Global Gevrey Regularity for the Bénard Convection in Porous Medium with Zero Darcy-Prandtl Number.

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Abstract

In this paper, we prove the existence and uniqueness for the three-dimensional Bénard convection model in porous medium with zero Darcy-Prandtl number using the Galerkin procedure. In addition, we show that the solutions to this problem are analytic in time with values in a Gevrey class regularity. We also prove that the solution of the standard Galerkin method converges exponentially fast, in the wave number, to the exact solution. This gives an analytical justification to the two-dimensional computational results of Graham, Steen and Titi [J. of Nonlinear Science **3** (1993), 153-167].

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1 Introduction

Isothermal fluid flows in porous media have appeared in many industrial applications such as extracting thermal energy, designing insulation system for energy conservation, enhancing recovery of oil by thermal methods, assessing risks for nuclear waste disposal, etc. In many cases, people rely on computational simulations for a safe and economical understanding of the physical problems before the actual experiments, which are not easy to perform in the case of porous media, and thus pose a great number of challenges to numerical analysts due to the high degree of difficulties of the problems.

In this paper, we study a Bénard convection model in porous medium equations (2.1)-(2.7). This is one of the few three-dimensional models in Fluid Dynamics which is known to be globally well-posed for all positive time (see Fabrie [1986]). Specifically, Fabrie [1982] used the Galerkin approximation procedure to show the global existence and uniqueness of strong solutions to the *two-dimensional* version of the Bénard convection in porous medium problem (2.22)-(2.28). His proof for the two-dimensional case follows the same steps as for the two-dimensional Navier-Stokes equations (see, e.g., Constantin and Foias [1988], Foias, Manley and Temam [1987], Lions [1969], Temam [1983 & 1984]). However, Fabrie [1986] indicated that one would *not* be able to use the Galerkin approximation procedure to show the global regularity for the three-dimensional Bénard convection in porous medium problem (2.22)-(2.28); and thus he presented an alternative proof to the global regularity in this case. Roughly speaking, he first showed the short-time existence using iterative linearization procedure, and then he took advantage of the maximum principle for the temperature to establish the global existence. In fact, the maximum principle for the temperature is the key point in the proof of the global regularity for the system (2.22)-(2.28); and since one could not show the maximum principle for the temperature at the Galerkin-approximation level, Fabrie [1986] made the aforementioned claim. As a result, one would *not* be able to justify the Galerkin procedure as a numerical scheme for approximating solutions to the three-dimensional Bénard convection in porous media. In this paper, however, we establish in Section 4 the global existence and uniqueness of solutions to the three-dimensional problem using the standard Galerkin method. Furthermore, following the work of Foias and Temam [1989] (see also Promislow [1991] and Ferrari and Titi [1998]) we show in Section 5, again by using the Galerkin procedure, that the solutions to the system (2.22)-(2.28) belong to a certain Gevrey class of regularity for all time $t > 0$; and in particular that the solutions are real analytic. As a consequence of this regularity result, we justify not only the use of the Galerkin procedure as a numerical scheme, but we also show that the Galerkin scheme converges exponentially fast, in the wave numbers, to the exact solutions. This result holds both in two and three-dimensional cases. In particular, it holds for solutions which lie on the global attractor in both two and three-dimensional cases. This exponential-convergence result gives the analytical justification for the computational results of Graham, Steen and Titi [1993] for the two-dimensional case. To be more specific, it is well-known that the Nonlinear Galerkin method improves the convergence rate of the standard Galerkin method by an algebraic power (see Devulder, Marion and Titi [1993] and references therein). Since in the case of the system (2.22)-(2.28) the convergence rate of standard Galerkin is already exponential it will be difficult to observe the algebraic improvement of the Nonlinear Galerkin approximation in practical computations, as it was remarked in Graham, Steen and Titi [1993] (see also Jones, Margolin and Titi [1995]).

2 The Physical Model

Let Ω be the box $\{(x,y,z)\,|\,0 \le x \le L,\ 0 \le y \le \ell, 0 \le z \le \mathcal{L}\}$ in $I\!R^3$ filled with fluid-saturated porous medium. In the presence of gravity, the walls of the box are insulated and the box is heated from below with the constant temperature T_0 and cooled from above with $T_1 < T_0$. Under Darcy's law and the Boussinesq approximation, the governing equations for convective flow through porous medium (see e.g. Beck [1972] and Joseph [1976] and references therein) are of the form

$$\epsilon^{-1}\frac{\partial \vec{\mathbf{q}}}{\partial t} = [1-\gamma(T-T_1)]g\hat{\mathbf{k}} - \varrho_0^{-1}\nabla p - \frac{\nu}{k'}\vec{\mathbf{q}}, \tag{2.1}$$

$$(c\varrho)_m\frac{\partial T}{\partial t} + c\varrho_0(\vec{\mathbf{q}}\cdot\nabla T) = \tilde{\kappa}\Delta T, \tag{2.2}$$

$$(c\varrho)_m = (1-\epsilon)(c\varrho)_s + \epsilon c\varrho_0,$$

with

$$\nabla\cdot\vec{\mathbf{q}} = 0, \tag{2.3}$$

subject to the boundary conditions

$$\vec{\mathbf{q}}\cdot\vec{\mathbf{n}}\Big|_{\partial\Omega} = 0, \tag{2.4}$$

$$T(x,y,0,t) = T_0, \qquad T(x,y,\mathcal{L},t) = T_1, \tag{2.5}$$

$$\frac{\partial T}{\partial x}(0,y,z,t) = \frac{\partial T}{\partial x}(L,y,z,t) = \frac{\partial T}{\partial y}(x,0,z,t) = \frac{\partial T}{\partial y}(x,\ell,z,t) = 0, \tag{2.6}$$

and initial conditions

$$T(x,y,z,0) = T^0(x,y,z), \qquad \vec{\mathbf{q}}(x,y,z,0) = \vec{\mathbf{q}}^0(x,y,z) \tag{2.7}$$

where $\vec{\mathbf{q}}$, T and p are the fluid velocity, temperature and pressure respectively; $\tilde{\kappa}$ stands for the thermal conductivity of the saturated medium, $\hat{\mathbf{k}}$ for the vertical unit vector, $\vec{\mathbf{n}}$ for the unit vector normal to the boundary surface, g for the gravity, γ for the volumetric coefficient of thermal expansion of the fluid, ϵ for the porosity of the medium, ϱ_0 for fluid density, ν for the kinematic viscosity of the fluid and k' for the permeability of the medium; $(c\varrho)_m$, $(c\varrho)_s$ and $c\varrho_0$ are the heat capacities per unit volume of the saturated medium, the solid material of the porous medium and of the fluid respectively.

We denote by $Pr = \frac{k'\tilde{\kappa}}{\nu\mathcal{L}^2\epsilon}$ the Darcy-Prandtl number and by $Ra = \frac{k'\gamma g\mathcal{L}}{\tilde{\kappa}\nu}(T_0 - T_1)$ the Rayleigh-Darcy number. Note that when the temperature gradient between the bottom and the top of the box Ω is small, the fluid is motionless and only heat conduction occurs. However for large gradient temperature (i.e. for large Rayleigh-Darcy number Ra), the fluid will be in motion and the heat convection will take place.

We next rescale the equations (2.1)-(2.7) to obtain the dimensionless ones. Let us start by performing the following substitutions

$$\begin{array}{lll} \vec{\mathbf{q}} = Ra^{1/2}(\frac{\tilde{\kappa}}{\mathcal{L}})\vec{\mathbf{v}} & T = (T_0 - T_1)\tilde{T} & p = Ra^{1/2}(\frac{\nu\tilde{\kappa}\varrho_0}{k'})\tilde{p} \\ x = \mathcal{L}\tilde{x} & y = \mathcal{L}\tilde{y} & z = \mathcal{L}\tilde{z} \\ \tilde{L} = \frac{L}{\mathcal{L}} & \tilde{\ell} = \frac{\ell}{\mathcal{L}} & t = \frac{\mathcal{L}^2}{\tilde{\kappa}}\tau. \end{array}$$

to equations (2.1)-(2.7) and rename the variables $[\tilde{x},\tilde{y},\tilde{z},\tilde{L},\tilde{\ell},\tilde{T},\tilde{p}]$ as $[x,y,z,L,\ell,T,p]$ respectively, to obtain the non-dimensionalized system:

$$Pr\frac{\partial \vec{\mathbf{v}}}{\partial \tau} + \vec{\mathbf{v}} + \nabla p + Ra^{1/2}T\hat{\mathbf{k}} = Ra^{1/2}\frac{1+\gamma T_1}{\gamma(T_0-T_1)}\hat{\mathbf{k}} \quad \text{in} \quad \Omega, \tag{2.8}$$

$$h\frac{\partial T}{\partial \tau} - \kappa\Delta T + Ra^{1/2}(\vec{\mathbf{v}}\cdot\nabla T) = 0 \quad \text{in} \quad \Omega, \tag{2.9}$$

$$\nabla\cdot\vec{\mathbf{v}} = 0 \quad \text{in} \quad \Omega, \tag{2.10}$$

$$T(x,y,z,0) = \frac{1}{T_0-T_1}T^0(x,y,z), \quad \vec{\mathbf{v}}(x,y,z,0) = \frac{\mathcal{L}}{\tilde{\kappa}Ra^{1/2}}\vec{\mathbf{q}}^0 \quad \text{in} \quad \Omega, \tag{2.11}$$

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}\Big|_{\partial\Omega} = 0, \tag{2.12}$$

with the boundary conditions

$$T(x,y,0,\tau) = \frac{T_0}{T_0-T_1}, \qquad T(x,y,1,\tau) = \frac{T_1}{T_0-T_1}, \tag{2.13}$$

$$\frac{\partial T}{\partial x}(0,y,z,\tau) = \frac{\partial T}{\partial x}(L,y,z,\tau) = \frac{\partial T}{\partial y}(x,0,z,\tau) = \frac{\partial T}{\partial y}(x,\ell,z,\tau) = 0, \tag{2.14}$$

where $h = \frac{(c\varrho)_m}{c\varrho_0}$ and $\kappa = \frac{1}{c\varrho_0}$. Note that in porous medium flows, the Darcy-Prandtl number Pr is commonly very small as $k' \sim 10^{-4}\text{cm}^2 - 10^{-8}\text{cm}^2$ (see Joseph [1976] p. 56 for more details), and it is reasonable to neglect it. The non-zero Darcy-Prandtl number case is treated in Oliver-Titi [1998]. For now we assume $Pr = 0$. Moreover, by rescaling the time with $\tau = \frac{c\varrho_0}{(c\varrho)_m}t$ equations (2.8)-(2.9) become

$$\vec{\mathbf{v}} + \nabla p + Ra^{1/2}\hat{\mathbf{k}}T = Ra^{1/2}\frac{1+\gamma T_1}{\gamma(T_0-T_1)}\hat{\mathbf{k}} \quad \text{in} \quad \Omega, \tag{2.15}$$

$$\frac{\partial T}{\partial t} - \kappa\Delta T + Ra^{1/2}(\vec{\mathbf{v}}\cdot\nabla T) = 0 \quad \text{in} \quad \Omega, \tag{2.16}$$

$$\nabla\cdot\vec{\mathbf{v}} = 0 \quad \text{in} \quad \Omega, \tag{2.17}$$

$$T(x,y,z,0) = \frac{1}{T_0-T_1}T^0(x,y,z) \quad \text{in} \quad \Omega, \tag{2.18}$$

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}\Big|_{\partial\Omega} = 0, \tag{2.19}$$

$$T(x,y,0,t) = \frac{T_0}{T_0-T_1}, \qquad T(x,y,1,t) = \frac{T_1}{T_0-T_1}, \tag{2.20}$$

$$\frac{\partial T}{\partial x}(0,y,z,t) = \frac{\partial T}{\partial x}(L,y,z,t) = \frac{\partial T}{\partial y}(x,0,z,t) = \frac{\partial T}{\partial y}(x,\ell,z,t) = 0. \tag{2.21}$$

By letting $T = \theta + \frac{T_0}{T_0-T_1} - z$ and replacing p by $p + Ra^{1/2}[\frac{z^2}{2} - z(1 - \frac{1}{\gamma(T_0-T_1)})]$, equations (2.15)-(2.21) can be rewritten with the new variables as:

$$\vec{\mathbf{v}} + \nabla p + Ra^{1/2}\hat{\mathbf{k}}\theta = 0 \quad \text{in} \quad \Omega, \tag{2.22}$$

$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + Ra^{1/2}(\vec{\mathbf{v}}\cdot\nabla\theta) - Ra^{1/2}(\vec{\mathbf{v}}\cdot\hat{\mathbf{k}}) = 0 \quad \text{in} \quad \Omega, \tag{2.23}$$

$$\nabla\cdot\vec{\mathbf{v}} = 0 \quad \text{in} \quad \Omega, \tag{2.24}$$

$$\theta(x,y,z,0) = \theta^0(x,y,z) \quad \text{in} \quad \Omega, \tag{2.25}$$

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}\Big|_{\partial\Omega} = 0, \tag{2.26}$$

$$\theta(x,y,0) = 0, \qquad \theta(x,y,1) = 0, \tag{2.27}$$

$$\frac{\partial\theta}{\partial x}(0,y,z) = \frac{\partial\theta}{\partial x}(L,y,z) = \frac{\partial\theta}{\partial y}(x,0,z) = \frac{\partial\theta}{\partial y}(x,l,z) = 0. \tag{2.28}$$

Notice that for this system we have to specify the initial data only for θ, i.e. θ^0, and that the initial velocity $\vec{\mathbf{v}}^0$ can be recovered from θ^0 by solving equation (2.22). This is unlike the non-zero Darcy-Prandtl number case where one has to specify both the initial conditions for θ_0 and $\vec{\mathbf{v}}_0$ (see, for instance, Oliver-Titi [1998] for details and other related results).

3 Preliminaries and Functional Setting

Let $L^p(\Omega)$ and $H^k(\Omega)$ denote the usual L^p-Lebesgue space of integrable functions and H^k-Sobolev spaces respectively for $1 \le p \le \infty$ and $k \in I\!R$. The inner product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. Let

$$\mathcal{V} := \{\theta \in C^\infty(\Omega) \ \mid \ \theta \text{ satisfies (2.27) and (2.28) }\}$$

$$V := \text{Closure of } \ \mathcal{V} \text{ in the } H^1 - \text{norm}$$

$$H := \text{Closure of } \ \mathcal{V} \text{ in the } L^2 - \text{norm},$$

and let us denote the L^2-norm of H by $\|\cdot\|_H$, and the norm of V by $\|\cdot\|_V$. The inner product of H is exactly the inner product of $L^2(\Omega)$. Notice that due to the boundary conditions (2.27), the Poincaré inequality implies that the V-norm and the H^1-Sobolev norm are equivalent and thus $\|\theta\|_V = \|\nabla\theta\|_H$.

We set the operator $A = -\Delta$ subject to the boundary conditions (2.27) and (2.28) with domain

$$\mathcal{D}(A) = \{\psi \in H^2(\Omega) \ \mid \ \psi \text{ satisfies (2.27) and (2.28) }\}.$$

One can show using the Lax-Milgram Theorem and the elliptic regularity for the box domain (see Grisvard [1985] p. 147) that $A^{-1} : L^2(\Omega) \longrightarrow \mathcal{D}(A)$ exists and is a self-adjoint positive operator. Furthermore, by the Rellich Lemma (see, e.g., [1]) $A^{-1} : L^2(\Omega) \longrightarrow L^2(\Omega)$ is compact, and therefore $L^2(\Omega)$ has a complete set of orthonormal eigenfunctions of A. Indeed, the eigenfunctions are

$$\theta_{i,j,k}(\vec{\mathbf{x}}) = \sqrt{\frac{8}{\ell L}}\cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\sin(k\pi z), \tag{3.1}$$

for $i,j = 0,1,2,\ldots$; $k = 1,2,\ldots$ with the corresponding eigenvalues

$$\lambda_{i,j,k} = (\frac{i\pi}{L})^2 + (\frac{j\pi}{\ell})^2 + (k\pi)^2. \tag{3.2}$$

Let $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_m < \cdots$ denote the set of distinct eigenvalues $\lambda_{i,j,k}$'s ordered by their magnitude. We set $H_m = \text{span}\{\theta_{i,j,k} \mid \lambda_{i,j,k} \le \Lambda_m\}$.

Let us define

$$\tilde{\mathcal{V}} := \{\vec{\phi} \in (C^\infty(\Omega))^3 \ \mid \ \vec{\phi}\cdot\vec{\mathbf{n}}\Big|_{\partial\Omega} = 0 \ \text{ and } \ \nabla\cdot\vec{\phi} = 0 \ \text{ on } \ \Omega\},$$

$$\mathbf{V} := \text{Closure of } \tilde{\mathcal{V}} \text{ in the } H^1 - \text{norm},$$

$$\mathbf{H} := \text{Closure of } \tilde{\mathcal{V}} \text{ in the } L^2 - \text{norm}.$$

Let $\mathrm{P}_\sigma : (L^2(\Omega))^3 \to \mathbf{H}$ be the Helmholtz-Leray orthogonal projection. By applying P_σ to (2.22) and using (2.24) and (2.26), we have

$$\vec{\mathbf{v}} + Ra^{1/2}\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta) = 0. \tag{3.3}$$

Proposition 3.1 *Given a function $\theta \in V$, there exists a unique solution $\vec{\mathbf{v}}$ to the problem (2.22), (2.24) subject to boundary conditions (2.26). Moreover, $\vec{\mathbf{v}}$ satisfies*

$$\|\vec{\mathbf{v}}\|_{\mathbf{V}} \le c_2 Ra^{1/2}\|\theta\|_V. \tag{3.4}$$

If, in addition, $\theta = \theta_{i,j,k}$ as in (3.1), then the corresponding solution $\vec{\mathbf{v}}$ is given by

$$\vec{\mathbf{v}} = \frac{Ra^{1/2}\pi^2}{\lambda_{i,j,k}}\sqrt{\frac{8}{\ell L}}\vec{\mathbf{v}}_{i,j,k},$$

where $\lambda_{i,j,k}$ is given in (3.2) *and*

$$\vec{\mathbf{v}}_{i,j,k}(\vec{\mathbf{x}}) = \begin{bmatrix} \frac{ik}{L}\sin(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ \frac{jk}{\ell}\cos(\frac{i\pi}{L}x)\sin(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ -(\frac{i^2}{L^2}+\frac{j^2}{\ell^2})\cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\sin(k\pi z) \end{bmatrix}, \tag{3.5}$$

for $i, j = 0, 1, 2, \ldots; k = 1, 2, \ldots$.

Proof. By applying the div operator, i.e. $\nabla\cdot$, to equation (2.22) and taking scalar product of (2.22) with $\vec{\mathbf{n}}$ at the boundary of Ω, we get

$$-\Delta p = Ra^{1/2}\frac{\partial\theta}{\partial z} \tag{3.6}$$

$$\frac{\partial p}{\partial \vec{\mathbf{n}}}\Big|_{\partial\Omega} = 0. \tag{3.7}$$

It is known that the solution p to the above Neumann boundary value problem can be determined uniquely up to a constant. Thus from (2.22), $\vec{\mathbf{v}}$ is uniquely determined. The uniqueness is also clear from equation (3.3). In addition, the elliptic regularity estimates for the above Neumann boundary value problem in certain nonsmooth domains (see e.g. Grisvard [1985], p. 126 & 150) give

$$\Lambda_1^{1/2}\|\nabla p\|_{L^q(\Omega)} + \sum_{i,j=1}^{3}\|D_{i,j}p\|_{L^q(\Omega)} \le c_1 Ra^{1/2}\|\nabla\theta\|_{L^q(\Omega)} \tag{3.8}$$

for $1 < q < \infty$, (see also the Helmholtz decomposition for general $L^q(\Omega)$ spaces, for $1 < q < \infty$, in Fujiwara and Morimoto [1977]). But from (2.22) and (3.8), we get

$$\|\vec{\mathbf{v}}\|_{W^{1,q}(\Omega)} \le Ra^{1/2}\|\theta\|_{W^{1,q}(\Omega)} + \|\nabla p\|_{W^{1,q}(\Omega)} \le c_2 Ra^{1/2}\|\theta\|_{W^{1,q}(\Omega)}.$$

In particular, for $q = 2$ we have (3.4).

Now suppose $\theta = \theta_{i,j,k}$ for some $i, j = 0, 1, 2, \ldots;\ k = 1, 2, \ldots$, then by the method of separation of variables we can find the corresponding solution to Neumann boundary value problem (3.6)-(3.7). In fact, the exact solution is

$$p \equiv C + \frac{Ra^{1/2}(k\pi)}{\lambda_{i,j,k}}\sqrt{\frac{8}{\ell L}}\cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z).$$

Therefore it follows from (2.22) that

$$\begin{aligned}
\vec{\mathbf{v}} &= \frac{Ra^{1/2}(k\pi)}{\lambda_{i,j,k}}\sqrt{\frac{8}{\ell L}}\begin{bmatrix} \frac{i\pi}{L}\sin(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ \frac{j\pi}{\ell}\cos(\frac{i\pi}{L}x)\sin(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ (k\pi)\cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\sin(k\pi z) \end{bmatrix} - \\
&\qquad -Ra^{1/2}\sqrt{\frac{8}{\ell L}}\begin{bmatrix} 0 \\ 0 \\ \cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\sin(k\pi z) \end{bmatrix} \\
&= \frac{Ra^{1/2}\pi^2}{\lambda_{i,j,k}}\sqrt{\frac{8}{\ell L}}\begin{bmatrix} \frac{ik}{L}\sin(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ \frac{jk}{\ell}\cos(\frac{i\pi}{L}x)\sin(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ -(\frac{i^2}{L^2}+\frac{j^2}{\ell^2})\cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\sin(k\pi z) \end{bmatrix} \\
&= \frac{Ra^{1/2}\pi^2}{\lambda_{i,j,k}}\sqrt{\frac{8}{\ell L}}\vec{\mathbf{v}}_{i,j,k}.
\end{aligned}$$

■

Remark 3.2 *Observe that* $\vec{\mathbf{v}}_{0,0,k} = 0$ *for all* $k > 0$.

Due to the linearity of the problem (2.22), (2.24) subject to boundary conditions (2.26), one can easily verify the following Corollary.

Corollary 3.3 *Suppose*

$$\theta = \sum_{\substack{i,j=0 \\ k=1}}^{N_i,N_j,N_k} C_{i,j,k}\theta_{i,j,k},$$

for some positive integers N_i, N_j, N_k*, then the solution* $\vec{\mathbf{v}}$ *corresponding to the problem (2.22), (2.24) subject to boundary conditions (2.26) is*

$$\vec{\mathbf{v}} = \sum_{\substack{i,j=0 \\ k=1}}^{N_i,N_j,N_k} \frac{C_{i,j,k}Ra^{1/2}\pi^2}{\lambda_{i,j,k}}\sqrt{\frac{8}{\ell L}}\vec{\mathbf{v}}_{i,j,k}.$$

Since $\{\theta_{i,j,k}$'s$\}$ form a basis for H, then by Proposition (3.1) and Corollary (3.3), we conclude that any solution $\vec{\mathbf{v}}$ of the system (2.22)-(2.28) lie in the L^2-closure of the span of $\{\vec{\mathbf{v}}_{i,j,k} : i,j = 0,1,2,\ldots;\ k = 1,2,\ldots\}$. This does *not* mean, however, that $\{\vec{\mathbf{v}}_{i,j,k}$'s$\}$ form a basis for $\mathbf{H}$. Indeed, in the next proposition, we will complement $\{\vec{\mathbf{v}}_{i,j,k}\}$ to obtain an orthonormal basis for H.

Proposition 3.4 *(Fabrie [1989]) Let*

$$\vec{\mathbf{w}}_{i,j,k} = \begin{bmatrix} \frac{j}{\ell}\sin(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ \frac{-i}{L}\cos(\frac{i\pi}{L}x)\sin(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ 0 \end{bmatrix}$$

for $i,j,k = 0,1,2,\ldots$, then $\{\vec{\mathbf{v}}_{i,j,k}$'s$, \vec{\mathbf{w}}_{i,j,k}$'s$\}$ form an orthogonal basis for $\mathbf{H}$.

Proof. First notice that the functions

$$\vec{\mathbf{u}}_{i,j,k} = \begin{bmatrix} \sin(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ \cos(\frac{i\pi}{L}x)\sin(\frac{j\pi}{\ell}y)\cos(k\pi z) \\ \cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\sin(k\pi z) \end{bmatrix}$$

for $i,j,k = 0,1,2,\ldots$ form a complete orthogonal basis in $(L^2(\Omega))^3$, which by Helmholtz Decomposition Theorem (see, e.g., Fujirawa & Morimoto [1977]), can be decomposed into $G_2 \oplus \mathbf{H}$, where

$$G_2 = \{\nabla\Pi \in (L^2(\Omega))^3 :\ \Pi \in W^{1,2}(\Omega)\}.$$

But each function $\vec{\mathbf{u}}_{i,j,k}$ is a sum of $\nabla\Pi_{i,j,k}$ and the linear combinations of $\{\vec{\mathbf{v}}_{i,j,k}\}$'s and $\{\vec{\mathbf{w}}_{i,j,k}\}$'s, where

$$\Pi_{i,j,k} = \cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{\ell}y)\cos(k\pi z).$$

We then conclude that $\{\vec{\mathbf{v}}_{i,j,k}$'s$, \vec{\mathbf{w}}_{i,j,k}$'s$\}$ are dense in $\mathbf{H}$, and since they are orthogonal, they form an an orthogonal basis for $\mathbf{H}$. ■

Although the basis for $\mathbf{H}$, $\{\vec{\mathbf{v}}_{i,j,k}$'s$, \vec{\mathbf{w}}_{i,j,k}$'s$\}$, are found explicitly, the solution space for the velocity is spanned only by $\{\vec{\mathbf{v}}_{i,j,k}$'s$\}$. We summarize this observation in the following lemma:

Lemma 3.5 *Let $\vec{\mathbf{v}} \in \mathbf{H}$ satisfy*

$$\vec{\mathbf{v}} = \sum_{i,j,k}\alpha_{i,j,k}\vec{\mathbf{v}}_{i,j,k} + \sum_{i,j,k}\beta_{i,j,k}\vec{\mathbf{w}}_{i,j,k}.$$

Suppose $\vec{\mathbf{v}}$ solves (2.22)-(2.28) for $\theta \in V$, then

$$\beta_{i,j,k} = 0,$$

for $i,j = 0,1,2,\ldots; k = 1,2,\ldots$.

As a result, one finds that $\{\theta_{i,j,k}\text{'s}\}$ and $\{\vec{\mathbf{v}}_{i,j,k}\text{'s}\}$ are the right bases for the spaces of the solutions for the temperature θ and the velocity $\vec{\mathbf{v}}$, respectively. Therefore, one can employ these functions as bases for a Galerkin approximation (or numerical) procedure.

Remark 3.6 *The bases $\{\theta_{i,j,k}\text{'s}\}$ and $\{\vec{\mathbf{v}}_{i,j,k}\text{'s}\}$ also appear in a different context, developed by Beck [1972]. Namely, while studying the linear stability of the steady state, $\theta = 0$ and $\vec{\mathbf{v}} = 0$, of the system (2.22)-(2.28), Becks finds that $\{\theta_{i,j,k}\text{'s}\}$ and $\{\vec{\mathbf{v}}_{i,j,k}\text{'s}\}$ are the eigenfunctions of the corresponding linearized system with the eigenvalues different from $\{\lambda_{i,j,k}\text{'s}\}$ given in (3.2).*

Next, we define the bilinear form $B(\cdot,\cdot) : \mathbf{V} \times \mathcal{D}(A) \longrightarrow H$, such that $B(\vec{\mathbf{v}}, \phi) = \vec{\mathbf{v}} \cdot \nabla\phi$ for every $\vec{\mathbf{v}} \in \mathbf{V}$ and $\phi \in \mathcal{D}(A)$. Also, by using the Generalized Stokes Formula (see, e.g., Constantin and Foias [1988] and Temam [1984]) and the boundary conditions (2.27)-(2.28), one can easily check that for any $\vec{\mathbf{v}} \in \mathbf{V}$ and $\theta \in \mathcal{D}(A)$

$$(B(\vec{\mathbf{v}}, \theta), \theta) = 0. \tag{3.9}$$

By using the Cauchy-Schwarz inequality and the Sobolev Embedding Theorem (see, e.g., Adams [1975], Constantin & Foias [1988], Renardy & Rogers [1993], Temam [1998]), for the three-dimensional spatial space, one can easily show the following interpolation inequalities. Namely, that there is a constant c_3, which is scale invariant, such that for any function $u \in H^1(\Omega)$ we have:

$$\|u\|_{L^6(\Omega)} \le c_3 \|u\|_{H^1(\Omega)}, \tag{3.10}$$

$$\|u\|_{L^4(\Omega)} \le \|u\|_{L^2(\Omega)}^{1/4} \|u\|_{L^6(\Omega)}^{3/4} \le c_3^{3/4} \|u\|_{L^2(\Omega)}^{1/4} \|u\|_{H^1(\Omega)}^{3/4} \quad \text{by (3.10)}, \tag{3.11}$$

$$\|u\|_{L^3(\Omega)} \le \|u\|_{L^2(\Omega)}^{1/3} \|u\|_{L^4(\Omega)}^{2/3} \le c_3^{1/2} \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \quad \text{by (3.11)}. \tag{3.12}$$

Lemma 3.7 *For any $\theta \in \mathcal{D}(A)$, let $\vec{\mathbf{v}} \in \mathbf{V}$ satisfying (3.3), then there exists a constant a_1 such that*

$$\|B(\vec{\mathbf{v}}, \theta)\|_H \le a_1 Ra^{1/2} \|\theta\|_V^{3/2} \; \|A\theta\|_H^{1/2}. \tag{3.13}$$

Proof. One can easily verify the following estimates using the Hölder inequality, (3.4), (3.10), (3.12) and (3.4)

$$\begin{aligned} \|B(\vec{\mathbf{v}}, \theta)\|_H &\le \|\vec{\mathbf{v}}\|_{L^6} \|\nabla\theta\|_{L^3} \\ &\le a_0 \|\vec{\mathbf{v}}\|_{\mathbf{V}} \|\theta\|_V^{1/2} \; \|A\theta\|_H^{1/2} \\ &\le a_1 Ra^{1/2} \|\theta\|_V^{3/2} \|A\theta\|_H^{1/2}. \end{aligned}$$ ■

Definition 3.8 *Let $\theta^0 \in V$ be given, and let $\mathcal{T} > 0$. A strong solution of* (2.22)-(2.28) *in the interval* $[0, \mathcal{T}]$ *is a function* $\theta \in C([0,\mathcal{T}]; V) \cap L^2([0,\mathcal{T}]; \mathcal{D}(A))$ *such that* $\frac{d\theta}{dt} \in L^2([0,\mathcal{T}]; H)$, *and*

$$\frac{d\theta}{dt} + \kappa A\theta + Ra^{1/2} B(\vec{\mathbf{v}}, \theta) - Ra^{1/2} \vec{\mathbf{v}} \cdot \hat{\mathbf{k}} = 0 \qquad \text{in } H \tag{3.14}$$

for every $t \in [0,T]$*, with* $\vec{\mathbf{v}}$ *given in equation* (3.3)*. That is for every* $\omega \in H$ *we have*

$$(\theta(t_2),\omega) - (\theta(t_1),\omega) + \kappa \int_{t_1}^{t_2} (A\theta(\tau),\omega)d\tau$$
$$+Ra^{1/2}\int_{t_1}^{t_2} (B(\vec{\mathbf{v}}(\tau),\theta(\tau)),\omega)d\tau - Ra^{1/2}\int_{t_1}^{t_2} (\vec{\mathbf{v}}(\tau)\cdot\hat{\mathbf{k}},\omega)d\tau = 0, \qquad (3.15)$$

for every $t_1, t_2 \in [0,T]$*, where* $\vec{\mathbf{v}}$ *is given by* (3.3)*.*

4 Global Existence and Uniqueness

Following the proof of existence of strong solutions to the Navier-Stokes equations (see, e.g., Constantin and Foias [1988], Lions [1969], and Temam [1983,1984]) we first introduce the Galerkin approximating system to the dimensionless system (2.22)-(2.28) along with some a-priori estimates to obtain the short-time existence of strong solutions. Then we will prove a version of the Maximum Principle to guarantee the global existence of solutions. Let us start with the short-time existence of strong solution theorem.

Theorem 1 *If* $\theta^0 \in V$*, then there exists* $T^* = T^*(\|\theta^0\|_V, Ra) > 0$*, such that the system* (2.22)-(2.28) *has a unique strong solution* θ *on* $[0,T^*]$*.*

Proof. Let us now consider the Galerkin approximation system and prove the short-time existence of solutions as it has been done for the Navier-Stokes equations in Constantin-Foias [1988] and Temam [1984]

$$\frac{d\theta_m}{dt} + \kappa A\theta_m + Ra^{1/2}P_m B(\vec{\mathbf{v}}_m,\theta_m) - Ra^{1/2}P_m(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}}) = 0, \qquad (4.1)$$

$$\vec{\mathbf{v}}_m = -Ra^{1/2}\mathrm{P}_m\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta_m), \qquad (4.2)$$

with initial value

$$\theta_m(\vec{\mathbf{x}},0) = \theta_m^0(\vec{\mathbf{x}}) = P_m\theta^0(\vec{\mathbf{x}}) \qquad (4.3)$$

where $\vec{\mathbf{v}}_m \in \mathbf{H}_m$ and $\theta_m \in H_m$. The ordinary differential system (4.1)-(4.2) has short-time unique solution because it has a Lipschitz nonlinearity. It is well-known that for such systems, the solution exists as long as it is finite. Therefore to estimate the interval of existence we will estimate the interval for which the solution is finite. By taking the L^2-inner product of (4.1) with θ_m, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_m\|_H^2 + \kappa\|A^{1/2}\theta_m\|_H^2 + Ra^{1/2}(B(\vec{\mathbf{v}}_m,\theta_m),\theta_m) - Ra^{1/2}(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}},\theta_m) = 0. \qquad (4.4)$$

It follows from (3.9) and (4.2) that $(B(\vec{\mathbf{v}}_m,\theta_m),\theta_m) = 0$ and $Ra^{1/2}|(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}},\theta_m)| \le Ra\|\theta_m\|_H^2$, respectively. So that equation (4.4) becomes

$$\frac{1}{2}\frac{d}{dt}\|\theta_m\|_H^2 + \kappa\|A^{1/2}\theta_m\|_H^2 \le Ra\|\theta_m\|_H^2. \qquad (4.5)$$

Neglecting the non-negative term $\kappa\|A^{1/2}\theta_m\|_H^2$ and using Gronwall's inequality, we get

$$\|\theta_m(t)\|_H^2 \le \|\theta_m(0)\|_H^2\, e^{2Rat} \le \|\theta^0\|_H^2\, e^{2Ra\mathcal{T}} = \rho_0'(\mathcal{T}, \|\theta^0\|_H)^2, \tag{4.6}$$

for any $\mathcal{T} > 0$ and for all $t \in [0, \mathcal{T}]$. Next we take the L^2-inner product of (2.23) with $A\theta_m$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta_m\|_V^2 + \kappa\|A\theta_m\|_H^2 \le Ra^{1/2}|(B(\vec{\mathbf{v}}_m, \theta_m), A\theta_m)| + Ra^{1/2}|(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}}, A\theta_m)|. \tag{4.7}$$

We are going to estimate the first term on the right hand side of (4.7) as follows,

$$\begin{aligned} Ra^{1/2}|(B(\vec{\mathbf{v}}_m, \theta_m), A\theta_m)| &\le Ra^{1/2}\, \|B(\vec{\mathbf{v}}_m, \theta_m)\|_H\, \|A\theta_m\|_H \\ &\le a_1 Ra\|A^{1/2}\theta_m\|_H^{3/2}\, \|A\theta_m\|_H^{3/2} \ \text{ (using (3.13))} \\ &\le \frac{\kappa}{4}\|A\theta_m\|_H^2 + \frac{a_3 Ra^4}{\kappa^3}\|\theta_m\|_V^6 \ \text{ (by Young's ineq.).} \end{aligned} \tag{4.8}$$

Similarly,

$$Ra^{1/2}|(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}}, A\theta_m)| \le \frac{\kappa}{4}\|A\theta_m\|_H^2 + \frac{Ra^2}{\kappa}\|\theta_m\|_H^2 \le \frac{\kappa}{4}\|A\theta_m\|_H^2 + \frac{Ra^2}{\kappa\Lambda_1}\|\theta_m\|_V^2. \tag{4.9}$$

Inserting (4.8) (4.9) in (4.7), we obtain

$$\frac{d}{dt}\|\theta_m\|_V^2 + \kappa\|A\theta_m\|_H^2 \le 2[\frac{a_3 Ra^4}{\kappa^3}\|\theta_m\|_V^6 + \frac{Ra^2}{\kappa\Lambda_1}\|\theta_m\|_V^2] \le a_4\frac{Ra^4}{\kappa^3}(\|\theta_m\|_V^2 + \sigma_1)^3, \tag{4.10}$$

where $a_4 = 2\max\{a_3, 1\}$ and $\sigma_1 = \frac{\kappa}{Ra\Lambda_1^{1/2}}$. Now we let $y(t) = \sigma_1 + \|\theta_m(t)\|_V^2$. Then $y'(t) \le a_4 y^3(t)$ and by integrating this inequality we get

$$y^2(t) \le \frac{y^2(0)}{\sigma_1 - 2y^2(0)a_4 t} \quad \text{for } 0 \le t < \frac{\sigma_1}{2y^2(0)a_4}.$$

Since $y(0) \le \sigma_1 + \|\theta^0\|_V^2$, we have

$$\|\theta_m(t)\|_V^2 \le y(t) \le \frac{\sigma_1 + \|\theta^0\|_V^2}{\sqrt{\sigma_1 - 2(\sigma_1 + \|\theta^0\|_V^2)^2 a_4 t}} \qquad \text{for } 0 \le t < \frac{\sigma_1}{2(\sigma_1 + \|\theta^0\|_V^2)^2 a_4}.$$

In particular, for all $m \ge 1$ and for all $t \in [0, T^*]$

$$\|\theta_m(t)\|_V^2 \le \sqrt{2}(\sigma_1 + \|\theta^0\|_V^2) := (\rho_1')^2, \tag{4.11}$$

where $T^*(\|\theta^0\|_V) := \frac{\sigma_1}{4(\sigma_1 + \|\theta^0\|_V^2)^2 a_4}$. Integrating (4.10) over the interval $[0, T^*]$ and using (4.11), we get

$$\kappa\int_0^{T^*} \|A\theta_m(\tau)\|_H^2 d\tau \le \|\theta^0\|_V^2 + a_4 T^*[(\rho_1')^2 + \sigma_1]^3 := \kappa(\rho_2')^2. \tag{4.12}$$

Consequently we have from (4.6), (4.11) and (4.12) that

$$\{\theta_m\} \subset L^\infty([0,\infty); H) \cap L^\infty([0, T^*]; V) \cap L^2([0, T^*]; \mathcal{D}(A)), \tag{4.13}$$

and that the sequence $\{\theta_m\}$ is bounded in the corresponding norms. Moreover it follows from (4.2) that

$$\{\vec{\mathbf{v}}_m\} \subset L^\infty([0,\infty);\mathbf{H}) \cap L^\infty([0,T^*];\mathbf{V}) \cap L^2([0,T^*];\mathbf{H}^2), \tag{4.14}$$

and that the sequence $\{\vec{\mathbf{v}}_m\}$ is bounded in the corresponding norms.

Next we will show that $\frac{d\theta_m}{dt} \in L^2([0,T^*];H)$ and the corresponding norm in this space is bounded uniformly in m. In fact, we will show that the norm of $\frac{d\theta_m}{dt}$ is bounded uniformly in $L^2([0,T^*];H)$ due to (4.13) and (4.14). However it is sufficient, according to equation (4.1), to show that $\int_0^{T^*} \|Ra^{1/2}P_m(B(\vec{\mathbf{v}}_m,\theta_m))\|_H^2 dt$ is bounded uniformly in m; particularly,

$$\begin{aligned}
\int_0^{T^*} \|Ra^{1/2}P_m(B(\vec{\mathbf{v}}_m,\theta_m))\|_H^2 dt &\le \int_0^{T^*} Ra\|(B(\vec{\mathbf{v}}_m,\theta_m))\|_H^2 dt \\
&\le a_1^2 Ra^2 \int_0^{T^*} \|\theta_m\|_V^3 \, \|A\theta_m\|_H dt \qquad \text{(using (3.13))} \\
&\le a_1^2 Ra^2 (\int_0^{T^*} \|\theta_m\|_V^6 dt)^{1/2} (\int_0^{T^*} \|A\theta_m\|_H^2 dt)^{1/2} \\
&\le a_1^2 Ra^2 (T^*)^{1/2} (\rho_1')^3 \rho_2'.
\end{aligned} \tag{4.15}$$

Now we are ready to pass to the limits. Note that the sequence $\{\theta_m\}$ is bounded in $L^2([0,T^*];\mathcal{D}(A))$, then by the weak compactness theorem, there exists a $\theta \in L^2([0,T^*];\mathcal{D}(A))$ and a subsequence $\{\theta_{m'}\}$ of $\{\theta_m\}$ such that $\{\theta_{m'}\}$ converges weakly to θ in $L^2([0,T^*];\mathcal{D}(A))$. In addition, since by the Rellich Lemma (see, e.g., [1]) $\mathcal{D}(A)$ is compactly embedded in V, and since the sequence $\{\frac{d\theta_m}{dt}\}$ is bounded in $L^2([0,T^*];H)$, one can employ the Aubin's Compactness Theorem (see, e.g., Constantin and Foias [1988] p. 69, Lions [1969] and Temam [1984]) to show that $\{\theta_{m'}\}$ converges strongly to θ in $L^2([0,T^*];V)$. Let us from now on denote all the subsequences of $\{\theta_m\}$ by itself by extracting and relabeling. Thus, $\{\theta_m\}$ also converges strongly to θ in V almost everywhere in $[0,T^*]$. In particular,

$$\|\theta_m(t)\|_V \longrightarrow \|\theta(t)\|_V \quad \text{pointwise everywhere on } E \subseteq [0,T^*]; \quad |[0,T^*]\backslash E| = 0. \tag{4.16}$$

Similarly, one can establish the uniform bound in $L^2([0,T^*];\mathbf{H})$ for $\{\frac{d\vec{\mathbf{v}}_m}{dt}\}$ through equation (4.2) and then use (4.14) to show that there exists a function $\vec{\mathbf{v}} \in L^2([0,T^*];\mathbf{H}^2)$ such that $\{\vec{\mathbf{v}}_m\}$ converges strongly to $\vec{\mathbf{v}}$ in $L^2([0,T^*];\mathbf{V})$. In summary, we have

$$\begin{aligned}
A\theta_m &\longrightarrow A\theta \quad \text{weakly in} \quad L^2([0,T^*];H) \\
\theta_m &\longrightarrow \theta \quad \text{strongly in} \quad L^2([0,T^*];V) \\
\frac{d\theta_m}{dt} &\longrightarrow \frac{d\theta}{dt} \quad \text{weakly in} \quad L^2([0,T^*];H), \\
\vec{\mathbf{v}}_m &\longrightarrow \vec{\mathbf{v}} \quad \text{strongly in} \quad L^2([0,T^*];\mathbf{V}),
\end{aligned}$$

respectively. Thus it follows that for every $t_2, t_1 \in [0,T^*]$ and every $\omega \in H$

$$\kappa \int_{t_1}^{t_2} (A\theta_m,\omega)d\tau \longrightarrow \kappa \int_{t_1}^{t_2} (A\theta,\omega)d\tau, \tag{4.17}$$

and

$$\int_{t_1}^{t_2}(Ra^{1/2}P_m(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}}),\omega)d\tau \longrightarrow \int_{t_1}^{t_2}(Ra^{1/2}(\vec{\mathbf{v}}\cdot\hat{\mathbf{k}}),\omega)d\tau. \tag{4.18}$$

Moreover, since the sequence $\{\theta_m\}$ is bounded in $L^\infty([0,T^*];H)$ and $\{\frac{d\theta_m}{dt}\}$ is bounded in $L^2([0,T^*];H)$, we conclude from the Arzela-Ascoli Theorem that $\{\theta_m\}$ converges to θ in $C([0,T^*];H)$, hence in $C_W([0,T^*];H)$ and $C_W([0,T^*];V)$. In particular, we have

$$(\theta_m(t_2),\omega)-(\theta_m(t_1),\omega)\longrightarrow(\theta(t_2),\omega)-(\theta(t_1),\omega), \tag{4.19}$$

for every $t_2,t_1\in[0,T^*]$ and every $\omega\in H$.

We then take the L^2-inner product of (4.1) with $\omega\in H$ and integrate over the interval $[t_1,t_2]\subset[0,\mathcal{T}]$ to get

$$\begin{aligned}(\theta_m(t_2),\omega)-(\theta_m(t_1),\omega)&+\kappa\int_{t_1}^{t_2}(A\theta_m,\omega)d\tau\\ &= -Ra^{1/2}\int_{t_1}^{t_2}(P_m(B(\vec{\mathbf{v}}_m,\theta_m)),\omega)d\tau+Ra^{1/2}\int_{t_1}^{t_2}(P_m(\vec{\mathbf{v}}_m\cdot\hat{\mathbf{k}}),\omega)d\tau.\end{aligned} \tag{4.20}$$

Next we will show that

$$\int_{t_1}^{t_2}(P_m(B(\vec{\mathbf{v}}_m,\theta_m)),\omega)d\tau-\int_{t_1}^{t_2}(B(\vec{\mathbf{v}},\theta),\omega)d\tau\longrightarrow 0, \tag{4.21}$$

as $m\to\infty$ for every $t_2,t_1\in[0,\mathcal{T}]$. But first notice that

$$\begin{aligned}\|P_m(B(\vec{\mathbf{v}}_m,\theta_m))-B(\vec{\mathbf{v}},\theta)\|_H &\le \|B(\vec{\mathbf{v}}_m-\vec{\mathbf{v}},\theta_m)\|_H+\|B(\vec{\mathbf{v}},(\theta_m-\theta))\|_H\\ &\le \|\vec{\mathbf{v}}_m-\vec{\mathbf{v}}\|_{L^6}\ \|\nabla\theta_m\|_{L^3}+\|\vec{\mathbf{v}}\|_{L^6}\ \|\nabla(\theta_m-\theta)\|_{L^3}\\ &\le a_6[\|\vec{\mathbf{v}}_m-\vec{\mathbf{v}}\|_{\mathbf{V}}\|\theta_m\|_V^{1/2}\|A\theta_m\|_H^{1/2}+\\ &\qquad \|\vec{\mathbf{v}}\|_{\mathbf{V}}\|(\theta_m-\theta)\|_V^{1/2}\|A(\theta_m-\theta)\|_H^{1/2}],\end{aligned}$$

hence

$$\begin{aligned}&|\int_{t_1}^{t_2}(P_m(B(\vec{\mathbf{v}}_m,\theta_m))-B(\vec{\mathbf{v}},\theta),\omega)d\tau|\le\int_{t_1}^{t_2}\|P_m(B(\vec{\mathbf{v}}_m,\theta_m))-\vec{\mathbf{v}}\cdot\nabla\theta\|_H\ \|\omega\|_H d\tau\\ &\le a_6\|\omega\|_H\int_{t_1}^{t_2}[\|\vec{\mathbf{v}}_m-\vec{\mathbf{v}}\|_{\mathbf{V}}\|\theta_m\|_V^{1/2}\|A\theta_m\|_H^{1/2}+\|\vec{\mathbf{v}}\|_{\mathbf{V}}\|(\theta_m-\theta)\|_V^{1/2}\|A(\theta_m-\theta)\|_H^{1/2}]d\tau\\ &\le a_6\|\omega\|_H\Big[\Big\{\int_{t_1}^{t_2}\|\vec{\mathbf{v}}_m-\vec{\mathbf{v}}\|_{\mathbf{V}}^2d\tau\Big\}^{1/2}\Big\{\int_{t_1}^{t_2}\|\theta_m\|_V^2d\tau\Big\}^{1/4}\Big\{\int_{t_1}^{t_2}\|A\theta_m\|_H^2d\tau\Big\}^{1/4}\\ &\qquad+\Big\{\int_{t_1}^{t_2}\|\vec{\mathbf{v}}\|_{\mathbf{V}}^2d\tau\Big\}^{1/2}\Big\{\int_{t_1}^{t_2}\|\theta_m-\theta\|_V^2d\tau\Big\}^{1/4}\Big\{\int_{t_1}^{t_2}\|A(\theta_m-\theta)\|_H^2d\tau\Big\}^{1/4}\Big].\end{aligned} \tag{4.22}$$

Now since $\{\theta_m\}$ converges weakly to θ in $L^2([0,T^*];\mathcal{D}(A))$ and $\{\vec{\mathbf{v}}_m\}$ converges to $\vec{\mathbf{v}}$ strongly in $L^2([0,T^*];\mathbf{V})$, condition (4.21) is attained. Next we use (4.18)-(4.19) and (4.21) to pass to the limit as $m\to\infty$ for (4.20) to get

$$\begin{aligned}(\theta(t_2),\omega)-(\theta(t_1),\omega)+\kappa\int_{t_1}^{t_2}(A\theta,\omega)d\tau &= -Ra^{1/2}\int_{t_1}^{t_2}(B(\vec{\mathbf{v}},\theta),\omega)d\tau\\ &\quad+Ra^{1/2}\int_{t_1}^{t_2}(\vec{\mathbf{v}}\cdot\hat{\mathbf{k}},\omega)d\tau,\end{aligned} \tag{4.23}$$

for every $\omega \in H$ and for every $t_1, t_2 \in [0, T^*]$. Moreover since θ_m converges to θ weakly in $L^2([0,T^*];\mathcal{D}(A))$ and strongly in $L^2([0,T^*];V)$, we have from (4.6), (4.11) and (4.12) that

$$\|\theta(t)\|_H \leq \liminf_{m\to\infty} \|\theta_m(t)\|_H \quad \leq \rho_0 \quad \text{for all} \quad t \in [0,\infty), \tag{4.24}$$

$$\|\theta(t)\|_V \leq \liminf_{m\to\infty} \|\theta_m(t)\|_V \quad \leq \rho_1' \quad \text{for a.e.} \quad t \in [0,T^*], \tag{4.25}$$

and

$$\int_0^{T^*} \|A\theta(\tau)\|_H^2 d\tau \leq \liminf_{m\to\infty} \int_0^{T^*} \|A\theta_m(\tau)\|_H^2 d\tau \leq (\rho_2')^2, \tag{4.26}$$

respectively. Consequently we have

$$\theta \in L^\infty([0,\infty);H) \cap C([0,T^*];H) \cap L^\infty([0,T^*];V) \cap L^2([0,T^*];\mathcal{D}(A)), \tag{4.27}$$

and

$$\vec{\mathbf{v}} \in L^\infty([0,\infty);\mathbf{H}) \cap C([0,T^*];\mathbf{H}) \cap L^\infty([0,T^*];\mathbf{V}) \cap L^2([0,T^*];\mathbf{H}^2). \tag{4.28}$$

By integrating (4.10) over $[s,t] \subset [0,T^*]$ for $s < t$, we get

$$\|\theta_m(t)\|_V^2 + \kappa \int_s^t \|A\theta_m(\tau)\|_H^2 d\tau \leq \|\theta_m(s)\|_V^2 + a_4 \frac{Ra^4}{\kappa^3} \int_s^t (\|\theta_m(\tau)\|_V^2 + \sigma_1)^3 d\tau. \tag{4.29}$$

Due to (4.16),(4.25), (4.26) and the following property

$$\limsup_{n\to\infty}(a_n + b_n) \geq \limsup_{n\to\infty}(a_n) + \liminf_{n\to\infty}(b_n),$$

for any sequences a_n, $b_n \geq 0$, inequality (4.29) becomes

$$\|\theta(t)\|_V^2 + \kappa \int_s^t \|A\theta(d\tau)\|_H^2 d\tau \leq \|\theta(s)\|_V^2 + a_4 \frac{Ra^4}{\kappa^3}((\rho_1')^2 + \sigma_1)^3 (t-s), \tag{4.30}$$

for every s and $t \in E$, $s < t$, E is given in (4.16). By rearranging (4.30), we obtain:

$$\Big| \|\theta(t)\|_V^2 - \|\theta(s)\|_V^2 \Big| \leq \kappa \Big| \int_s^t \|A\theta(d\tau)\|_H^2 d\tau \Big| + a_4 \frac{Ra^4}{\kappa^3}((\rho_1')^2 + \sigma_1)^3 (t-s), \tag{4.31}$$

which one can extend to all s and t in $[0,T^*]$ by manipulating sequences in E that converge to elements in $[0,T^*]\backslash E$. Now using (4.31) together with the fact that $\theta \in C_{\mathrm{W}}([0,T^*];V)$, we get $\theta \in C([0,T^*];V)$.

To prove the uniqueness, let θ_1 and θ_2 be any two strong solutions of the system (3.3) along with (2.23)-(2.28) and let $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ be the corresponding velocities. Thus

$$(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2) + Ra^{1/2} \mathrm{P}_\sigma[\hat{\mathbf{k}}(\theta_1 - \theta_2)] = 0, \tag{4.32}$$

and

$$\begin{aligned} \frac{d}{dt}(\theta_1 - \theta_2) \quad &+ \quad \kappa A(\theta_1 - \theta_2) + Ra^{1/2}(\vec{\mathbf{v}}_1 \cdot \nabla)(\theta_1 - \theta_2) \\ &+ \quad Ra^{1/2}(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2) \cdot \nabla\theta_2 - Ra^{1/2}(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2) \cdot \hat{\mathbf{k}} = 0, \end{aligned} \tag{4.33}$$

where the above equation holds in $L^2([0,T^*];H)$. Let us set $\delta\theta=(\theta_1-\theta_2)$ and $\delta\vec{\mathbf{v}}=(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2)$. Insert (4.32) into (4.33) and act the result on $A\delta\theta\in L^2([0,T^*];H)$, to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta\theta\|_V^2+\kappa\|A\delta\theta\|_H^2 &\leq Ra^{1/2}|(B(\vec{\mathbf{v}}_1,\delta\theta),A\delta\theta)|+Ra^{1/2}|(B(\delta\vec{\mathbf{v}},\theta_2),A\delta\theta)| \\ &\quad +Ra|([\mathrm{P}_\sigma(\hat{\mathbf{k}}\delta\theta)]\cdot\hat{\mathbf{k}},A\delta\theta)|, \end{aligned} \tag{4.34}$$

for all $t>0$. But note that the three terms on the right hand side of equation (4.34) can be approximated as follows

$$\begin{aligned} Ra^{1/2}|(B(\vec{\mathbf{v}}_1,\delta\theta),A\delta\theta)| &\leq Ra^{1/2}\|\vec{\mathbf{v}}_1\|_{L^6(\Omega)}\ \|\nabla\delta\theta\|_{L^3(\Omega)}\|A\delta\theta\|_H \\ &\leq a'_{11}Ra\|\theta_1\|_{L^6(\Omega)}\ \|\delta\theta\|_V^{1/2}\|A\delta\theta\|_H^{3/2} \\ &\leq a'_{11}Ra\|\theta_1\|_V\ \|\delta\theta\|_V^{1/2}\|A\delta\theta\|_H^{3/2} \\ &\leq \frac{\kappa}{6}\|A\delta\theta\|_H^2+\frac{a_7Ra^4}{\kappa^3}\|\theta_1\|_V^4\ \|\delta\theta\|_V^2 \\ &\leq \frac{\kappa}{6}\|A\delta\theta\|_H^2+\frac{a_7Ra^4(\rho'_1)^4}{\kappa^3}\ \|\delta\theta\|_V^2, \end{aligned}$$

$$\begin{aligned} Ra^{1/2}|(B(\delta\vec{\mathbf{v}},\theta_2),A\delta\theta)| &\leq Ra^{1/2}\|\delta\vec{\mathbf{v}}\|_{L^6(\Omega)}\ \|\nabla\theta_2\|_{L^3(\Omega)}\ \|A\delta\theta\|_H \\ &\leq a'_{12}Ra\|\delta\theta\|_{L^6(\Omega)}\ \|\theta_2\|_V^{1/2}\ \ \|A\delta\theta_2\|_H^{1/2}\ \|A\delta\theta\|_H \\ &\leq a'_{12}Ra\|\delta\theta\|_V\ \|A\delta\theta_2\|_H\ \|A\delta\theta\|_H \\ &\leq \frac{\kappa}{6}\|A\delta\theta\|_H^2+\frac{a_8Ra^2}{\kappa}\|\delta\theta\|_V^2\ \|A\theta_2\|_H^2, \end{aligned}$$

and

$$Ra|([\mathrm{P}_\sigma(\hat{\mathbf{k}}\delta\theta)]\cdot\hat{\mathbf{k}},A\delta\theta)|\leq\frac{\kappa}{6}\|A\delta\theta\|_H^2+a_9Ra^2\frac{1}{\kappa}\|\delta\theta\|_H^2\leq\frac{\kappa}{6}\|A\delta\theta\|_H^2+a_9Ra^2\frac{1}{\kappa}\|\delta\theta\|_V^2.$$

So (4.34) becomes

$$\frac{d}{dt}\|\delta\theta\|_V^2+\kappa\|A\delta\theta\|_H^2\leq 2[\frac{a_7Ra^4(\rho'_1)^4}{\kappa^3}+\frac{a_8Ra^2}{\kappa}\ \|A\theta_2\|_H^2+a_9Ra^2\frac{1}{\kappa}]\ \|\delta\theta\|_V^2.$$

and therefore by Gronwall's inequality, we have

$$\begin{aligned} \|\delta\theta(t)\|_V^2 &\leq \|\delta\theta^0\|_V^2\ e^{2\int_0^t[\frac{a_7Ra^4(\rho'_1)^4}{\kappa^3}+\frac{a_8Ra^2}{\kappa}\ \|A\theta_2\|_H^2+a_9Ra^2\frac{1}{\kappa}]d\tau} \\ &\leq \|\delta\theta^0\|_H^2\ e^{2[\frac{a_7Ra^4(\rho'_1)^4}{\kappa^3}+a_9Ra^2\frac{1}{\kappa}]t+\frac{a_8Ra^2}{\kappa}(\rho'_2)^2}, \end{aligned}$$

for all $t\in[0,T^*]$. Since $\|\delta\theta^0\|_V=\|\theta_1(0)-\theta_2(0)\|_V\equiv 0$, the above inequality implies that $\delta\theta(t)=0$ for a.e. $t\in[0,T^*]$. ■

As a result of the above proof one can also conclude the continuous dependence of the solutions on the initial data. Therefore, we have the following:

Corollary 4.1 *The initial-boundary value problem* (2.22)-(2.28) *is well-posed in* V.

We would like to remark here that since $\theta \in \mathcal{D}(A) \subset H^2(\Omega)$ a.e. in $(0,T^*]$, by the Sobolev Embedding Theorem we have $\theta \in C(\overline{\Omega})$ a.e. in $(0,T^*]$. Now let $t_0 \in (0,T^*)$ such that $\|\theta(t_0)\|_{L^\infty(\Omega)}$ finite. Let us recall that $T(t) = \theta(t) + \frac{T_0}{T_0-T_1} - z$. Thus at $t = t_0$, we have $\underline{\kappa} \leq T(\vec{\mathbf{x}},t_0) \leq \overline{\kappa}$ for some constants $\underline{\kappa}$ and $\overline{\kappa}$. We are going to prove that T actually satisfies the Maximum Principle from t_0 onwards. Namely, we will prove

Proposition 4.2 *Let $\theta^0 \in V$ and $\theta(\vec{\mathbf{x}},t)$ be the strong solution of* (2.22)-(2.28) *on $\Omega \times [0,\mathcal{T}]$. Let $t_0 \in [0,\mathcal{T}]$ be such that $\underline{\kappa} \leq T(\vec{\mathbf{x}},t_0) \leq \overline{\kappa}$, for almost every $\vec{\mathbf{x}} \in \Omega$, for some finite numbers $\underline{\kappa}$ and $\overline{\kappa}$. Then we have $\underline{\kappa} \leq T(\vec{\mathbf{x}},t) \leq \overline{\kappa}$ for almost every $\vec{\mathbf{x}} \in \Omega$ and for almost every $t \in [t_0,\mathcal{T}]$.*

Proof. The technique employed in this proof has been adopted ¿from Foias, Manley and Temam [1987] (see also Ly, Mease and Titi [1996]). We first let

$$M(\vec{\mathbf{x}},t) := [T(\vec{\mathbf{x}},t) - \overline{\kappa}]_+ \equiv \max\{T(\vec{\mathbf{x}},t) - \overline{\kappa}, 0\} = \begin{cases} T(\vec{\mathbf{x}},t) - \overline{\kappa} & T(\vec{\mathbf{x}},t) > \overline{\kappa} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\theta \in L^2([0,\mathcal{T}];\mathcal{D}(A))$, we have $T \in L^2([0,\mathcal{T}];H^2(\Omega))$, so that $M(\vec{\mathbf{x}},t) \in H^1(\Omega)$ for almost every $t \in [0,\mathcal{T}]$ (see Gilbarg and Trudinger [1983] p. 153 for more details). By rewriting equation (2.16) in the following form

$$\frac{\partial}{\partial t}(T - \overline{\kappa}) - \kappa\Delta(T - \overline{\kappa}) + Ra^{1/2}(\vec{\mathbf{v}}\cdot\nabla(T - \overline{\kappa})) = 0 \quad \text{in} \quad \Omega. \tag{4.35}$$

Note that (4.35) holds in H for a.e. $t \in (0,\mathcal{T}]$. Now we take the L^2-inner product of (4.35) with $M \in H^1(\Omega) \subset H$ to get

$$(\frac{\partial}{\partial t}(T - \overline{\kappa}), M) - (\Delta[T - \overline{\kappa}], M) + Ra^{1/2}(\vec{\mathbf{v}}\cdot\nabla(T - \overline{\kappa}), M) = 0, \tag{4.36}$$

for almost every $t \in [t_0,\mathcal{T}]$. Note that,

$$(\frac{\partial}{\partial t}(T - \overline{\kappa}), M) = \frac{1}{2}\frac{d}{dt}\|M\|_H^2;$$

also observe that from the generalized Stokes Theorem (see, e.g., Constantin Foias [1988] and Temam [1984])

$$\begin{aligned} -(\Delta[T - \overline{\kappa}], M) &= \|\nabla M\|_H^2 - \int_{\partial\Omega} \frac{\partial(T - \overline{\kappa})}{\partial \vec{\mathbf{n}}} M\, dS \\ &= \|\nabla M\|_H^2 \quad \text{(using (2.13)-(2.14))} \\ &\geq \Lambda_1 \|M\|_H^2, \end{aligned}$$

and

$$Ra^{1/2}(\vec{\mathbf{v}}\cdot\nabla(T - \overline{\kappa}), M) = Ra^{1/2}[\int_{\partial\Omega} \frac{M^2}{2}\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}\, dS - (\nabla\cdot\vec{\mathbf{v}}, (\frac{M^2}{2}))] = 0 \quad \text{(by using (2.12)-(2.10))}.$$

Therefore (4.36) implies that

$$\frac{d}{dt}\|M\|_H^2 \leq -2\Lambda_1\|M\|_H^2 \quad \text{and hence} \quad \|M(t)\|_H^2 \leq \|M(t_0)\|_H^2\, e^{-2\Lambda_1(t - t_0)},$$

for any $t \geq t_0$. But $M(t_0) \equiv 0$, it follows that $\|M(t)\|_H \leq 0$ and thus $M(\vec{\mathbf{x}},t) \equiv 0$ and $T(\vec{\mathbf{x}},t) \leq \overline{\kappa}$ for almost every $\vec{\mathbf{x}} \in \Omega$ and every $t \in [t_0, \mathcal{T}]$.

Similarly, by letting $m(\vec{\mathbf{x}},t) := [T(\vec{\mathbf{x}},t) - \underline{\kappa}]_- \equiv \min\{T(\vec{\mathbf{x}},t) - \underline{\kappa}, 0\}$ one can repeat the above argument to show that $T(\vec{\mathbf{x}},t) \geq \underline{\kappa}$ for almost every $\vec{\mathbf{x}} \in \Omega$ and every $t \in [t_0, \mathcal{T}]$. ■

Since $T(\vec{\mathbf{x}},t) = \theta(\vec{\mathbf{x}},t) + \frac{T_0}{T_0 - T_1} - z$ we conclude from the previous proposition the following Corollary:

Corollary 4.3 *Let $\theta^0 \in V$ and $\theta(\vec{\mathbf{x}},t)$ be the strong solution of* (2.22)-(2.28) *on $\Omega \times [0,\mathcal{T}]$. Then for a.e. $t_0 > 0$,*

$$\|\theta(t)\|_{L^\infty(\Omega)} \leq \rho_\infty(t_0) := \|T(t_0)\|_{L^\infty(\Omega)} + 1 + \left|\frac{T_0}{T_0 - T_1}\right| \tag{4.37}$$

for almost every $t \in [t_0, \mathcal{T}]$.

Recall from Theorem 1 that we have short time existence of the strong solution for any initial data $\theta^0 \in V$. Next we show that the strong solution exists globally. Let us assume that $[0, T_{\max})$ is the maximal interval of existence. Then either $T_{\max} = \infty$ and we are done or $T_{\max} < \infty$ and $\limsup_{t \to T_{\max}^-} \|\theta(t)\|_V = \infty$. We suppose the latter and show that it is impossible. ¿From the definition of strong solution, we have $A\theta$ belongs to H for almost every $t \in (0, T_{\max})$. Let $t_0 \in (0, T_{\max})$ such that $\theta(t_0) \in \mathcal{D}(A) \subset L^\infty(\Omega)$. We also have $\frac{d\theta}{dt} \in L^2([t_0, T_{\max}]; H)$ and thus we can let equation (2.23) act on $A\theta \in L^2([t_0, T_{\max}]; H)$, its dual space, to get

$$\int_{t_0}^t (\frac{d\theta}{d\tau}, A\theta) d\tau + \kappa \int_{t_0}^t \|A\theta\|_H^2 d\tau \leq Ra^{1/2} \int_{t_0}^t |(B(\vec{\mathbf{v}},\theta), A\theta)| d\tau + Ra^{1/2} \int_{t_0}^t |(\vec{\mathbf{v}}\cdot\hat{\mathbf{k}}, A\theta)| d\tau, \tag{4.38}$$

for every $t \in (t_0, T_{\max})$. It follows from Chapter III Lemma 1.2 of Temam [1984] that

$$(\frac{d\theta}{d\tau}, A\theta) = \frac{1}{2}\frac{d}{d\tau}\|\theta\|_V^2.$$

Also by justifying similarly as in (4.15), one can obtain the following estimates

$$\begin{aligned} Ra^{1/2} \int_{t_0}^t |(B(\vec{\mathbf{v}},\theta), A\theta)| d\tau &\leq a_9 Ra \rho_\infty(t_0) \int_{t_0}^t \|\theta\|_V^{1/2} \|A\theta\|_H^{3/2} d\tau \\ &\leq \frac{\kappa}{4} \int_{t_0}^t \|A\theta\|_H^2 d\tau + \frac{a_9 Ra^4 \rho_\infty^4(t_0)}{\kappa^3} \int_{t_0}^t \|\theta(t)\|_V^2 d\tau \end{aligned} \tag{4.39}$$

and

$$Ra^{1/2} \int_{t_0}^t |(\vec{\mathbf{v}}\cdot\hat{\mathbf{k}}, A\theta)| d\tau \leq \frac{\kappa}{4} \int_{t_0}^t \|A\theta\|_H^2 d\tau + \frac{Ra^2}{\kappa} \int_{t_0}^t \|\theta\|_H^2 d\tau. \tag{4.40}$$

Thus, putting together equations (4.38)-(4.40), we have

$$\int_{t_0}^t \frac{d}{d\tau}\|\theta\|_V^2 d\tau + \kappa \int_{t_0}^t \|A\theta\|_H^2 d\tau \leq a_{10} \int_{t_0}^t \|\theta\|_V^2 d\tau, \tag{4.41}$$

for every $t \in (t_0, T_{\max})$, where $a_{10} = \frac{Ra^2}{\kappa}[\frac{a_9 Ra^2 \rho_\infty^4(t_0)}{\kappa^2} + \frac{1}{\Lambda_1}]$. It follows from Gronwall's Lemma that

$$\|\theta(t)\|_V^2 \leq \|\theta(t_0)\|_V^2 \, e^{a_{10}\cdot(t - t_0)} \tag{4.42}$$

for any $t \in (t_0, T_{\max})$ and hence,

$$\limsup_{t \to T_{\max}^-} \|\theta(t)\|_V \le \|\theta(t_0)\|_V^2 e^{a_{10}\cdot(T_{\max}-t_0)} < \infty$$

which contradicts the assumption. As a result we have established the global existence of strong solution.

Theorem 2 *Let $\theta^0 \in V$, then for any $\mathcal{T} > 0$ the system (2.22)-(2.28) has a unique strong solution θ on $[0, \mathcal{T}]$. Moreover, there exist two positive constants ρ_0 and ρ_1 depending on θ^0, κ, Ra and $|\Omega|$, but not on $\mathcal{T}$ such that*

$$\|\theta(t)\|_H \le \rho_0, \tag{4.43}$$

and

$$\|\theta(t)\|_V \le \rho_1 \tag{4.44}$$

for every $t \in [0, \mathcal{T}]$.

Proof. It follows from the Corollary 4.3 that for almost any $t_0 > 0$,

$$\|\theta(t)\|_H \le |\Omega|^{1/2} \|\theta(t)\|_{L^\infty(\Omega)} \le |\Omega|^{1/2} \rho_\infty(t_0)$$

for every $t \ge t_0$. However if we take the L^2-inner product of (2.23) with $\theta(t)$ and follow similar steps as in the proof of Theorem 1, especially (4.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_H^2 + \kappa\|\theta\|_V^2 \le Ra\|\theta\|_H^2. \tag{4.45}$$

So that $\|\theta(t)\|_H^2 \le \|\theta^0\|_H^2 e^{2Ra\cdot t}$. Thus by letting $\rho_0(\theta^0) := \max\{\|\theta^0\|_H \ e^{2Ra\cdot t_0}, |\Omega|^{1/2}\rho_\infty(t_0)\}$, inequality (4.43) is achieved. To show (4.44), we integrate (4.45) with respect to τ over the interval $[t, (t+1)]$ for $t \ge t_0$ to get

$$\int_t^{t+1} \|\theta\|_V^2 d\tau \le \frac{1}{\kappa}[\|\theta(t)\|_H^2 + Ra\rho_0^2] \le \frac{1 + Ra}{\kappa}\rho_0^2. \tag{4.46}$$

Recall from (4.41) that

$$\begin{aligned} \frac{d}{dt}\|\theta(t)\|_V^2 + \kappa\|A\theta(t)\|_H^2 &\le \frac{Ra^2}{\kappa}[a_9^2\rho_\infty^2(t_0)\|\theta(t)\|_V^2 + \|\theta(t)\|_H^2] \\ &\le \frac{Ra^2}{\kappa}a_9^2\rho_\infty^2(t_0)\|\theta(t)\|_V^2 + \frac{Ra^2}{\kappa}\rho_0^2. \end{aligned} \tag{4.47}$$

By the Generalized Gronwall's inequality (see Temam [1988] p. 88) we have

$$\begin{aligned} \|\theta(t)\|_V^2 &\le \left(\int_t^{t+1} \|\theta(\tau)\|_V^2 d\tau + \frac{Ra^2}{\kappa}\rho_0^2\right) \cdot \exp[\frac{Ra^2}{\kappa}a_9^2\rho_\infty^2(t_0)] \\ &\le \frac{1 + Ra + Ra^2}{\kappa}\rho_0^2 \cdot \exp[\frac{Ra^2}{\kappa}a_9^2\rho_\infty^2(t_0)], \quad \text{by (4.46)} \end{aligned} \tag{4.48}$$

for all $t \geq t_0$. Recall also from (4.25) that $\|\theta(t)\|_V^2 \leq {\rho'_1}^2 \equiv \sqrt{2}(1+\|\theta^0\|_V^2)$ for all $t \in [0, T^*]$, where $T^* := \dfrac{1}{4(1+\|\theta^0\|_V^2)^2 a_4}$ and from (4.42) that $\|\theta(t)\|_V^2 \leq \|\theta(t_0)\|_V^2 \cdot \exp[a_{10}\cdot(t-t_0)]$ for all $t \geq t_0$. Thus by letting $t_0 = T^*$ and

$$\rho_1^2 := \max\{\sqrt{2}(1+\|\theta^0\|_V^2), \|\theta(t_0)\|_V^2 \ \exp[a_{10}\cdot(t_0+1)], \frac{1+Ra+Ra^2}{\kappa}\rho_0^2\cdot\exp[\frac{Ra^2}{\kappa}a_9^2\rho_\infty^2(t_0)]\},$$

inequality (4.44) is attained.

Let us denote by $\theta(x,t) = S(t)\theta_0(x)$ the solution operator. Then based on the definition of strong solution, for any $\theta_0 \in V$ and any $T > 0$ we have $S(t)\theta_0 \in L^2([0,T];\mathcal{D}(A))$ and hence $S(t)\theta_0 \in \mathcal{D}(A)$ for almost every $t > 0$. In fact we will show in the next section that $S(t)\theta_0$ is analytic in time with value in $\mathcal{D}(A)$ so that $S(t)\theta_0 \in \mathcal{D}(A)$ for all $t > 0$. Moreover since $\mathcal{D}(A)$ is compactly embedded in V, we conclude that $S(t) : V \longrightarrow V$ is a compact operator. Thus the ball $B(0,\rho_1) = \{\theta \in H^1(\Omega) \mid \|\theta\| < \rho_1\}$ is an absorbing set in $H^1(\Omega)$ and the ω-limit set of $B(0,\rho_1)$ under $S(t)$, which we denote $\mathcal{A}$,

$$\mathcal{A} = \bigcap_{s>0}(\overline{\bigcup_{t\geq s} S(t)B(0,\rho_1)}),$$

where the closure is taken in the H^1 topology, is the global attractor (see Temam [1988] and references therein). The global attractor $\mathcal{A}$ is necessarily a nonempty compact subset of $H^1(\Omega)$. We would like to remark here that the existence and the dimension estimates of the global attractor for the two and three-dimensional Bénard convection in porous medium with zero Darcy-Prandtl number have been established by Fabrie [1990].

5 Analyticity in Time and a Gevrey Class Spatial Regularity

It is well known that the Nonlinear Galerkin method converges algebraically faster, to the exact solution of dissipative systems, than the standard Galerkin method (see, for instance, Devulder, Marion and Titi [1993]). However, Graham, Steen and Titi [1993] did not observe this advantage of the Nonlinear Galerkin method in their computations of the two-dimensional Bénard convection in porous medium. Their explanation of these computational result was that for this system the standard Galerkin method is already converging exponentially fast and that is why one does not observe the algebraic improvement of the Nonlinear Galerkin method. We will present here a rigorous justification to the explanation given by Graham, Steen and Titi [1997].

Following Foias and Temam [1989] (see also Ferrari and Titi [1998]) let us start with the definition of the following Gevrey class.

Definition 5.1 *Let $\sigma > 0$, we denote by $G_\sigma = \mathcal{D}(e^{\sigma A^{1/2}})$ the Gevrey class of all functions $\theta \in L^2(\Omega)$*

$$\theta(\vec{\mathbf{x}}) = \sum_{\substack{i,j=0\\k=1}}^{\infty} \Theta_{i,j,k}\ \theta_{i,j,k}(\vec{\mathbf{x}}) \quad \text{satisfying} \quad \sum_{\substack{i,j=0\\k=1}}^{\infty} e^{2\sigma\lambda_{i,j,k}^{1/2}}\ |\Theta_{i,j,k}|^2 < \infty.$$

Let $\theta_1, \theta_2 \in G_\sigma$ we denote the inner product in G_σ by

$$(\theta_1,\theta_2)_{G_\sigma} = (e^{\sigma A^{1/2}}\theta_1, e^{\sigma A^{1/2}}\theta_2)$$

and the corresponding norm

$$\|\theta\|_{G_\sigma}^2 = (\theta,\theta)_{G_\sigma} = (e^{\sigma A^{1/2}}\theta, e^{\sigma A^{1/2}}\theta) = \|e^{\sigma A^{1/2}}\theta\|_H^2.$$

It also follows immediately from the Cauchy-Schwarz inequality that

$$|(\theta_1,\theta_2)_{G_\sigma}| = (e^{\sigma A^{1/2}}\theta_1, e^{\sigma A^{1/2}}\theta_2) \le \|e^{\sigma A^{1/2}}\theta_1\|_H \; \|e^{\sigma A^{1/2}}\theta_2\|_H \tag{5.1}$$

One can see that elements in this Gevrey class will have high-mode coefficients decay exponentially in wave number to zero. Thus if we can show that the strong solution belong to some Gevrey class of regularity, then the result of the standard Galerkin converging exponentially can be easily achieved. The Gevrey regularity for the two-dimensional and the short-time three-dimensional Navier-Stokes equations have been established earlier by Foias and Temam [1989]. Similar results regarding the Navier-Stokes equations can be found in Henshaw, Kreiss and Reyna [1990]; however, their approach is quite different from the one by Foias and Temam [1989]. Promislow [1991] has generalized the result of Foias and Temam [1989] to parabolic equations with polynomial nonlinearities and Ferrari and Titi [1998] gave a general proof for parabolic systems with analytic nonlinearities. Along with the Gevrey regularity, Foias and Temam [1979] showed, by complexifying the time domain, that the solutions to the Navier-Stokes equations are analytic in time. Here we will indeed follow the work of Foias and Temam [1989] and [1979] to show that the solutions are analytic in time with values in a Gevrey class of regularity, but first let us show the following lemma:

Lemma 5.2 *If $\theta \in G_\sigma$ and $\vec{\mathbf{v}}$ is the corresponding solution of* (2.22)-(2.28)*, then for any $\varphi \in H$,*

$$|(e^{\sigma A^{1/2}}B(\vec{\mathbf{v}},\theta),\varphi)| \le d_1 Ra^{1/2}\|e^{\sigma A^{1/2}}\theta\|_V^{3/2} \; \|e^{\sigma A^{1/2}}A\theta\|_H^{1/2}\|\varphi\|_H, \tag{5.2}$$

and

$$\|B(\vec{\mathbf{v}},\theta)\|_{G_\sigma} \le d_1 Ra^{1/2}\|A^{1/2}\theta\|_{G_\sigma}^{3/2} \; \|A\theta\|_{G_\sigma}^{1/2}. \tag{5.3}$$

Proof. To show (5.2), we first express θ, φ, and $\vec{\mathbf{v}}$ in terms of Fourier expansions instead of $\{\theta_{i,j,k}$'s$\}$ and $\{\vec{\mathbf{v}}_{i,j,k}$'s$\}$, which can be achieved because the solution spaces, as shown in Corollary (3.3), are spanned by sines and cosines. Then we use the estimates similar to Foias and Temam [1989] as for the case of the Navier-Stokes equations to achieve (5.2). Namely, we first write θ, φ and $\vec{\mathbf{v}}$ as

$$\theta = \sum_{\vec{\mathbf{k}}} \mathbf{C}_{\vec{\mathbf{k}}} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}})}; \quad \varphi = \sum_{\vec{\mathbf{m}}} \mathbf{E}_{\vec{\mathbf{m}}} e^{i(\vec{\mathbf{m}}\cdot\vec{\mathbf{x}})}; \quad \vec{\mathbf{v}} = \sum_{\vec{\mathbf{n}}} \vec{\mathbf{D}}_{\vec{\mathbf{n}}} e^{i(\vec{\mathbf{n}}\cdot\vec{\mathbf{x}})};$$

where $\mathbf{C}_{\vec{\mathbf{k}}}$, $\mathbf{E}_{\vec{\mathbf{m}}} \in \mathbb{C}$, $\bar{\mathbf{C}}_{\vec{\mathbf{k}}} = \mathbf{C}_{-\vec{\mathbf{k}}}$, $\vec{\mathbf{D}}_{\vec{\mathbf{k}}} \in \mathbb{C}^3$, and $\vec{\mathbf{k}}, \vec{\mathbf{m}}, \vec{\mathbf{n}} \in \mathbf{Z}^3$. Note that $|\mathbf{C}_{\vec{\mathbf{k}}}| \le d_1'|\Theta_{i,j,k}|$ and $|\vec{\mathbf{D}}_{\vec{\mathbf{n}}}| \le d_2'|\Theta_{i,j,k}|$. Also,

$$\begin{aligned}
|(e^{\sigma A^{1/2}}B(\vec{\mathbf{v}},\theta),\varphi)| &= |(B(\vec{\mathbf{v}},\theta), e^{\sigma A^{1/2}}\varphi)| \\
&\le (L\ell)Ra^{1/2}\sum_{\vec{\mathbf{k}}+\vec{\mathbf{n}}=\vec{\mathbf{m}}} |\vec{\mathbf{D}}_{\vec{\mathbf{n}}}\cdot\vec{\mathbf{k}}\mathbf{C}_{\vec{\mathbf{k}}}\bar{\mathbf{E}}_{\vec{\mathbf{m}}}|e^{\sigma|\vec{\mathbf{k}}+\vec{\mathbf{n}}|} \\
&\le (L\ell)Ra^{1/2}\sum_{\vec{\mathbf{k}}+\vec{\mathbf{n}}=\vec{\mathbf{m}}} |\vec{\mathbf{D}}_{\vec{\mathbf{n}}}| \; e^{\sigma|\vec{\mathbf{n}}|} \; |\vec{\mathbf{k}}| \; |\mathbf{C}_{\vec{\mathbf{k}}}| \; e^{\sigma|\vec{\mathbf{k}}|} \; |\mathbf{E}_{\vec{\mathbf{m}}}| \\
&= \int_\Omega \xi_1(\vec{\mathbf{x}})\xi_2(\vec{\mathbf{x}})\xi_3(\vec{\mathbf{x}})d\vec{\mathbf{x}} \le \|\xi_1\xi_2\|_H\|\xi_3\|_H,
\end{aligned} \tag{5.4}$$

where $\xi_1(\vec{\mathbf{x}}) = \sum_{\vec{\mathbf{n}}} |\vec{\mathbf{D}}^*_{\vec{\mathbf{n}}}| e^{i(\vec{\mathbf{n}}\cdot\vec{\mathbf{x}})}$; $\xi_2(\vec{\mathbf{x}}) = \sum_{\vec{\mathbf{k}}} |\vec{\mathbf{k}}|\ |\mathbf{C}^*_{\vec{\mathbf{k}}}| e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}})}$; $\xi_3(\vec{\mathbf{x}}) = \sum_{\vec{\mathbf{m}}} |\mathbf{E}_{\vec{\mathbf{m}}}| e^{i(\vec{\mathbf{m}}\cdot\vec{\mathbf{x}})}$; with $\vec{\mathbf{D}}^*_{\vec{\mathbf{n}}} = \vec{\mathbf{D}}_{\vec{\mathbf{n}}}\ e^{\sigma|\vec{\mathbf{n}}|}$; $\mathbf{C}^*_{\vec{\mathbf{k}}} = \mathbf{C}_{\vec{\mathbf{k}}}\ e^{\sigma|\vec{\mathbf{k}}|}$. First notice that $\|\xi_3\|_H = \varphi\|_H$. The term $\|\xi_1\xi_2\|_H$ on the right hand side of (5.4) is estimated using procedures similar to those in the proof of Lemma 3.7, Proposition 3.1 and Corollary 3.3 to obtain

$$\begin{aligned}
\|\xi_1\xi_2\|_H &\leq \|\xi_1\|_{L^6}\|\xi_2\|_{L^3} \leq d_1'\|\xi_1\|_V\|\xi_2\|_V^{1/2}\|A\xi_2\|_H^{1/2} \\
&\leq d_1 Ra^{1/2}\|e^{\sigma A^{1/2}}\theta\|_V\|e^{\sigma A^{1/2}}\theta\|_V^{1/2}\ \|e^{\sigma A^{1/2}}A\theta\|_H^{1/2} \\
&= d_1 Ra^{1/2}\|e^{\sigma A^{1/2}}\theta\|_V^{3/2}\ \|e^{\sigma A^{1/2}}A\theta\|_H^{1/2}.
\end{aligned}$$

Consequently, (5.2) is obtained and (5.3) follows immediately. ■

Theorem 3 *Let $\mathcal{T} > 0$, $\theta^0 \in V$ and θ be the strong solution of* (2.22)-(2.28) *on* $\Omega \times [0, \mathcal{T}]$. *There exists a $\tau = \tau(\|\theta^0\|_V, Ra, \kappa) > 0$ such that for any $t_0 \in (0, \tau)$ the strong solution $\theta(t)$ is analytic, in time, in the interval $[t_0, \mathcal{T}]$ with values in $\mathcal{D}(A^{1/2}e^{t_0 A^{1/2}})$.*

Proof. Let $H_{\mathbb{C}} := \{\eta + i\xi \mid \eta \in H,\ \xi \in H\}$ be the complexification of the space H. For any $\theta_1, \theta_2 \in H_{\mathbb{C}}$ the usual complex inner product in $H_{\mathbb{C}}$ is defined as $(\theta_1, \theta_2) = \int_\Omega \theta_1(\vec{\mathbf{x}})\overline{\theta_2}(\vec{\mathbf{x}})d\mathbf{x}$, where $\overline{\theta_2}$ denotes the complex conjugate of θ_2. Similarly $V_{\mathbb{C}}$ and $\mathcal{D}_{\mathbb{C}}(A)$ are the complexification spaces of V and $\mathcal{D}(A)$, respectively.

Let us consider the complexified system of (4.1), (4.2) and (4.3). In particular, we consider the following complexified Galerkin system

$$\frac{d\theta_m}{d\zeta}(\zeta) + \kappa A\theta_m(\zeta) + Ra^{1/2}P_m(B(\vec{\mathbf{v}}_m(\zeta), \theta_m(\zeta))) - Ra^{1/2}P_m(\vec{\mathbf{v}}_m(\zeta)\cdot\hat{\mathbf{k}}) = 0, \tag{5.5}$$

$$\vec{\mathbf{v}}_m(\zeta) = -Ra^{1/2}\mathrm{P}_m\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta_m(\zeta)), \tag{5.6}$$

$$\theta_m(\vec{\mathbf{x}}, 0) = \theta^0_m(\vec{\mathbf{x}}) = P_m\theta^0(\vec{\mathbf{x}}) \tag{5.7}$$

where $\zeta = se^{i\alpha} \in \mathbb{C}$ with $s \geq 0$ and $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so that $\cos\alpha$ is always positive. We would like to remark here that the system (5.5)-(5.7) is an analytic system of O.D.E. which admits a unique analytic solution $\theta_m(\zeta)$ in the neighborhood of 0 of the complex plane. Moreover, since θ^0 is real-valued, the solutions $\theta_m(\zeta)$ with ζ restricted to the neighborhood of $(0, \mathcal{T}]$ in the real line will coincide with the usual Galerkin solutions. As before we will show first some estimates on the various norms which are uniformly independent of m. Then we will pass to the limit to obtain the time-complexified solution.

Let us take the scalar product of (5.5) with $A\theta_m(\zeta)$ in $G_{s\cos\alpha}$, then multiply by $e^{i\alpha}$ and take the real part to get

$$\begin{aligned}
&\mathrm{Re}\Big\{e^{i\alpha}(e^{s\cos\alpha A^{1/2}}\frac{d\theta_m}{d\zeta}(\zeta), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\} + \kappa\cos\alpha\|e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta)\|_H^2 \\
&= Ra^{1/2}\mathrm{Re}\Big\{e^{i\alpha}(e^{s\cos\alpha A^{1/2}}P_m(\vec{\mathbf{v}}_m(\zeta)\cdot\hat{\mathbf{k}}), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\} - \\
&\quad Ra^{1/2}\mathrm{Re}\Big\{e^{i\alpha}(e^{s\cos\alpha A^{1/2}}P_mB(\vec{\mathbf{v}}_m(\zeta), \theta_m(\zeta)), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\}.
\end{aligned} \tag{5.8}$$

But observe that the terms in (5.8) can be estimated by using inequalities (5.1) and (5.3)

$$\begin{aligned}
&\mathrm{Re}\Big\{e^{i\alpha}(e^{s\cos\alpha A^{1/2}}\frac{d\theta_m}{d\zeta}(\zeta), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\} \\
&\qquad = \mathrm{Re}\Big\{(\frac{d}{ds}(e^{s\cos\alpha A^{1/2}}A^{1/2}\theta_m(\zeta)), A^{1/2}\theta_m(\zeta))\Big\} - \\
&\qquad\qquad \cos\alpha\mathrm{Re}\Big\{(e^{s\cos\alpha A^{1/2}}A^{1/2}\theta_m(\zeta), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\} \\
&\qquad = \frac{1}{2}\frac{d}{ds}\|A^{1/2}\theta_m(\zeta)\|^2_{G_{s\cos\alpha}} - \cos\alpha\mathrm{Re}\Big\{(A^{1/2}\theta_m(\zeta), A\theta_m(\zeta))_{G_{s\cos\alpha}}\Big\},
\end{aligned}$$

$$|\cos\alpha\mathrm{Re}\ (A^{1/2}\theta_m(\zeta), A\theta_m(\zeta))_{G_{s\cos\alpha}}| \le \frac{\kappa\cos\alpha}{4}\|A\theta_m(\zeta)\|^2_{G_{s\cos\alpha}} + a_{11}\|A^{1/2}\theta_m(\zeta)\|^2_{G_{s\cos\alpha}},$$

$$\begin{aligned}
&|Ra^{1/2}\mathrm{Re}\Big\{e^{i\alpha}(e^{s\cos\alpha A^{1/2}}P_m(\vec{\mathbf{v}}_m(\zeta)\cdot\hat{\mathbf{k}}), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\}| \\
&\qquad = |Ra^{1/2}\mathrm{Re}\ e^{i\alpha}(P_m(\vec{\mathbf{v}}_m(\zeta)\cdot\hat{\mathbf{k}}), A\theta_m(\zeta))_{G_{s\cos\alpha}}| \\
&\qquad \le \frac{\kappa\cos\alpha}{8}\|A\theta_m(\zeta)\|^2_{G_{s\cos\alpha}} + a_{12}\|\theta_m(\zeta)\|^2_{G_{s\cos\alpha}},
\end{aligned}$$

and

$$\begin{aligned}
&|Ra^{1/2}\mathrm{Re}\Big\{e^{i\alpha}(e^{s\cos\alpha A^{1/2}}P_mB(\vec{\mathbf{v}}_m(\zeta),\theta_m(\zeta)), e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta))\Big\}| \\
&\qquad = |Ra^{1/2}\mathrm{Re}\ e^{i\alpha}(P_m(\vec{\mathbf{v}}_m(\zeta)\cdot\nabla\theta_m(\zeta)), A\theta_m(\zeta))_{G_{s\cos\alpha}}| \\
&\qquad \le \|Ra^{1/2}B(\vec{\mathbf{v}}_m,\theta_m)\|_{G_{s\cos\alpha}}\ \ \|A\theta_m\|_{G_{s\cos\alpha}} \\
&\qquad \le a_0 Ra\ \|A^{1/2}\theta_m\|^{3/2}_{G_{s\cos\alpha}}\ \|A\theta_m\|^{3/2}_{G_{s\cos\alpha}} \quad \text{(by (5.3))} \\
&\qquad \le \frac{\kappa\cos\alpha}{8}\|A\theta_m(\zeta)\|^2_{G_{s\cos\alpha}} + a_{13}\|A^{1/2}\theta_m(\zeta)\|^6_{s\cos\alpha},
\end{aligned}$$

where $a_{11} = \frac{\cos\alpha}{\kappa}$, $a_{12} = \frac{2Ra^2}{\kappa\cos\alpha}$ and $a_{13} = \frac{54a_0^4Ra^4}{\kappa^3\cos^3\alpha}$. So that (5.8) becomes

$$\begin{aligned}
&\frac{d}{ds}\|A^{1/2}\theta_m\|^2_{G_{s\cos\alpha}} + \kappa\cos\alpha\|A\theta_m\|^2_{G_{s\cos\alpha}} \\
&\qquad \le a_{11}\|A^{1/2}\theta_m\|^2_{G_{s\cos\alpha}} + a_{12}\|\theta_m\|^2_{G_{s\cos\alpha}} + a_{13}\|A^{1/2}\theta_m\|^6_{G_{s\cos\alpha}} \\
&\qquad \le a_{11}\|A^{1/2}\theta_m\|^2_{G_{s\cos\alpha}} + \frac{a_{12}}{\Lambda_1}\|A^{1/2}\theta_m\|^2_{G_{s\cos\alpha}} + a_{13}\|A^{1/2}\theta_m\|^6G_{s\cos\alpha} \\
&\qquad \le a_{13}[\sigma_2 + \|A^{1/2}\theta_m\|^2_{G_{s\cos\alpha}}]^3,
\end{aligned}$$

where $\sigma_2 = \sqrt{\frac{a_{11}+a_{12}\Lambda_1^{-1}}{a_{13}}}$ Let $y(\zeta) = \sigma_2 + \|A^{1/2}\theta_m(\zeta)\|^2_{G_{s\cos\alpha}} = \sigma_2 + \|e^{s\cos\alpha A^{1/2}}A^{1/2}\theta_m(\zeta)\|^2_H$ and proceed similarly to the analysis performed in Section 4, we attain

$$\|e^{s\cos\alpha A^{1/2}}\theta_m(\zeta)\|^2_V = \|A^{1/2}\theta_m(\zeta)\|^2_{G_{s\cos\alpha}} \le 2[\sigma_2 + \|\theta^0\|^2_V] := (\rho_1'')^2 \tag{5.9}$$

for all $\zeta = se^{i\alpha}$ such that $0 \le s \le \frac{\sigma_2}{4a_{13}(\sigma_2+\|\theta^0\|^2_V)}$. Hence we obtain a domain of analyticity D_τ of $\theta_m(\zeta)$, where $D_\tau = \{\zeta = se^{i\alpha}\ |\ \ |\alpha| < \frac{\pi}{2}\ \&\ 0 \le s \le \tau\}$ and $0 < \tau(\|\theta^0\|_V, Ra, \kappa) =$

$\frac{\sigma_2}{4a_{13}(\sigma_2+\|\theta^0\|_V^2)}$. Let Γ be the boundary of the closed disk of radius r in D_τ and centered at ζ. Then by the Cauchy integral formula

$$\frac{d^k\theta_m}{d\zeta^k}(\zeta) = \frac{k!}{2\pi i}\int_\Gamma \frac{\theta_m}{(z-\zeta)^{k+1}}dz. \tag{5.10}$$

Thus

$$\begin{aligned}\|\frac{d^k\theta_m}{d\zeta^k}(\zeta)\|_{G_{s\cos\alpha}} &\leq \frac{k!}{2\pi}\int_\Gamma \frac{\|\theta_m(\zeta)\|_{G_{s\cos\alpha}}}{|z-\zeta|_{\mathbb{C}}{}^{k+1}}|dz| \\ &\leq \frac{k!}{2\pi\Lambda_1^{1/2}}\int_\Gamma \frac{\|A^{1/2}\theta_m(\zeta)\|_{G_{s\cos\alpha}}}{|z-\zeta|_{\mathbb{C}}{}^{k+1}}|dz| \leq \frac{k!}{r^k\Lambda_1^{1/2}}\rho_1''.\end{aligned} \tag{5.11}$$

Let $\mathcal{S}$ be compact subset of D_τ and $r_{\mathcal{S}}$ = distance$(\mathcal{S}, \partial D_\tau)$. Particularly for $k=1$ and r smaller than $r_{\mathcal{S}}$, we have from (5.11) that

$$\|\frac{d}{d\zeta}\theta_m(\zeta)\|_{G_{s\cos\alpha}} \leq \frac{1}{r\Lambda_1^{1/2}}\rho_1'' \qquad \text{for all} \quad \zeta\in\mathcal{S}. \tag{5.12}$$

Then by using inequalities (5.12), (5.3), the Young's inequality and (5.9), equation (5.5) yields

$$\begin{aligned}\kappa\|A\theta_m(\zeta)\|_{G_{s\cos\alpha}} &\leq \|\frac{d\theta_m(\zeta)}{d\zeta}\|_{G_{s\cos\alpha}} + Ra\|\theta_m(\zeta)\hat{\mathbf{k}}\cdot\nabla\theta_m(\zeta)\|_{G_{s\cos\alpha}} + Ra\|\theta_m(\zeta)\|_{G_{s\cos\alpha}} \\ &\leq \frac{1}{r\Lambda_1^{1/2}}\rho_1'' + Ra\|A^{1/2}\theta_m\|^{3/2}_{G_{s\cos\alpha}}\ \|A\theta_m\|^{1/2}_{G_{s\cos\alpha}} + Ra\|\theta_m(\zeta)\|_{G_{s\cos\alpha}} \\ &\leq \frac{1}{r\Lambda_1^{1/2}}\rho_1'' + \frac{Ra^2}{2\kappa}(\rho_1'')^3 + \frac{\kappa}{2}\|A\theta_m\|_{G_{s\cos\alpha}} + Ra\rho_1'',\end{aligned}$$

so that

$$\kappa\|e^{s\cos\alpha A^{1/2}}A\theta_m(\zeta)\|_H = \kappa\|A\theta_m(\zeta)\|_{G_{s\cos\alpha}} \leq \frac{2}{r\sigma_2}\rho_1'' + \frac{Ra^2}{\kappa}(\rho_1'')^3 + 2Ra\rho_1'' := \kappa\rho_2'', \tag{5.13}$$

for all $\zeta\in\mathcal{S}$. For any $\zeta\in D_\tau$ and $\vec{\mathbf{x}}\in\Omega$

$$\begin{aligned}|e^{s\cos\alpha A^{1/2}}\theta_m(\vec{\mathbf{x}},\zeta)|^2_{\mathbb{C}} &\leq \sup_{\vec{\mathbf{x}}\in\Omega}\{|\mathrm{Re}\ e^{s\cos\alpha A^{1/2}}\theta_m(\vec{\mathbf{x}},\zeta)|^2_{\mathbb{C}} + |\mathrm{Im}\ e^{s\cos\alpha A^{1/2}}\theta_m(\vec{\mathbf{x}},\zeta)|^2_{\mathbb{C}}\} \\ &\qquad\qquad \text{(by using Agmon's ineq.)} \\ &\leq a_{14}\|e^{s\cos\alpha A^{1/2}}\theta_m(\vec{\mathbf{x}},\zeta)\|_{H^1}\ \|e^{s\cos\alpha A^{1/2}}\theta_m(\vec{\mathbf{x}},\zeta)\|_{H^2} \\ &\leq a_{14}\rho_1''\rho_2''\ \text{(by (5.9) and (5.13))}.\end{aligned}$$

Since $\theta_m(\zeta)$ is analytic in D_τ, it follows ¿from the vector-valued version of the Vitali's Convergence Theorem (see Marsden and Hoffman [1987] p. 450) that $\theta_m(\zeta)$ has a subsequence that converges uniformly to $\theta(\zeta)$ on every compact subset of D_τ and $\theta(\zeta)$ is analytic in D_τ. Note that $\theta(\zeta)$ satisfies the same estimates as $\theta_m(\zeta)$ in D_τ. In particular, by using (5.13) and a simple weak-compactness argument, we obtain

$$\|A^{1/2}e^{s\cos\alpha A^{1/2}}\theta(se^{i\alpha})\|_H \leq \liminf_{m\to\infty}\|A^{1/2}e^{s\cos\alpha A^{1/2}}\theta_m(se^{i\alpha})\|_H \leq \rho_1'',$$

for $se^{i\alpha} \in D_\tau$ with $s \geq 0$ and $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore for $t_0 \in (0, \tau)$ and $\alpha = 0$, $\theta(t)$ is analytic and

$$\theta(t) \in \mathcal{D}(A^{1/2} e^{\sigma A^{1/2}}) \text{ with } \|A^{1/2} e^{\sigma A^{1/2}} \theta(t)\|_H \leq \rho_1'',$$

for all $t \in [t_0, \tau]$, where $\sigma = t_0$. Finally, it follows from (4.44) that $\|\theta(t)\|_V \leq \rho_1(\|\theta^0\|_V)$ uniformly in time and hence we can repeat the above argument to get

$$\|A^{1/2} e^{t_0 A^{1/2}} \theta(t)\|_H \leq 2[\sigma_2 + \rho_1^2(\|\theta^0\|_V)] := \mathcal{G}_1(\|\theta^0\|_V, Ra, \kappa), \tag{5.14}$$

for all $t \in [t_0, \mathcal{T}]$. ■

Corollary 5.3 *There exists a constant $t_0 > 0$ which depends on $\|\theta^0\|_V$, Ra and κ such that the global attractor $\mathcal{A} \subset G_{t_0}$.*

6 Rate of Convergence of the Standard Galerkin

As a result of Theorem 3, we now can show that the standard Galerkin solution θ_N of equations (4.1)-(4.3) converge to the exact solution exponentially fast as a function of N (see also Doelman and Titi [1993] for similar result concerning the complex Ginzburg-Landau equation). In particular, we will prove:

Theorem 4 *Let $\mathcal{T} > 0$, $\theta^0 \in V$ and θ be the strong solution of* (3.3) *and* (2.23)-(2.28) *on $\Omega \times [0, \mathcal{T}]$. Let t_0 be defined as in Theorem 3 and $\theta_N(\vec{\mathbf{x}}, t)$ be the Galerkin approximation of equations* (4.1)-(4.3) *with the initial condition $\theta_N(\vec{\mathbf{x}}, t_0) = P_N(\theta(\vec{\mathbf{x}}, t_0))$, then there exists a constant K_1 depending on θ^0, κ, Ra, Λ_1 and $\mathcal{T}$ such that for any $t \in [t_0, \mathcal{T}]$*

$$\|\theta(t) - \theta_N(t)\|_V \leq K_1 \, e^{-t_0 \Lambda_{N+1}^{1/2}}. \tag{6.1}$$

Proof. Let $q := \theta - P_N\theta = (I - P_N)\theta$, then it follows from the previous theorem and (5.14) that

$$\|q(t)\|_V^2 \leq a_{15} \sum_{\lambda_{i,j,k} \geq \Lambda_{N+1}} \lambda_{i,j,k}^{1/2} |\Theta_{i,j,k}(t)|^2 \leq a_{15} e^{-2t_0 \Lambda_{N+1}^{1/2}} \|A^{1/2}\theta\|_{G_{t_0}}^2 \leq a_{15} \mathcal{G}_1^2 e^{-2t_0 \Lambda_{N+1}^{1/2}}. \tag{6.2}$$

We also let $\delta := P_N\theta - \theta_N$, hence $\theta = \delta + q + \theta_N$ and

$$\frac{d\delta}{dt} + \kappa A\delta + Ra^{1/2} P_N(B(\vec{\mathbf{v}}, \theta) - B(\vec{\mathbf{v}}_N, \theta_N)) - Ra^{1/2} P_N(\vec{\mathbf{v}} \cdot \hat{\mathbf{k}} - \vec{\mathbf{v}}_N \cdot \hat{\mathbf{k}}) = 0. \tag{6.3}$$

Let us take the L^2-inner product of (6.3) with $A\delta$ to get

$$\frac{1}{2}\frac{d}{dt}\|\delta\|_V^2 + \kappa \|A\delta\|_H^2 \leq Ra^{1/2} |(B(\vec{\mathbf{v}}, \theta) - B(\vec{\mathbf{v}}_N, \theta_N), A\delta)| + Ra^{1/2} |(\{\vec{\mathbf{v}} - \vec{\mathbf{v}}_N\} \cdot \hat{\mathbf{k}}, A\delta)|. \tag{6.4}$$

Note that the first and the second terms on the right hand side of (6.4) can be estimated as follows

$$\begin{aligned}
Ra^{1/2}|(B(\vec{\mathbf{v}},\theta)-B(\vec{\mathbf{v}}_N,\theta_N),A\delta)| &= Ra|(\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta)\cdot\nabla\theta-\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta_N)\cdot\nabla\theta_N,A\delta)| \\
&= Ra||(\mathrm{P}_\sigma\{\hat{\mathbf{k}}(\delta+q+\theta_N)\}\cdot\nabla\theta \\
&\quad -\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta_N)\cdot\nabla(\theta-\delta-q),A\delta)||_H \\
&= Ra||\{\mathrm{P}_\sigma(\hat{\mathbf{k}}\delta)+\mathrm{P}_\sigma(\hat{\mathbf{k}}q)\}\cdot\nabla\theta \\
&\quad +\mathrm{P}_\sigma(\hat{\mathbf{k}}\theta_N)\cdot(\nabla\delta+\nabla q)||_H\ ||A\delta||_H \\
&\le Ra[||\delta+q||_{L^6}\ ||\nabla\theta||_{L^3}\ ||A\theta||_H \\
&\quad +||\theta_N||_{L^\infty}\ ||\nabla\delta+\nabla q||_H\ ||A\theta||_H] \\
&\le a_{16}Ra[||\delta+q||_V\ ||\theta||_V^{1/2}\ ||A\theta||_H^{1/2}\ ||A\theta||_H \\
&\quad +||\theta_N||_V^{1/2}\ ||A\theta_N||_H^{1/2}\ ||\delta+q||_V\ ||A\theta||_H] \\
&\quad \text{by (3.10)(3.12) and Agmon ineq.} \\
&\le 2a_{16}Ra[||\theta||_V^{1/2}\ ||A\theta||_H^{1/2}+||\theta_N||_V^{1/2}\ ||A\theta_N||_H^{1/2}] \\
&\quad \times[\ ||\delta||_V+||q||_V\ ]\ ||A\delta||_H \\
&\le \frac{\kappa}{4}||A\delta||_H^2+\frac{4a_{16}^2Ra^2}{\kappa}\times[||\theta||_V\ ||A\theta||_H \\
&\quad +||\theta_N||_V\ ||A\theta_N||_H\ ][\ ||\delta||_V^2+||q||_V^2\ ],
\end{aligned}$$

$$\begin{aligned}
Ra^{1/2}|(\{\vec{\mathbf{v}}-\vec{\mathbf{v}}_N\}\cdot\hat{\mathbf{k}},A\delta)| &\le Ra||\delta||_H\ ||A\delta||_H \\
&\le \frac{\kappa}{4}||A\delta||_H^2+\frac{Ra^2}{\kappa}||\delta||_H^2 \\
&\le \frac{\kappa}{4}||A\delta||_H^2+\frac{Ra^2}{\kappa\Lambda_1}||\delta||_V^2,
\end{aligned}$$

so that (6.4) can be rewritten as

$$\begin{aligned}
\frac{d}{dt}||\delta||_V^2+\kappa||A\delta||_H^2 &\le \frac{Ra^2}{\kappa}[4a_{16}^2(||\theta||_V\ ||A\theta||_H+||\theta_N||_V\ ||A\theta_N||_H)+\frac{1}{\Lambda_1}]\ ||\delta||_V^2 \\
&\quad +\frac{4a_{16}^2Ra^2}{\kappa}[||\theta||_V\ ||A\theta||_H\ +\ ||\theta_N||_V\ ||A\theta_N||_H]||q||_V^2. \qquad (6.5)
\end{aligned}$$

By letting

$$g=\frac{Ra^2}{\kappa}[4a_{16}^2(||\theta||_V\ ||A\theta||_H+||\theta_N||_V\ ||A\theta_N||_H)+\frac{1}{\Lambda_1}]$$

and

$$h=\frac{4a_{16}^2Ra^2}{\kappa}[||\theta||_V\ ||A\theta||_H\ +\ ||\theta_N||_V\ ||A\theta_N||_H]||q||_V^2,$$

inequality (6.5) can be written as $\frac{d}{dt}||\delta||_V^2\le g||\delta||_V^2+h$. Now we apply the Generalized Gronwall Lemma (see Temam [1988] p. 88) to inequality (6.5) along with the assumption that $\delta(0)=\theta_N(\vec{\mathbf{x}},0)-P_N(\theta^0(\vec{\mathbf{x}}))=0$ and the fact that $g\ge 0$ to get

$$\begin{aligned}
||\delta(t)||_V^2 &\le \Big[\int_{t_0}^t h(s)\exp\{-\int_{t_0}^s g(\tau)d\tau\}ds\Big]\exp\{\int_{t_0}^t g(\tau)d\tau\} \\
&\le \Big[\int_{t_0}^t h(s)ds\Big]\exp\{\int_{t_0}^t g(\tau)d\tau\}. \qquad (6.6)
\end{aligned}$$

One can show that for each fixed integer N, the global bounds for the norms of the Galerkin solution θ_N, defined by

$$\frac{d\theta_N}{dt} + \kappa A\theta_N + RaP_N(B(\mathrm{P}_\sigma(\hat{\mathrm{k}}\theta_N),\theta_N) - RaP_N(\mathrm{P}_\sigma(\hat{\mathrm{k}}\theta_N)\cdot\hat{\mathrm{k}}) = 0$$

$$\theta_N(0) = P_N(\theta(0)),$$

can be found in the same approach as we did for θ in Section 4. In particular, one would obtain, without the loss of generality, the same bounds for θ_N (Temam [1977, 1983], Doelman and Titi [1993]) and thus

$$\begin{aligned}\int_{t_0}^t h(s)ds &= \frac{4a_{16}^2Ra^2}{\kappa}\int_{t_0}^t[\|\theta\|_V\ \|A\theta\|_H + \|\theta_N\|_V\ \|A\theta_N\|_H]\|q\|_V^2ds\\ &\le \frac{4a_{16}^2Ra^2}{\Lambda_1^{1/2}\kappa}a_{15}\mathcal{G}_1^2e^{-2t_0\Lambda_{N+1}^{1/2}}\int_0^{\mathcal{T}}(\|A\theta\|_H^2ds + \|A\theta_N\|_H^2)ds\\ &\le \frac{8a_{15}a_{16}^2\mathcal{G}_1^2Ra^2}{\Lambda_1^{1/2}\kappa}\rho_2^2e^{-2t_0\Lambda_{N+1}^{1/2}},\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}\int_{t_0}^t g(s)ds &= \frac{Ra^2}{\kappa}\int_{t_0}^t[4a_{16}^2(\|\theta\|_V\ \|A\theta\|_H + \|\theta_N\|_V\ \|A\theta_N\|_H) + \frac{1}{\Lambda_1}]ds\\ &\le \frac{Ra^2}{\kappa}\int_0^{\mathcal{T}}[\frac{4a_{16}^2}{\Lambda_1^{1/2}}(\|A\theta\|_H^2 + \|A\theta_N\|_H^2) + \frac{1}{\Lambda_1}]ds\\ &\le \frac{Ra^2}{\kappa}[\frac{8a_{16}^2\rho_2^2}{\Lambda_1^{1/2}} + \frac{\mathcal{T}}{\Lambda_1}],\end{aligned} \tag{6.8}$$

where, by substituting (4.43) and (4.44) into (4.47) to have

$$\rho_2^2 \equiv \int_0^{\mathcal{T}}\|A\theta\|_H^2ds \le \frac{1}{\kappa}\|\theta_0\|_V^2 + \frac{Ra^2}{\kappa^2}[a_9^2\rho_\infty^2(t_0)\rho_1^2 + \rho_0^2]\mathcal{T}.$$

By inserting (6.7) and (6.8) into (6.6), we have for any $t\in[t_0,\mathcal{T}]$

$$\|\delta(t)\|_V^2 \le \frac{8a_{15}a_{16}^2\mathcal{G}_1^2Ra^2}{\Lambda_1^{1/2}\kappa}\rho_2^2e^{-2t_0\Lambda_{N+1}^{1/2}}\exp\frac{Ra^2}{\kappa}[\frac{8a_{16}^2\rho_2^2}{\Lambda_1^{1/2}} + \frac{\mathcal{T}}{\Lambda_1}] \le a_{17}\ e^{-2t_0\Lambda_{N+1}^{1/2}}, \tag{6.9}$$

where $a_{17} = \frac{8a_{15}a_{16}^2\mathcal{G}_1^2Ra^2}{\Lambda_1^{1/2}\kappa}\rho_2^2\exp\frac{Ra^2}{\kappa}[\frac{4a_{16}^2\rho_2^2}{\Lambda_1^{1/2}} + \frac{\mathcal{T}}{\Lambda_1}]$. Finally, by using the definition of δ, (6.2) and (6.9) we get $t\in[t_0,\mathcal{T}]$

$$\|\theta(t)-\theta_N(t)\|_V^2 = \|P_N\theta(t) + q - \theta_N(t)\|_V^2 \le \|\delta(t)\|_V^2 + \|q\|_V^2 \le K_1^2\ e^{-2t_0\Lambda_{N+1}^{1/2}},$$

with $K_1^2 = (a_{17} + a_{15}\mathcal{G}_1^2)$. ■

Corollary 6.1 *Let $\mathcal{T}>0$ be given, then there exist positive constants $K = K(Ra,\kappa,\mathcal{T})$ and $t_0 = t_0(Ra,\kappa)$ such that for every initial data θ^0 in the global attractor $\mathcal{A}$ we*

$$\|\theta(t)-\theta_N(t)\|_V \le K\ e^{-t_0\Lambda_{N+1}^{1/2}},$$

for all $t\in[0,\mathcal{T}]$.

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