.

Let $\bar{\rho}$ be the linear mass density which varies with θ due to the presence of the patches. If we take ρ^e, ρ^{ve}, ρ^c, ρ^p be the volume densities of the elastic, viscoelastic, constraining, and piezoceramic layers respectively, then we have

$$\bar{\rho} = \rho^e hb + \rho^{ve} 2\Delta b + \rho^c 2\tau b + \rho^p (Tb\chi_1^p + Tb\chi_2^p)$$

$$2\Delta = \sum_{ij} \Delta_{ij}.$$

The tangential and radial displacements v and w of a curved beam are given by [BSW],[BSW2], [M] by the equations

$$\bar{\rho}\partial_t^2 v - \frac{b}{R}\partial N_\theta - \frac{b}{R}Q_\theta = b\hat{q}_\theta \tag{2.51}$$

$$\bar{\rho}\partial_t^2 w + \frac{b}{R}N_\theta - \frac{b}{R}\partial Q_\theta = b\hat{q}_n + \frac{b}{R}\partial \hat{m}_x \tag{2.52}$$

where b is the width of the beam. We shall supplement the system (2.51) and (2.52) by a set of equations for

$$\bar{V} = (\bar{V}_{11}, \bar{V}_{12}, \bar{V}_{21}, \bar{V}_{22})^T, \tag{2.53}$$

where (see (2.4), (2.5), (2.16), (2.17))

$$\bar{V}_{ij}(t,\theta) = V_{ij}^{ve}(t,\theta,0). \tag{2.54}$$

We consider the layer L_{ij} for which the equation of motion locally is given by

$$\rho^{ve}\partial_t^2 V_{ij}^{ve} = \frac{1}{R_{ij}+\tilde{z}}\,\partial\sigma_{\theta,ij}^{ve} + \frac{2}{R_{ij}+\tilde{z}}\,\sigma_{\theta\tilde{z},ij}^{ve} + \partial_{\tilde{z}}\sigma_{\theta\tilde{z},ij}^{ve}. \tag{2.55}$$

The equation (2.55) is obtained by taking an infinitesimal element and writing force balance [M]. Again we remark that $\tilde{z}$ measures the normal distance of the point $(\theta,\tilde{z})$ from the geometrical centerline of the layer L_{ij}.

In (2.55) set $\tilde{z}=0$ to obtain the equation

$$\begin{aligned}
\rho^{ve}\partial_t^2\bar{V}_{ij} &= \frac{E^{ve}}{R_{ij}^2(1-\nu_{ve}^2)}\left[\partial^2\bar{V}_{ij} + \partial w - \int_{-r}^{0} g(s)(\partial^2\bar{V}_{ij}(t+s,\theta) + \partial w(t+s,\theta))ds\right] \\
&+ \frac{E^{ve}}{2(1+\nu_{ve})R_{ij}}\left[\frac{\partial w}{R_{ij}} - \frac{1}{R_{ij}}\bar{V}_{ij} + (-1)^{i+j}S_{ij}\right. \\
&- \left.\int_{-r}^{0} g(s)\left(\frac{\partial w(t+s,\theta)}{R_{ij}} - \frac{\bar{V}_{ij}(t+s,\theta)}{R_{ij}} + (-1)^{i+j}S_{ij}(t+s,\theta)\right)ds\right].
\end{aligned} \tag{2.56}$$

Let

$$\Lambda = \frac{E^{ve}}{1-\nu_{ve}^2}diag(R_{11}^{-2}, R_{12}^{-2}, R_{21}^{-2}, R_{22}^{-2}) \tag{2.57}$$

$$L = \frac{E^{ve}}{1-\nu_{ve}^2}(R_{11}^{-2}, R_{12}^{-2}, R_{21}^{-2}, R_{22}^{-2})^T \tag{2.58}$$

$$\tilde{P} = diag(1,-1,-1,1). \tag{2.59}$$

Then, using (2.57), (2.58), (2.59) we rewrite (2.56) as

$$\begin{aligned}
\rho^{ve}\partial_t^2\bar{V} &= \partial^2\Lambda\bar{V} + \partial wL - \textstyle\int_{-r}^{0} g(s)(\partial^2\Lambda\bar{V} + \partial wL)ds \\
&+ \frac{1}{2}(1-\nu_{ve})\Big[\partial wL - \Lambda\bar{V} + \tilde{P}\Lambda S \\
&- \int_{-r}^{0} g(s)(\partial wL - \Lambda\bar{V} + \tilde{P}\Lambda S)ds\Big].
\end{aligned} \tag{2.60}$$

In (2.60) $\bar{V}$ is as in (2.53) and $S = (S_{11}, S_{12}, S_{21}, S_{22})^T$.

Recalling that $\bar{V}_{ij}(t,\theta) = V_{ij}^{ve}(t,\theta,0)$ and using the no-slip condition we have

$$\bar{V}_{11} = v + \frac{h}{2R}v - \frac{h}{2R}\partial w + \frac{1}{2}\Delta_{11}S_{11}$$

$$\begin{aligned}\bar{V}_{12} &= v + \frac{h}{2R}\,v - \frac{h}{2R}\,\partial w + \Delta_{11}S_{11} - \frac{1}{2}\,\Delta_{12}S_{12} \\ \bar{V}_{21} &= v - \frac{h}{2R}\,v + \frac{h}{2R}\,\partial w - \frac{1}{2}\,\Delta_{21}S_{21} \\ \bar{V}_{22} &= v - \frac{h}{2R}\,v + \frac{h}{2R}\,\partial w - \Delta_{21}S_{21} + \frac{1}{2}\,\Delta_{22}S_{22}. \end{aligned} \tag{2.61}$$

Let

$$\tilde{J}_1 = \left(1 + \frac{h}{2R}\,, 1 + \frac{h}{2R}\,, 1 - \frac{h}{2R}\,, 1 - \frac{h}{2R}\right)^T \tag{2.62}$$

$$\tilde{J}_2 = \left(-\frac{h}{2R}\,, -\frac{h}{2R}\,, \frac{h}{2R}\,, \frac{h}{2R}\right)^T. \tag{2.63}$$

Then, we rewrite (2.61) as

$$\bar{V} = T_1 S + v\tilde{J}_1 + \partial w \tilde{J}_2. \tag{2.64}$$

Observing that T_1 is invertible due to our basic assumption that $\Delta_{ij} \neq 0$, we then have

$$S = T_1^{-1}\bar{V} - vT_1^{-1}\tilde{J}_1 - \partial w T_1^{-1}\tilde{J}_2. \tag{2.65}$$

Next, let

$$P = \tilde{P}\Lambda T_1^{-1} - \Lambda \tag{2.66}$$

$$J_2 = L - \tilde{P}\Lambda T_1^{-1}\tilde{J}_2. \tag{2.67}$$

Using (2.65), (2.66), and (2.67) we rewrite (2.60) as follows

$$\begin{aligned}\rho^{ve}\partial_t^2\bar{V} &= \partial^2\Lambda\bar{V} + \partial w L - \int_{-r}^0 g(s)(\partial^2\Lambda\bar{V} + \partial w L)ds \\ &\quad + \frac{1}{2}\,(1 - \nu_{ve})[P\bar{V} - v\tilde{P}\Lambda T_1^{-1}\tilde{J}_1 + \partial w J_2 \\ &\quad - \int_{-r}^0 g(s)(P\bar{V} - v\tilde{P}\Lambda T_1^{-1}\tilde{J}_1 + \partial w J_2)ds]. \end{aligned} \tag{2.68}$$

From (2.51) and (2.52) finally we have the equations

$$\begin{aligned}\bar{\rho}\partial_t^2 v &- \frac{b}{R}\partial[\xi_1(\partial v + w) + \xi_2\partial^2 w + \langle D(\theta), \partial S\rangle] \\ &+ \frac{b}{R}\,\partial\int_{-r}^0 g(s)\{a_1(\partial v + w) + b_1\partial^2 w + \langle J^{ve}, T_1\partial S\rangle\}ds \\ &- \frac{b}{R}[a_2(\partial w - v) + \langle D_1, S\rangle + \frac{1}{R}\,\partial M_\theta] \\ &+ \frac{b}{R}\int_{-r}^0 g(s)\{a_2(\partial w - v) + \langle D_1, S\rangle\}ds = b\hat{q}_\theta \end{aligned} \tag{2.69}$$

$$\bar{\rho}\partial_t^2 w + \frac{b}{R}[\xi_1(\partial v + w) + \xi_2\partial^2 w + \langle D(\theta), \partial S\rangle]$$

$$
\begin{aligned}
&- \frac{b}{R}\int_{-r}^{0} g(s)\{a_1(\partial v + w) + b_1\partial^2 w + \langle J^{ve}, T_1\partial S\rangle\}ds \\
&- \frac{b}{R}\partial[a_2(\partial w - v) + \langle D_1, S\rangle + \frac{1}{R}\partial M_\theta] \\
&+ \frac{b}{R}\int_{-r}^{0} g(s)\partial\{a_2(\partial w - v) + \langle D_1, S\rangle\}ds = b\hat{q}_n + \frac{b}{R}\partial\hat{m}_x.
\end{aligned}
\tag{2.70}
$$

The equations (2.68), (2.69), (2.70) give a system of equations governing the motion of the beam in terms of the variables v, w, $\bar{V}$. Note that in (2.69) and (2.70) the variable vector S is to be replaced according to (2.65).

We rewrite (2.69) and (2.70) by substituting for S. First we let

$$\bar{D}(\theta) = bR^{-1}(T_1^t)^{-1}D(\theta) \tag{2.71}$$

$$\bar{\xi}_1 = bR^{-1}\xi_1 - \langle \bar{D}(\theta), \tilde{J}_1\rangle \tag{2.72}$$

$$\bar{a}_1 = bR^{-1}(a_1 - \langle J^{ve}, \tilde{J}_1\rangle) \tag{2.73}$$

$$\bar{a}_2 = (a_2 + \langle (T_1^t)^{-1}D_1, \tilde{J}_1\rangle)bR^{-1} \tag{2.74}$$

$$\bar{D}_1 = bR^{-1}(T_1^t)^{-1}D_1 \tag{2.75}$$

$$D_2 = bR^{-1}(\xi_2 - a_2) + \langle \bar{D}_1 - \bar{D}(\theta), \tilde{J}_2\rangle \tag{2.76}$$

$$D_3 = bR^{-1}(\langle J^{ve}, \tilde{J}_2\rangle - (a_2 + b_1)) + \langle \bar{D}_1\tilde{J}_2\rangle. \tag{2.77}$$

Then,

$$
\begin{aligned}
\bar{\rho}\partial_t^2 v &- \partial\bar{\xi}_1\partial v + (bR^{-1}a_2 + \langle \bar{D}_1, \tilde{J}_1\rangle)v \\
&- bR^{-1}[\partial(\xi_1 w) + (a_2 + \langle \bar{D}_1, \tilde{J}_2\rangle)\partial w] + \underline{\partial\langle \bar{D}(\theta), \tilde{J}_2\rangle\partial^2 w} \\
&- bR^{-1}\underline{\partial(\xi_2\partial^2 w)} - \underline{bR^{-2}\partial M_\theta} + \langle \bar{D}_1, \bar{V}\rangle - \partial\langle \bar{D}(\theta), \partial\bar{V}\rangle \\
&+ \textstyle\int_{-r}^{0} g(s)\{\bar{a}_1\partial^2 v - \bar{a}_2 v - bR^{-1}(a_2 - a_1)\partial w \\
&- bR^{-1}\langle J^{ve}, \tilde{J}_2\rangle\partial^2 w + bR^{-1}b_1\underline{\partial^3 w} \\
&+ bR^{-1}\langle J^{ve}, \partial^2\bar{V}\rangle + \langle D_1, \bar{V}\rangle\}ds \\
&= b\hat{q}_\theta
\end{aligned}
\tag{2.78}
$$

$$
\begin{aligned}
\bar{\rho}\partial_t^2 w &+ (bR^{-1}a_2 + \bar{\xi}_1)\partial v + bR^{-1}\xi_1 w + D_2\partial^2 w + bR^{-1}\partial^2 M_\theta \\
&+ \langle \bar{D}(\theta) - \bar{D}_1, \partial\bar{V}\rangle \\
&+ \textstyle\int_{-r}^{0} g(s)\{(\bar{a}_1 + \bar{a}_2)\partial v - bR^{-1}a_1 w + D_3\partial^2 w \\
&+ \langle \bar{D}_1 - bR^{-1}J^{ve}, \partial\bar{V}\rangle\}ds = b\hat{q}_n + \tfrac{b}{R}\partial\hat{m}_x.
\end{aligned}
\tag{2.79}
$$

The system we should consider consists of the equations (2.68), (2.78), and (2.79). We ignore the underlined terms in (2.78) (see REMARK 3.1).

The system (2.68), (2.78), and (2.79) can be rewritten in the form

$$\partial_t^2\begin{pmatrix} v \\ w \\ \bar{V}\end{pmatrix} + A\begin{pmatrix} v \\ w \\ \bar{V}\end{pmatrix} + \int_{-r}^{0} g(s)B\begin{pmatrix} v(t+s,\theta) \\ w(t+s,\theta) \\ \bar{V}(t+s,\theta)\end{pmatrix}ds = \frac{b}{\bar{\rho}}\begin{pmatrix} \hat{q}_\theta \\ \hat{q}_n + \frac{1}{R}\partial\hat{m}_x \\ 0\end{pmatrix}. \tag{2.80}$$

Using (2.68) we consider the equation

$$\begin{aligned}\rho^{ve}\partial_t^2 Z &= \partial^2\Lambda Z + \partial^2 wL - \int_{-r}^{0} g(s)(\partial^2\Lambda Z + \partial^2 wL)ds \\ &\quad + \frac{1}{2}(1-\nu_{ve})[PZ - \partial v\tilde{P}\Lambda T_1^{-1}\tilde{J}_1 + \partial^2 wJ_2] \\ &\quad - \int_{-r}^{0} g(s)(PZ - \partial v\tilde{P}\Lambda T_1^{-1}\tilde{J}_1 + \partial^2 wJ_2)ds.\end{aligned} \tag{2.81}$$

In (2.78) and (2.79) we replace $\bar{V}$ according to the formula

$$\partial\bar{V}(t,\theta) = Z(t,\theta) \tag{2.82}$$

and obtain a system corresponding to (2.80):

$$\partial_t^2\begin{pmatrix} v \\ w \\ Z\end{pmatrix} + A_1\begin{pmatrix} v \\ w \\ Z\end{pmatrix} + \int_{-r}^{0} g(s)B_1\begin{pmatrix} v(t+s,\theta) \\ w(t+s,\theta) \\ Z(t+s,\theta)\end{pmatrix} ds = \frac{b}{\bar{\rho}}\begin{pmatrix}\hat{q}_\theta \\ \hat{q}_n + \frac{1}{R}\partial\hat{m}_x \\ 0\end{pmatrix} \tag{2.83}$$

where the operators A_1, B_1 (which are rather tedious to define) are given in the Appendix.

Next, let

$$\hat{A} = A_1 + \alpha B_1$$

where

$$\alpha = \int_{-r}^{0} g(s)ds.$$

Then, we rewrite (2.83) as

$$\partial_t^2\begin{pmatrix} v \\ w \\ Z\end{pmatrix} + \hat{A}\begin{pmatrix} v \\ w \\ Z\end{pmatrix} - \int_{-r}^{0} g(s)B_1\begin{pmatrix} v(t)-v_t(s) \\ w(t)-w_t(s) \\ Z(t)-Z_t(s)\end{pmatrix} ds = \frac{b}{\bar{\rho}}\begin{pmatrix}\hat{q}_\theta \\ \hat{q}_n + \frac{1}{R}\partial\hat{m}_x \\ 0\end{pmatrix}, \tag{2.84}$$

where $v_t(s) = v(t+s)$, $w_t(s) = w(t+s)$, $Z_t(s) = Z(t+s)$.

2.2.2 External Forces and Moments

We assume for our model that the external forces and moments are of a form such as those produced by piezo materials. Thus, we ignore any contribution from external air damping. The PZT patches can be used as both sensors and actuators, and as actuators they can generate significant external forces and moments. We follow the formulations in [BSW], and to aid in quantitative description of the PZT forces, we define the indicator function

$$\hat{S}_{1,2}(\theta) = \begin{cases} 1 & \theta > (\theta_1+\theta_2)/2 \\ 0 & \theta = (\theta_1+\theta_2)/2\ . \\ -1 & \theta < (\theta_1+\theta_2)/2 \end{cases}$$

Then, assuming that the two piezo patches have the same Young's modulus, E^p, the same Poisson ratio, ν_p, and the same strain constant d_{31}, the total extended force from the piezo patches is given by

$$\begin{aligned} N_\theta^p &= (N_{\theta,1}^p \chi_1^p + N_{\theta,2}^p \chi_2^p)\hat{S}_{1,2} \\ &= -\frac{E^p}{1-\nu_p} d_{31}(V_1\chi_1^p + V_2\chi_2^p)\hat{S}_{1,2}. \end{aligned}$$

Here V_1 and V_2 are the voltages applied to the patches to produce deformations. Similarly, the total external moment due to the patches is found to be

$$\begin{aligned} M_\theta^p &= M_{\theta,1}^p \chi_1^p + M_{\theta,2}^p \chi_2^p \\ &= -\frac{E^p}{1-\nu_p}\frac{d_{31}}{2T}(V_2\chi_2^p - V_1\chi_1^p)\left[\left(\frac{h}{2}+T+\Delta+\tau\right)^2 - \left(\frac{h}{2}+\Delta+\tau\right)^2\right]. \end{aligned}$$

Finally the external forces in (2.51) and (2.52) are given by (for details, see [BSW],[BSW2])

$$\begin{aligned} \hat{q}_\theta &= -\hat{S}_{1,2}(\theta)\frac{1}{R}\partial N_\theta^p\,, \\ \hat{m}_x &= -\frac{1}{R}\partial M_\theta^p. \end{aligned}$$

3 Semigroup Formulation and Well-Posedness

3.1 System Formulation

Let

$$\mathcal{V} = H_0^1(\theta_1,\theta_2) \times H_0^2(\theta_1,\theta_2) \times [H_0^1(\theta_1,\theta_2)]^4.$$

and introduce an inner-product on $\mathcal{V}$ by

$$\begin{aligned} \left\langle \begin{pmatrix} v_1 \\ w_1 \\ Z_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ w_2 \\ Z_2 \end{pmatrix} \right\rangle_{\mathcal{V}} &= \langle (\bar{\xi}_1 - \alpha\bar{a}_1)\partial v_1, \partial v_2\rangle \\ &+ \langle \xi_2 \partial^2 w_1, \partial^2 w_2\rangle + \langle (1-\alpha)\Lambda \partial Z_1, \partial Z_2\rangle, \end{aligned} \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ is an L_2-inner-product with respect to Lebesgue measure. We require

$$\bar{\xi}_1 - \alpha\bar{a}_1 > 0 \qquad 1-\alpha > 0$$

in (3.1) in order for this to define an inner product for $\mathcal{V}$.

Let

$$\mathcal{H} = L_2(\theta_1,\theta_2) \times L_2(\theta_1,\theta_2) \times [L_2(\theta_1,\theta_2)]^4.$$

Introduce an inner-product on $\mathcal{H}$ by

$$\left\langle \begin{pmatrix} v_1 \\ w_1 \\ Z_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ w_2 \\ Z_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle \bar{\rho} v_1, v_2 \rangle + \left\langle \frac{R}{b} \bar{\rho} w_1, w_2 \right\rangle + \langle \rho^{ve} Z_1, Z_2 \rangle. \tag{3.2}$$

Next, let

$$\mathcal{W} = L^2_G(-r, 0; \mathcal{V}).$$

We introduce an inner-product on $\mathcal{W}$ by

$$\left\langle \begin{pmatrix} v_1 \\ w_1 \\ Z_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ w_2 \\ Z_2 \end{pmatrix} \right\rangle_{\mathcal{W}} = \int_{-r}^{0} g(s) \{ \langle \bar{a}_1 \partial v_1, \partial v_2 \rangle + \langle \partial^2 w_1, \partial^2 w_2 \rangle + \langle \Lambda \partial Z_1, \partial Z_2 \rangle \} ds. \tag{3.3}$$

We regard the operators $\hat{A}$ and B_1 given in (2.83) and (2.84) as elements of $\mathcal{L}(\mathcal{V}, \mathcal{V}^*)$.

Define

$$\hat{K} : \mathcal{W} \to \mathcal{V}^*$$

by

$$\hat{K}\eta = \int_{-r}^{0} g(s) B_1 \eta(s) ds.$$

Let

$$\phi = \begin{pmatrix} v \\ w \\ Z \end{pmatrix} \in \mathcal{V}, \qquad \psi = \begin{pmatrix} \dot{v} \\ \dot{w} \\ \dot{Z} \end{pmatrix} \in \mathcal{H}, \qquad \gamma = \begin{pmatrix} v(t) - v_t(\cdot) \\ w(t) - w_t(\cdot) \\ Z(t) - Z_t(\cdot) \end{pmatrix} \in \mathcal{W}.$$

Let

$$\mathcal{D}\eta = \frac{d}{ds}\eta \quad \text{and} \quad \mathcal{D} : \text{dom } \mathcal{D} \to \mathcal{W}$$

and

$$\text{dom } \mathcal{D} = \{ \eta \in H^1(-r, 0; \mathcal{V}), \eta(0) = 0 \}.$$

Then,

$$\frac{d}{dt} \begin{pmatrix} \phi \\ \psi \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -\hat{A} & 0 & \hat{K} \\ 0 & I & \mathcal{D} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \gamma \end{pmatrix} + \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}.$$

3.2 Abstract Cauchy Problem

We can verify that

$$\langle \mathcal{D}\eta, \eta \rangle_{\mathcal{W}} \leq 0$$

provided that $g(-r) \geq 0$ and $\frac{dg}{ds} \geq 0$.

Let

$$\begin{aligned} x &= \langle v, w, Z, \dot{v}, \dot{w}, \dot{Z}, \eta(t+\cdot,\cdot)\rangle^T \\ \mathcal{X} &= (\mathcal{V}, |\cdot|_{\mathcal{V}} \times (\mathcal{H}, |\cdot|_{\mathcal{H}}) \times (\mathcal{W}, |\cdot|_{\mathcal{W}}). \end{aligned}$$

Thus, we have the formal abstract Cauchy problem

$$\dot{x}(t) = \mathcal{A}x(t) + F \quad \text{in } \mathcal{X}, \tag{3.4}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ -\hat{A} & 0 & \hat{K} \\ 0 & I & \mathcal{D} \end{pmatrix}$$

with

$$\text{dom } \mathcal{A} = \left\{ (\phi, \psi, \eta)^T \in \mathcal{X} : \begin{array}{c} \psi \in \mathcal{V}, \eta \in \text{dom } \mathcal{D} \\ -\hat{A}\phi + \hat{K}\eta \in \mathcal{H} \end{array} \right\}.$$

Remark 3.1 In (2.78) the underlined terms have coefficients of the same order in h/R if, for example, $\tau + T + \Delta \sim h/\sqrt{R}$. The term $\partial M_\theta/R$ is the sum of shear stresses in the elastic, constraining, and piezo layers. Since these layers are considered elastic, the contribution of $\partial M/R$ to the in-plane forces in these layers is considered negligible and ignored (see [BSW], [M], [BSW2]). Thus, the underlined terms, of which $\partial M_\theta/R$ is one, are ignored in what follows. In subsequent work we will consider these terms as perturbations and determine the extent of their effect on the dynamics of the beam.

We now proceed with Remark 3.1 in mind. We observe that $\mathcal{A}$ as defined above is the restriction of $\tilde{\mathcal{A}} \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}^*)$, where $\mathcal{Y} \equiv \mathcal{V} \times \mathcal{V} \times \mathcal{W}$, defined by the sesquilinear form: $\sigma(\Phi, \Psi) = \langle \tilde{\mathcal{A}}\Phi, \Psi \rangle_{\mathcal{Y}^*, \mathcal{Y}}$ so that $\sigma(\Phi, \Psi) = \langle \mathcal{A}\Phi, \Psi \rangle_{\mathcal{X}}$ for $\Phi \in \text{dom } \mathcal{A}$, $\psi \in \mathcal{X}$.

We can verify that

$$\text{Re}\left\langle \mathcal{A}\begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix} \right\rangle_{\mathcal{X}} \leq \lambda_0 \left(|\phi|^2_{\mathcal{V}} + |\psi|^2_{\mathcal{H}} + |\eta|^2_{\mathcal{W}} \right)$$

for some positive constant λ_0. Here ψ can be thought of as a constant function in $\mathcal{W}$. Now, it is clear that $\mathcal{A} - \lambda_0 I$ is dissipative.

Let $(f, g, h)^T \in \mathcal{X}$ and consider the equation

$$(\lambda - \mathcal{A})\begin{pmatrix} \phi \\ \psi \\ \eta \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \tag{3.5}$$

for $\lambda > \lambda_0$. This equation is equivalent to

$$\begin{aligned} &\lambda\phi - \psi = f \\ &\lambda\psi + \hat{A}\phi - \hat{K}\eta = g \\ &-\psi + (\lambda - \mathcal{D})\eta = h. \end{aligned} \tag{3.6}$$

Rearranging and solving for η in terms of other variables we obtain

$$\lambda^2\phi + \hat{A}\phi - (\hat{K}(1-e^{\lambda s})\phi) = \hat{K}[(\lambda - \mathcal{D})^{-1}(h-f)] + g + \lambda f. \tag{3.7}$$

Define a sesquilinear form $(\phi,\psi) \mapsto \mu_\lambda(\phi,\psi)$ on $\mathcal{V}\times\mathcal{V}$ by

$$\mu_\lambda(\phi,\psi) = \lambda^2\langle\phi,\psi\rangle + \langle\hat{A}\phi,\psi\rangle - \langle\hat{K}(1-e^{\lambda s})\phi,\psi\rangle.$$

It is readily seen that μ_λ is coercive. Using Lax-Milgram's lemma we solve (3.7) for ϕ. In turn we have

$$\begin{aligned} \psi &= \lambda\phi - f \in \text{dom } \hat{A} = \mathcal{V} \\ \eta &= (\lambda - \mathcal{D})^{-1}(h+\psi) \in \text{dom } \mathcal{D} \\ -\hat{A} + \hat{K}\eta &= g - \lambda\psi \in \mathcal{H}. \end{aligned}$$

Thus, we can solve, (3.6) for $(\phi,\psi,\eta)^T \in \text{dom } \mathcal{A}$. Thus, $\mathcal{A}$ generates a C_0-semigroup $T(t)$ on $\mathcal{X}$ which satisfies $|T(t)| \le \bar{M}e^{\omega t}$.

The above theory is adequate to treat the piezo embedded constrained layer curved beam of Section 2 so long as there are no input voltages to the patches and the normal external forces $\hat{q}_n$ are sufficiently smooth (i.e., in $L_2(\theta_1,\theta_2)$). However, when the patches are activated, the forces $\hat{q}_\theta$ and $\partial\hat{m}_x$ are not in $L_2(\theta_1,\theta_2)$ (in this case $(\hat{q}_\theta, \partial\hat{m}_x)$ is in $\mathcal{V}^*$) and a theory that extends the sense of equation (3.4) and the semigroup $T(t)$ to a larger space containing $\mathcal{Y}^*$ is necessary. The ideas to carry out this extension are by now rather straightforward although technically somewhat tedious. The generic arguments for this procedure (often referred to as "Haraux extrapolation") are given in [BKW] for general examples and specific arguments for second order systems written as first order abstract Cauchy systems (such as the example of this paper) are given in detail in [BSW2]. This leads directly to an extension of $T(t)$ on $\mathcal{X}$ to a C_0-semigroup $\hat{T}(t)$ on a space $\tilde{\mathcal{Y}}^*$ containing $\mathcal{Y}^* \equiv \mathcal{V}\times\mathcal{V}^*\times L^2(-r,0;\mathcal{V}^*)$ where we have assumed the usual Gelfand triple framework $\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}^* \hookrightarrow \mathcal{V}^*$, and $\mathcal{Y} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}^*$ where $\mathcal{Y} = \mathcal{V}\times\mathcal{V}\times L^2(-r,0;\mathcal{V})$ and $\mathcal{X} = \mathcal{V}\times\mathcal{H}\times L^2(-r,0;\mathcal{V})$. Thus, the theory above is readily extended to treat the general case of active patches in the models of Section 2. We shall not give the details here.

4 Approximation

The equations (2.78), (2.79), and (2.81) represent an infinite dimensional dynamical system, and thus whether interested in control, parameter estimation, or just simply simulation, we desire a finite dimensional computational algorithm. For this purpose, we need an approximation theory to serve as basis for computational schemes. In this section we describe one class of approximations that provides a good basis for computations.

4.1 The Approximation Spaces

After considering the function spaces in Section 3.1 we have chosen linear B-splines and cubic B-splines to approximate the tangential and the radial motions, $v(\cdot,\theta)$ and $w(\cdot,\theta)$ respectively. We shall use N to denote the number of subintervals used in partitioning $[\theta_1,\theta_2]$. We take the standard splines (see the Appendix of [BK] as well as references [Sh], [Sch]) corresponding to the partition $\{\theta_i\}_{i=0}^N$ and we obtain $N+1$ linear splines and $(N+3)$ cubic splines. Looking at the function spaces chosen we see that we obtain $N+1-2=N-1$ linear splines, and $N+3-4=N-1$ cubic splines. We denote these linear and cubic splines, $\{b_{1,i}^N\}_{i=1}^{N-1}$, $\{b_{2,i}^N\}_{i=1}^{N-1}$, respectively. Using them as basis elements, we define the following space

$$\mathcal{V}^N=\left\{\left(\sum_{i=1}^{N-1}\alpha_i b_{1,i}^N,\sum_{i=1}^{N-1}\beta_i b_{2,i}^N\sum_{i=1}^{N-1}\alpha_i^1 b_{1,i}^N,\ldots,\sum_{i=1}^{N-1}\alpha_i^4 b_{2,i}^N\right)\mid \alpha_i,\alpha_i^j,\beta_i\in\mathcal{R}\right\}. \tag{4.1}$$

For each N, $\mathcal{V}^N\subset\mathcal{V}\subset\mathcal{H}$, and the sequence $\{\mathcal{V}^\mathcal{N}\}$ provides an approximation space to $\mathcal{V}$. Letting $s_j^M=-jr/M$, $j=0,1,\ldots,M$, set

$$\chi_j^M(s)=\begin{cases}1 & s_j^M\le s<s_{j-1}^M\\ 0 & \text{otherwise}\end{cases}. \tag{4.2}$$

Let

$$\begin{aligned}\mathcal{W}^{M,N} &= \Big\{\Big(\sum_{j=1}^{M}\sum_{i=1}^{N-1}p_{ij}b_{1,i}^N(\theta)\chi_j^M(s),\sum_{j=1}^{M}\sum_{i=1}^{N-1}r_{ij}b_{2,i}^N(\theta)\chi_j^M(s),\\ &\quad \sum_{j=1}^{M}\sum_{i=1}^{N-1}\left(q_{ij}^1 b_{1,i}^N,\ldots,q_{ij}^4 b_{1,i}^N\right)\chi_j^M(s):p_{ij},q_{ij}^N,r_{ij}\in\mathcal{R}\Big)\Big\}.\end{aligned} \tag{4.3}$$

As a result, the approximation spaces to the state space $\mathcal{X}$ are given by

$$\mathcal{X}^{M,N}=\mathcal{V}^N\times\mathcal{V}^N\times\mathcal{W}^{M,N}.$$

Let

$$P_\mathcal{H}^N:\mathcal{H}\to\mathcal{V}^N,\qquad P_\mathcal{V}^N:\mathcal{V}\to\mathcal{V}^\mathcal{N},\qquad \mathcal{P}_\mathcal{G}^{\mathcal{M},\mathcal{N}}:\mathcal{W}\to\mathcal{W}^{\mathcal{M},\mathcal{N}}$$

denote the respective orthogonal projections. Thus, for $x=(\phi,\psi,\eta)^T\in\mathcal{X}$, we have $x^{M,N}=(\phi^N,\psi^N,\eta^{M,N})\in\mathcal{X}^{M,N}$, with $\phi^N=P_\mathcal{V}^N\phi$, $\psi^N=P_\mathcal{H}^N\psi$, and $\eta^{M,N}=P_G^{M,N}\eta$. In particular

$$\eta^{M,N} = (\eta_1^{M,N},\eta_2^{M,N},\eta_3^{M,N})^T$$

$$
\begin{aligned}
\eta_1^{M,N} &= \sum_{j=1}^{M}\sum_{i=1}^{N-1} \eta_{ij}^1 b_{1,i}^N(\theta)\chi_j^M(s), \qquad \eta_{ij}^1 \in \mathcal{R} \\
\eta_2^{M,N} &= \sum_{j=1}^{M}\sum_{i=1}^{N-1} \eta_{ij}^2 b_{2,i}^N(\theta)\chi_j^M(s), \qquad \eta_{ij}^2 \in \mathcal{R} \\
\eta_3^{M,N} &= \sum_{j=1}^{M}\sum_{i=1}^{N-1} \eta_{ij}^3 b_{1,i}^N(\theta)\chi_j^M(s), \qquad \eta_{ij}^3 \in \mathcal{R}^4.
\end{aligned}
$$

We define the operator $D^{M,N} : \mathcal{W}^{M,N} \to \mathcal{W}^{M,N}$ by

$$
\begin{aligned}
D^{M,N}\eta^{M,N} = & \left(\sum_{i=1}^{N-1}\sum_{j=1}^{M} \frac{M}{r}\left(\eta_{i,j-1}^1 - \eta_{ij}^1\right)\chi_j^M(s) b_{1,i}^N(\theta), \right. \\
& \sum_{i=1}^{N-1}\sum_{j=1}^{M} \frac{M}{r}\chi_j^M(s) b_{1,i}^N(\theta)\left(\eta_{i,j-1}^2 - \eta_{i,j}^2\right), \\
& \left. \sum_{i=1}^{N-1}\sum_{j=1}^{M} \frac{M}{r}\left(\eta_{i,j-1}^3 - \eta_{i,j}^3\right)\chi_j^M(s) b_{1,i}^N(\theta) \right),
\end{aligned}
\tag{4.4}
$$

with $\eta_{i,0}^1 = \eta_{i,0}^2 = 0$, $\eta_{i,0}^3 = (0,0,0,0)^T$.

We can now define the approximate system dynamics state operator $\mathcal{A}^{M,N}$ on

$$
\text{dom } \mathcal{A}^{M,N} = \left\{ (\phi^N, \psi^N, \eta^{M,N})^T \in \mathcal{X}^{M,N} : \eta^{M,N} \in \mathcal{W}^{M,N} \right\}
$$

$$
\mathcal{A}^{M,N}\begin{pmatrix} \phi^N \\ \psi^N \\ \eta^{M,N} \end{pmatrix} = \begin{pmatrix} \psi^N \\ -\hat{A}\phi^N + \hat{K}\eta^{M,N} \\ \psi^N + D^{M,N}\eta^{M,N} \end{pmatrix}.
$$

4.2 Convergence Analysis

Lemma 4.1 *The operator $D^{M,N} : \mathcal{W}^{M,N} \to \mathcal{W}^{M,N}$ is dissipative.*

Proof. Let

$$
\begin{aligned}
\omega_j^k &= \sum_{i=1}^{N-1} \eta_{ij}^k \partial^k b_{k,i}^N(\theta), \qquad k = 1, 2 \\
\omega_j^3 &= \sum_{i=1}^{N-1} \eta_{ij}^3 \partial b_{1,i}^N(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\langle \partial^k D^{M,N}\eta_k^{M,N}, \partial^k \eta_k^{M,N} \rangle &= \Big\langle \partial^k \textstyle\sum_{i=1}^{N-1}\sum_{j=1}^{M} \frac{M}{r}(\eta_{i,j-1}^1 - \eta_{i,j}^1) b_{k,i}^N \chi_j^M, \\
&\qquad \partial^k \textstyle\sum_{i=1}^{N-1}\sum_{j=1}^{M} \eta_{i,j}^1 b_{k,i}^N \chi_j^M \Big\rangle \\
&= \Big\langle \frac{M}{r}(\omega_{j-1}^1 - \omega_j^1)\chi_j^M, \omega_l \chi_l^M \Big\rangle.
\end{aligned}
$$

We thus have
$\int_{-r}^{0} g(s)\langle \partial^k D^{M,N}\eta_k^{M,N}, \partial^k \eta_k^{M,N}\rangle ds \le \frac{2M}{r}\sum_{j=1}^{M}\left(|\omega_{j-1}^k|_{L^2}^2 - |\omega_j^k|_L^2\right)\int_{-r}^{0} g(s)\chi_j^M(s) ds \le 0.$

We also note that

$$\int_{-r}^{0} g(s)\langle \partial D^{M,N}\Lambda\eta_2^{M,N}, \partial\eta_2^{M,N}\rangle ds$$
$$= \int_{-r}^{0} g(s)\langle \partial D^{M,N}\sqrt{\Lambda}\eta_2^{M,N}, \partial\sqrt{\Lambda}\eta_2^{M,N}\rangle ds \le 0.$$

Thus, $D^{M,N}$ is dissipative.
Let

$$\begin{aligned}\tilde{\mathcal{W}}^{M,N} &= \Bigg\{\Bigg(\sum_{j=1}^{M}\sum_{i=1}^{N-1} p_{ij}b_{1,i}^N(\theta)E_j^M(s), \sum_{j=1}^{M}\sum_{i=1}^{N-1} r_{ij}b_{2,i}^N(\theta)E_j^M(s),\\ &\sum_{j=1}^{M}\sum_{i=1}^{N-1}\left(q_{ij}^1 b_{1,i}^N(\theta),\ldots,q_{ij}^4 b_{1,i}^N(\theta)\right)E_j^M(s)\Bigg)\Bigg| p_{ij}, q_{ij}^n, r_{ij}\in\mathcal{R}\Bigg\},\end{aligned}$$

where

$$E_i^M(s) = \begin{cases} M(s-s_{i+1}^M)/r, & s_{i+1}^M \le s < s_i^M \\ M(s_{i-1}^M - s)/r, & s_i^M \le s < s_{i-1}^M \\ 0, & \text{otherwise} \end{cases}.$$

Define an isomorphism

$$i_{M,N} : \tilde{\mathcal{W}}^{M,N} \to \mathcal{W}^{M,N}$$

by the formula

$$\begin{aligned}&i_{M,N}\left(\sum_{j=1}^{M}\sum_{i=1}^{N-1} p_{ij}b_{1,i}^N E_j^M, \sum_{j=1}^{M}\sum_{i=1}^{N-1} M_{ij}b_{2,i}^N E_j^M, \sum_{j=1}^{M}\sum_{i=1}^{N-1}\left(q_{ij}^1 b_{1,i}^N,\ldots,q_{ij}^4 b_{1,i}^N\right)E_j^m\right)\\ &= \left(\sum_{j=1}^{M}\sum_{i=1}^{N-1} p_{ij}b_{1,i}^N\chi_j^M, \sum_{j=1}^{M}\sum_{i=1}^{N-1} M_{ij}b_{2,i}^N\chi_j^M, \sum_{j=1}^{M}\sum_{i=1}^{N-1}\left(q_{ij}^1 b_{1,i}^N,\ldots,q_{ij}^4 b_{1,i}^N\right)\chi_j^M\right),\end{aligned}$$

$p_{i,0} = q_{i,0}^l = r_{i,0} = 0$. For $\tilde{w}^{M,N} \in \tilde{\mathcal{W}}^{M,N}$, $D\tilde{w}^{M,N} = D^{M,N} i_{M,N}\tilde{w}^{M,N}$.

Lemma 4.2 *For $\lambda > 0$, $(\lambda I - D^{M,N})^{-1}P_G^{M,N}h \to (\lambda I - D)^{-1}h$, $h \in \mathcal{W}$.*

Proof. The proof of this lemma proceeds exactly in the same manner as in [BMZ]. We refer the reader to that paper for details.

The convergence scheme presented is adequate to treat approximate solutions of the homogeneous version of (2.78), (2.79) and (2.81). To obtain convergence for solutions when the right hand side of this system is in $\mathcal{Y}^*$, one must extend the convergence scheme to the extended semigroup $\hat{T}(t)$ on $\mathcal{Y}^*$ that was discussed at the end of Section 3. The necessary convergence scheme will be presented elsewhere.

5 Concluding Remarks

We have presented a general functional analytic framework that can be used to treat a wide class of curved active constrained layer structure models. This work extends our previous work ([BMZ]) where we did not consider the shear contribution in the viscoelastic layers and constraining layers were absent. The inclusion of shear effect and the constraining layers makes the model more realistic. We have illustrated how to treat one type of viscoelastic hysteresis in these models. Moreover, an approximation technique that can be used as a foundation for computational methods has been presented.

The ideas presented here are quite general and can be used (with minor modifications and/or extensions) to treat most of the active constrained layer structure models found in engineering and scientific research literature.

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6 Appendix

We present here the matrices A_1, B_1 used in (2.83). In (2.78), (2.79), M_θ, the sum of the internal moments of the elastic constraining, and piezo layers about the neutral line of the elastic core is approximated by $-\partial^2 \circ \xi_2 \circ \partial^2 w$, where ξ_2 is as in (2.46) and it is understood that derivatives are taken in the distributional sense where necessary. Some of the entries of the matrices need explanation, and we do that immediately after presenting the matrices. They are given by

$$A_1 = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where

$$\begin{aligned}
A_{11} &= -\tfrac{1}{\bar\rho}\partial \circ \bar\xi_1 \circ \partial + \tfrac{1}{\bar\rho}(bR^{-1}a_2 + \left\langle \bar D_1, \tilde J_1 \right\rangle), \\
A_{12} &= -\tfrac{bR^{-1}}{\bar\rho}\partial \circ \xi_1 - \tfrac{1}{\bar\rho}bR^{-1}(a_2 + \left\langle \bar D_1, \tilde J_2 \right\rangle)\partial + \partial \left\langle \bar D(\theta), \tilde J_2 \right\rangle \partial^2 - \tfrac{b}{R\bar\rho}\partial \circ \xi_2 \circ \partial^2 - \tfrac{b}{R^2}\tfrac{1}{\bar\rho}\partial \circ \xi_2 \circ \partial^2, \\
A_{13} &= \tfrac{1}{\bar\rho}\left\langle \bar D_1, I\cdot \right\rangle - \tfrac{1}{\bar\rho}\partial \left\langle \bar D(\theta), \cdot \right\rangle, \\
A_{21} &= \tfrac{1}{\bar\rho}(bR^{-1}a_2 + \bar\xi_1)\partial, \\
A_{22} &= \tfrac{1}{\bar\rho}(bR^{-1}\xi_1 + D_2\partial^2) + \tfrac{1}{\bar\rho}bR^{-1}\partial^2 \circ \xi_2 \circ \partial^2, \\
A_{23} &= \tfrac{1}{\bar\rho}\left\langle \bar D(\theta) - D_1, \cdot \right\rangle, \\
A_{31} &= \tfrac{1}{\rho^{ve}}E_1, \\
A_{32} &= \tfrac{1}{\rho^{ve}}E_2 + \tfrac{1}{\rho^{ve}}E_3, \\
A_{33} &= -\tfrac{1}{\rho^{ve}}\partial^2 - \tfrac{1}{\rho^{ve}}\tfrac{1-\nu_{ve}}{2}P,
\end{aligned}$$

and

$$B_1 = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

where

$$B_{11} = \frac{1}{\bar\rho}(\bar a_1 \partial^2 - a_2),$$
$$B_{12} = -\frac{1}{\bar\rho} bR^{-1}(a_2 - a_1)\partial - \frac{1}{\bar\rho} bR^{-1}\left\langle J^{ve}, \tilde J_2 \right\rangle \partial^2 + \frac{1}{\bar\rho} bR^{-1} b_1 \partial^3,$$
$$B_{13} = \frac{1}{\bar\rho} bR^{-1}\langle J^{ve}, \partial\cdot\rangle + \frac{1}{\bar\rho}\langle \bar D_1, I\cdot\rangle,$$
$$B_{21} = \frac{1}{\bar\rho}(\bar a_1 + \bar a_2)\partial,$$
$$B_{22} = -\frac{1}{\bar\rho} bR^{-1} a_1 + \frac{D_3}{\bar\rho}\partial^2,$$
$$B_{23} = \frac{1}{\bar\rho}\langle \bar D_1 - bR^{-1}J^{ve}, \cdot\rangle,$$
$$B_{31} = \frac{1}{\rho^{ve}} E_4,$$
$$B_{32} = \frac{1}{\rho^{ve}} E_5 + \frac{1}{\rho^{ve}} E_6,$$
$$B_{33} = \frac{1}{\rho^{ve}}\partial^2 \circ \Lambda + \frac{1}{\rho^{ve}} P.$$

We now explain the entries in A_1, B_1 as needed. From (2.82) recall $\partial \bar V(t,\theta) = Z(t,\theta)$.

In matrix A_1, $\langle \bar D_1, I\cdot\rangle(Z) = \langle \bar D_1, \bar V\rangle$, where $\langle\cdot,\cdot\rangle$ stands for inner product in $\mathcal{R}^4$. We remark that A_1 and B_1 act on the vector $(v, w, Z)^T$. Continuing,

$$\begin{aligned}
\langle \bar D(\theta) - D_1, \cdot\rangle(Z) &= \langle \bar D(\theta) - D_1, Z\rangle \\
\left\langle \bar D_1 - bR^{-1}J^{ve}, \cdot\right\rangle(Z) &= \left\langle \bar D_1 - bR^{-1}J^{ve}, Z\right\rangle \\
E_1(v) &= \frac{1}{2}(1 - \nu_{ve})(\partial v)\tilde P \Lambda T_1^{-1}\tilde J_1 \\
E_2(w) &= -(\partial^2 w)L \\
E_3(w) &= -\frac{1 - \nu_{ve}}{2}(\partial^2 w)J_2.
\end{aligned}$$

Next, we deal with matrix B_1. In matrix B_1,

$$\begin{aligned}
\langle J^{ve}, \partial\cdot\rangle(Z) &= \langle J^{ve}, \partial Z\rangle \\
\langle D_1, I\cdot\rangle(Z) &= \langle D_1, \bar V\rangle \\
\left\langle \bar D_1 - bR^{-1}J^{ve}, \cdot\right\rangle(Z) &= \left\langle \bar D_1 - bR^{-1}J^{ve}, Z\right\rangle \\
E_4(v) &= -(\partial v)\tilde P \Lambda T_1^{-1}\tilde J_1 \\
E_5(w) &= (\partial^2 w)L \\
E_6(w) &= (\partial^2 w)J_2.
\end{aligned}$$

References

[ATW] B. Azvine, G. R. Tomlinson and R. J. Wynne, Initial studies in the use of active constrained layer damping for controlling resonant vibration, Proc. SPIE Conference on Smart Structures and Materials, Orlando, 1994; *Journal of Smart Materials and Structures*, submitted

[BFW1] H. T. Banks, R. H. Fabiano and Y. Wang, Estimation of Boltzmann damping coefficients in beam models, Proceedings of COMCON Workshop on Stabilization of Flexible Structures, A. V. Balakrishnan and J. P. Zolesio, eds., Optimization Software Inc., New York (1987) pp. 13–35.

[BFW2] H. T. Banks, R. H. Fabiano and Y. Wang, Inverse problem techniques for beams with tip body and time hysteresis damping, *Mathematical Applications and Computations* **8** (1989) pp. 101–118.

[BGM] T. Bailey, A. Gruzen and P. Madden, RCS/piezoelectric distributed actuator study, Air Force Armstrong Labs, AFAL-TR-88-038 (1988).

[BIW] H. T. Banks, K. Ito and Y. Wang, Computational methods for identification and feedback control in structures with piezoceramic actuators and sensors, in *Recent Advances in Adaptive and Sensor Materials and Their Applications*, (C. A. Rogers, and R. C. Rogers, ed.), Technomic Publ., pp. 111–119, 1992; *J. Intell. Mat. Sys. Structures* **4**, pp. 469–476, 1993.

[BK] H. T. Banks and K. Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Birkhaüser, Boston, 1989.

[BKW] H. T. Banks, F. Kappel and C. Wang, Weak solutions and differentiability for size structured population models, *International Series of Numerical Mathematics* **100**, Birkhaüser, Basel, 1991, pp. 35–50.

[BMZ] H. T. Banks, N. G. Medhin, Y. Zhang, A mathematical framework for curved active constrained layer structures: Well-posedness and approximations, *Numer. Funct. Anal. and Optimz.* **17** (1&2) (1996) pp. 1–22.

[BR] A. Baz and J. Ro, The concept and performance of active constrained layer damping treatments, *Sound and Vibration* **28**, March (1994) pp. 18–21.

[BSW] H. T. Banks, R. C. Smith and Y. Wang, The modeling of piezoceramic patch interactions with shells, plates, and beams, *Quarterly of Applied Mathematics* **53** (1995) pp. 353–381.

[BSW2] H. T. Banks, R. C. Smith and Y. Wang, *Smart Materials Structures: Modeling, Estimation and Control*, Masson/J.Wiley, Paris/ Chichester, 1996.

[BZ] H. T. Banks and Y. Zhang, Computational methods for a curved beam with piezoceramic patches, *J. Intel. Mat. Syst. Structures*, Vol. **8**, (1997) pp. 260-278.

[C] R. M. Christensen, *Theory of Viscoelasticity: An Introduction*, 2nd ed., Academic Press, New York, 1982.

[G] L. Grenestedt, Lamination parameters for Reissner-Mindlin plates, *AIAA Journal*, Vol. **32**, No. 11: Technical Notes (1994) pp. 2328–2331.

[HWT] R. Hetnarski, R. West, J. Torok, Damping of vibrations of layered elastic-viscoelastic beams, ASME Reprint No. AMR 134 (1993) pp. s305–s311.

[KKS] D. B. Koconis, L. P. Kollar, G. Springer, Shape control of composite plates and shells with embedded actuators. I. Voltages specified, *Jour. Composite Materials*, Vol. **28**, No. 5 (1994) pp. 415–457.

[JK] C. D. Johnson and D. A. Kienholz, Finite element prediction of damping in structures with constrained viscoelastic layers, *American Institute of Aeronautics and Astronautics Journal* **20** (1981) pp. 1284–1290.

[K] H. Kolsky, The measurement of the material damping of high polymers over ten decades of frequency and its interpretation, V. K. Kinra and A. Wolfenden, eds. *Mechanics and Mechanisms of Material Damping*, American Society for Testing and Materials, Philadelphia, 1992.

[LH] C. Lin, C. Hsu, Interlaminar stresses in general laminates with straight free edges, *AIAA Journal*, Vol. **33**, No. 8, Aug. (1995) pp. 1471–1476.

[LW] W. H. Liao and K. W. Wang, On the active-passive hybrid vibration control actions of structures with active constrained layer treatments, Proc. ASME 15th Biennial Conference on Mechanical Vibration and Noise, to appear.

[M] S. Markus, *The Mechanics of Vibrations of Cylindrical Shells*, Elsevier, New York, 1988.

[Ma] M. Mace, Damping of beam vibrations by means of a thin constrained viscoelastic layer: evaluation of a new theory, *Journal of Sound and Vibration* **172** (1994) pp. 577–591.

[MM] D. J. Mead and S. Markus, The forced vibration of a three-layer, damped sandwiched beam with arbitrary boundary conditions, *Journal of Sound and Vibration* **10** (1969) pp. 163–175.

[Pa] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag New York, Inc., 1983.

[R] D. K. Rao, Frequency and loss of factors of sandwich beams under various boundary conditions, *Journal of Mechanical Engineering Science* **20** (1978) pp 271–282.

[RJ] C. D. Rahn, Sandeep Joshi, Modeling and control of an electrorheological sandwich beam, DE-Vol. 75, Active Control of Vibration and Noise ASME 1994, pp. 159–165.

[RWTW] J. A. Rongong, J. R. Wright, G. R. Tomlinson and R. J. Wynne, Modeling of hybrid constrained layer/piezoceramic approach to active damping, 1995, preprint.

[RWTW1] J. A. Rongong, J. R. Wright, G. R. Tomlinson, R. J. Wynne, Modelling of active damping for the extensional vibration of a beam (private communication).

[SC] I. H. Shames and F. A. Cozzarelli, *Elastic and Inelastic Stress Analysis*, Prentice Hall, New Jersey, 1992.

[Sch] L. Schumaker, *Spline Functions: Basic Theory*, John Wiley & Sons, New York, 1981.

[Sh] M. H. Schultz, *Spline Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[TB1] H. S. Tzou, Y. Bao, Dynamics and control of adaptive shells with continuous curvature changes, DE-Vol. 75, Active Control of Vibration and Noise, ASME 1994, pp. 149–157.

[TB2] H. S. Tzou, Y. Bao, A theory on anisotropic piezothermoelastic shell laminates with sensor/actuator applications, DE-Vol. 75, Active Control of Vibration and Noise, ASME 1994, pp. 411–420.

[VKI] W. C. Van Norstrand, G. Knowles and D. Inman, Active constrained layer damping for micro-satellites, *Dynamics and Control of Structures in Space*, Vol. II (1993) pp. 667–681.

[VL] V. V. Vasilev, S. A. Lure, On refined theories of beams, plates, and shells, *Jour. of Composite Materials*, Vol. **26**, No. 4 (1992) pp. 546–557.

[W] J. A. Walker, *Dynamical Systems and Evolution Equations*, Plenum Press, New York, 1980.

[Wl] J. Wloka, *Partial Differential Equations*, Cambridge University Press, 1992.

[Y] F. G. Yuan, Cross-plied cylindrical composite shells under transverse end load, *Jour. Composite Materials*, Vol. **29**, No. 7 (1995) pp. 903–925.

[YD] M.-J. Yan and E. H. Dowell, Governing equations for vibrating constrained-layer damping sandwich plates and beams, *Journal of Applied Mechanics* **39** (1972) pp. 1041–1047.