

# Adaptive Compensators for Perturbed Positive Real Infinite Dimensional Systems

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## Abstract

The aim of this investigation is to construct an adaptive observer and an adaptive compensator for a class of infinite-dimensional plants having a known exogenous input and a structured perturbation with an unknown constant parameter, such as the case of static output feedback with an unknown gain. The adaptive observer uses the nominal dynamics of the unperturbed plant and an adaptation law based on the Lyapunov redesign method. We obtain conditions on the system to ensure uniform boundedness of the estimator dynamics and the parameter estimates, and convergence of the estimator error. For the case of a known periodic exogenous input we design an adaptive compensator which forces the system to converge to a unique periodic solution. We illustrate our approach with a delay example and a diffusion example for which we obtain convincing numerical results.

## 1 Introduction

In this paper, we construct adaptive observers for the following class of infinite-dimensional systems on a given Hilbert space  $X$  with inner product and norm  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|_X$ , respectively:

$$\frac{d}{dt}x(t) = A_p x(t) + Bu(t) + f(t); \quad x(0) = x_0 \in X \quad (1.1)$$

$$y(t) = Cx(t), \quad (1.2)$$

where

$$A_p = A_0 + B\Gamma C \quad (1.3)$$

and  $A_0$  is a generator of an exponentially stable  $C_0$ -semigroup  $T(t)$ , i.e. there exist constants  $M, \mu > 0$  such that

$$\|T(t)\| \leq Me^{-\mu t}. \quad (1.4)$$

The signals  $u(t)$  and  $y(t)$  are the vector-valued inputs and outputs, respectively.  $f(t)$  is an  $X$ -valued known exogenous input and  $B \in \mathcal{L}(\mathbb{C}^m, X)$ ,  $C \in \mathcal{L}(X, \mathbb{C}^m)$ . Consequently,

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$A_p$  generates a strongly continuous  $C_0$ -semigroup for any  $\Gamma$  and the system is well-posed. A key assumption is that the original system  $(A_0, B, C)$  satisfy a positive-real condition, where by positive-real we mean the following.

**Definition 1.1** *Suppose that  $G(\cdot) : \mathfrak{C}_0^+ \rightarrow \mathcal{L}(\mathfrak{C}^m)$ , where  $\mathfrak{C}_0^+ = \{s : \operatorname{Re} s > 0\}$ . If*

$$(i) \quad \overline{G(s)} = G(\bar{s}); \quad (1.5)$$

$$(ii) \quad G(s) \text{ is holomorphic on } \mathfrak{C}_0^+; \quad (1.6)$$

$$(iii) \quad G(s) + G(s)^* \geq 0 \quad \text{for all } s = j\omega, \omega \in \mathbb{R}, \quad (1.7)$$

then  $G$  is positive-real.

In fact, we require that  $G(s - \mu) = C((s - \mu)I - A_0)^{-1}B$  be positive-real for some  $\mu > 0$ .

The motivation is that the perturbation  $B\Gamma C$  arises as the unknown constant gain in the feedback law  $u = \Gamma y$ . Thus, the actual formulation is

$$\dot{x}(t) = A_0 x(t) + B\Gamma y(t) + Bu(t) + f(t); \quad x(0) = x_0. \quad (1.8)$$

Some preliminary results on adaptive observers of this type were given in Demetriou and Ito [10], but they only applied to exponentially stable, dissipative systems  $(A_0 + A_0^* \leq -2\mu I)$  under a co-location assumption:  $B = C^*$ . In this case,  $G(s - \mu)$  is automatically positive-real. In Curtain, Demetriou and Ito [6], the co-location assumption was circumvented by appealing to a (very special version) of the positive-real lemma. Unfortunately, this version only applies to a system for which  $A_0$  is self-adjoint, exponentially stable, and the inputs and outputs are scalar-valued. In this paper, we introduce a third type of positive-real lemma based on recent results in Curtain [5]. Although there are many results on the positive-real lemma in the infinite-dimensional literature (see [21],[22],[23],[24], and [25],[26],[27]), most are in terms of a certain Riccati equation. For our application, we need the following singular version, for which no corresponding Riccati equation exists. We require the existence of operators  $Q \in \mathcal{L}(X)$  and  $L : X \rightarrow \mathfrak{C}^m$  such that for  $x \in D(A)$

$$A^*Qx + QAx = -L^*Lx \quad (1.9)$$

$$B^*Qx = Cx. \quad (1.10)$$

The earlier results in the literature are too restrictive for our application. Balakrishnan [1] assumes that  $A$  is Riesz-spectral and that the scalar inputs and outputs are very smooth; both Curtain [3, 4] and Pandolfi [17] require exact controllability, which is never satisfied by our class of systems. In the recent results by Curtain [5] and Pandolfi [18], the latter assumption is removed and [5] provides the type of positive-real lemma suited to our application.

In section 2, we summarize and compare three distinct versions of the positive-real lemma from [10], [6] and [5]. In [10] and [6] the  $L$ -operator maps from  $X$  to  $X$ , in contrast to the situation in the usual positive-real lemma, where  $L$  maps from  $X$  to  $\mathfrak{C}^m$ . We discuss three examples which satisfy at least one of the three versions of the positive-real lemma.

In section 3, we propose the following adaptive observer for the structurally perturbed system (1.9) with the observation (1.2) under the assumption that  $G(s - \mu) = C((s - \mu)I - A_0)^{-1}B$  is positive-real for some  $\mu > 0$ .

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) + Bu(t) + B\hat{\Gamma}(t)y(t) + f(t); \quad \hat{x}(0) = \hat{x}_0 \quad (1.11)$$

$$\dot{\hat{\Gamma}}(t) = GCe(t)y^T(t); \quad \hat{\Gamma}(0) = \hat{\Gamma}_0. \quad (1.12)$$

$0 < G = G^T$  is an adaptation matrix gain and we only require that  $f$  be locally Bochner integrable. The main result is that if  $u, y \in L_\infty(0, \infty; \mathbb{C}^m)$ , then  $\hat{\Gamma}(t)$  and  $\hat{x}(t)$  and the observation error  $e(t) = x(t) - \hat{x}(t)$  are bounded in norm for  $t \geq 0$  and  $\|e^{\frac{\mu}{2}t} Q^{\frac{1}{2}}(x(t) - \hat{x}(t))\|_X \rightarrow 0$  as  $t \rightarrow \infty$ , where  $Q$  is the operator in the positive-real lemma (1.9) with  $A = A_0 + \mu I$ . If  $Q$  is invertible or if  $y \in L_\infty \cap L_2(0, \infty; \mathbb{C}^m)$ , we obtain  $\|e^{\frac{\mu}{2}t}(x(t) - \hat{x}(t))\|_X \rightarrow 0$  as  $t \rightarrow \infty$ . Under a persistently exciting type condition we can achieve parameter convergence:  $\hat{\Gamma}(t) \rightarrow \Gamma$  as  $t \rightarrow \infty$ .

In section 4, we propose an adaptive compensator design using the separation scheme

$$u(t) = u_2(t) - \hat{\Gamma}(t)y(t) \quad (1.13)$$

and using an LQR design on the resulting adaptive observer

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) + B u_2(t) + f(t). \quad (1.14)$$

Assuming a known periodic exogenous input  $f$ , we obtain an optimal controller  $u_2(t)$  of the form

$$u_2(t) = -R^{-1}B^* (P\hat{x}(t) + r(t)), \quad (1.15)$$

where  $P$  and  $r(t)$  depend on the LQR design parameters and on  $f(t)$ . The corresponding closed loop trajectory  $\hat{x}(t)$  converges exponentially to a periodic signal  $p(t)$ . The overall adaptive compensator (1.11) - (1.15) makes the closed loop trajectory of (1.8) converge to a unique periodic signal  $p(t)$  in the following sense

$$\|e^{\beta t} Q^{\frac{1}{2}}(x(t) - p(t))\|_X \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for some  $\beta > 0$ . If  $Q$  is invertible,  $x(t)$  converges exponentially fast to  $p(t)$ .

To illustrate the above results we present some numerical results on our three examples in section 5.

## 2 Positive Real Lemmas

The adaptive observer scheme is only applicable to positive-real systems and the key is a positive real lemma. Just as in finite dimensions (1.5)-(1.6), in infinite dimensions it is possible to have different versions corresponding to spectral factors of different dimensions. We have found three useful versions. The first version is particularly useful for dissipative systems with co-located actuators and sensors, and it was utilized in Demetriou and Ito [10]. These systems are always positive-real, and the following lemma is trivial.

**Lemma 2.1** *Suppose that  $A$  is the infinitesimal generator of a contraction semigroup on  $X$  and  $B \in \mathcal{L}(\mathbb{C}^m, X)$ . Then  $Q = I$  is a solution to the constrained Lyapunov equation for  $x \in D(A)$*

$$\begin{aligned} \langle Ax, Qx \rangle + \langle Qx, Ax \rangle &\leq 0 \\ B^*Q &= B^*. \end{aligned}$$

In the adaptive observer application, one also needs to suppose that  $A$  generates an exponentially stable  $C_0$ -semigroup. An example satisfying Lemma 2.1 is the following.

**Example 2.A** Consider the diffusion equation

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + b(x)u(t); & z(0, t) = 0 = z(1, t) \\ y(t) &= \int_0^1 b(x)z(x, t)dx\end{aligned}$$

where  $b \in L_2(0, 1) = X$ .

We let

$$D(A) = \left\{ \begin{array}{l} h \in L_2(0, 1) : h, \frac{dh}{dx} \text{ are absolutely continuous,} \\ \frac{d^2 h}{dx^2} \in L_2(0, 1) \text{ and } h(0) = 0 = h(1) \end{array} \right\}$$

and define

$$Ah = \frac{d^2 h}{dx^2} \quad \text{for } h \in D(A).$$

Then  $A$  has compact resolvent, eigenvalues  $\lambda_n = -n^2\pi^2$ ,  $n \in \mathbb{N}$  and eigenvectors  $\phi_n = \sqrt{2}\sin(n\pi x)$ ,  $n \in \mathbb{N}$ , which form an orthonormal basis for  $L_2(0, 1)$ .  $A$  is exponentially stable, self-adjoint and for  $x \in D(A)$

$$\langle x, Ax \rangle \leq -\|x\|^2,$$

and it generates a contraction semigroup.

Finally, note that  $y(t) = \langle b, z(\cdot, t) \rangle = z(\cdot, t)$  and  $B = C^*$ .

To treat systems for which the actuators and sensors were not co-located, the following version was proven in Curtain, Demetriou and Ito [6].

**Lemma 2.2** *Suppose that  $A$  is self-adjoint, has compact resolvent, its eigenvalues  $\{\lambda_n, n \in \mathbb{N}\}$  are simple and its eigenvectors  $\{\phi_n, n \in \mathbb{N}\}$  form an orthonormal basis for  $X$ . Suppose that  $b, c \in X$  satisfy*

$$\begin{aligned}\langle c, \phi_n \rangle \langle b, \phi_n \rangle &> 0, & n \in \mathbb{N} \\ \sup_{n \in \mathbb{N}} \left| \frac{\langle c, \phi_n \rangle}{\langle b, \phi_n \rangle} \right| &< \infty.\end{aligned}$$

Then there exist  $0 \leq Q = Q^* \in \mathcal{L}(X)$ ,  $L \in \mathcal{L}(D(A), X)$  and  $\mu > 0$  such that for  $x, y \in D(A)$

$$\begin{aligned}\langle (A + \mu I)x, Qy \rangle + \langle Qx, (A + \mu I)y \rangle &= -\langle Lx, Ly \rangle \\ \langle x, c \rangle &= \langle x, Qb \rangle.\end{aligned}\tag{2.1}$$

Proof Show by direct substitution that the following operators satisfy the constrained Lyapunov equation

$$\begin{aligned}Qx &= \sum_{n=1}^{\infty} \frac{\langle c, \phi_n \rangle}{\langle b, \phi_n \rangle} \langle x, \phi_n \rangle \phi_n \\ Lx &= \sum_{n=1}^{\infty} \left( \frac{-2(\mu + \lambda_n) \langle c, \phi_n \rangle}{\langle b, \phi_n \rangle} \right)^{\frac{1}{2}} \langle x, \phi_n \rangle \phi_n.\end{aligned}\tag{2.2}$$

□

This lemma applies to Example 2.A, where we can take  $\mu = \pi^2 - \epsilon$  for any  $\epsilon > 0$ .

The following example from Curtain, Demetriou and Ito [6] does not satisfy the conditions of Lemma 2.1, but Lemma 2.2 does apply.

**Example 2.B** Consider the diffusion equation

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} - \alpha \frac{\partial z}{\partial x} + b(x)u(t) + f \\ z(0, t) &= 0, \quad z(1, t) = 0; \end{aligned}$$

with output given by

$$y(t) = \int_0^1 e^{-\alpha x} z(x, t) dx,$$

$$\text{where } b(x) = \begin{cases} 1 & \text{on } [0, \frac{1}{2}) \\ 0 & \text{elsewhere} \end{cases}.$$

Take  $X = L_2(0, 1)$  to be the Hilbert space with the weighted inner product

$$\langle f, g \rangle = \int_0^1 e^{-\alpha x} f(x)g(x) dx.$$

Then, defining

$$D(A_0) = \left\{ \begin{array}{l} h : h, \frac{dh}{dx} \text{ are absolutely continuous} \\ \text{and } \frac{d^2h}{dx^2} \in X, h(0) = 0 = h(1) \end{array} \right\}$$

and

$$A_0 h = \frac{d^2h}{dx^2} - \alpha \frac{dh}{dx} \quad \text{for } h \in D(A_0),$$

it is straightforward to show that  $A_0$  is self-adjoint with eigenvalues  $\lambda_n = -\frac{\alpha^2}{4} - n^2\pi^2$  and normalized eigenvectors  $\phi_n(x) = \sqrt{2}e^{\alpha x/2} \sin(n\pi x)$ ,  $n \in \mathbb{N}$ . The set  $\{\phi_n, n \in \mathbb{N}\}$  forms an orthonormal basis for  $X$ . Let

$$c_n := \langle c, \phi_n \rangle = \frac{4n\pi\sqrt{2} \left(1 - e^{-\frac{\alpha}{2}}(-1)^n\right)}{4n^2\pi^2 + \alpha^2}, \quad n \in \mathbb{N}$$

$$b_n := \langle b, \phi_n \rangle = \frac{4n\pi\sqrt{2} \left(1 + e^{-\frac{\alpha}{4}} \cos\left(\frac{n\pi}{2}\right) - \frac{\alpha}{2n\pi} e^{-\frac{\alpha}{4}} \sin\left(\frac{n\pi}{2}\right)\right)}{4n^2\pi^2 + \alpha^2}.$$

So  $b_n c_n > 0$  for all  $n$  and for certain constants  $m$  and  $M$

$$m \leq \sup_{n \geq 1} \left| \frac{c_n}{b_n} \right| \leq M.$$

So the assumptions of Lemma 2.2 are satisfied,  $Q$  given by (2.1) satisfies the constrained Lyapunov equation (2.2) and it is boundedly invertible;  $L$  is also bounded.

Note that in both Lemmas 2.1 and 2.2 the  $L$  term will be unbounded in general, even though  $B$  and  $C$  are bounded, and that  $L$  maps into the state-space  $X$ . This is in contrast to the usual finite-dimensional version for which  $L$  maps into the output space  $\mathbb{C}^m$ . The latter version is much harder to prove for infinite-dimensional systems and earlier versions

in Balakrishnan [1], Curtain [3, 4] and Pandolfi [17] assumed very strong conditions on the system, such as exact controllability. Here we extract some useful results from Curtain [5], who assumes only mild conditions on the system operators  $(A, B, C)$ .

First we need some extra notation:

$$H_\infty(Z) = \left\{ \begin{array}{l} f : \mathbb{C}_0^+ \rightarrow Z \text{ and } f \text{ is holomorphic and} \\ \|f\|_\infty = \sup_{\omega \in \mathbb{R}} \|f(j\omega)\|_X < \infty \end{array} \right\}$$

$$H_2(Z) = \left\{ \begin{array}{l} f : \mathbb{C}_0^+ \rightarrow Z \text{ and } f \text{ is holomorphic and} \\ \|f\|_2^2 = \sup_{x>0} \int_{-\infty}^{\infty} \|f(x+j\omega)\|_X^2 d\omega < \infty \end{array} \right\}$$

$$L_2((-\infty, \infty); Z) = \left\{ \begin{array}{l} f : (-\infty, \infty) \rightarrow Z \text{ and } f \text{ is measurable and} \\ \|f\|_2 = \left( \int_{-\infty}^{\infty} \|f(j\omega)\|_X^2 d\omega \right)^{\frac{1}{2}} < \infty \end{array} \right\}.$$

$H_\infty(Z)$  is a Banach space under the sup norm and  $H_2(Z)$  and  $L_2((-\infty, \infty); Z)$  are Hilbert spaces under their  $\|\cdot\|_2$  norms.  $f \in H_2(Z)$  has a limit to a function  $\tilde{f} \in L_2((0, \infty); Z)$  and  $\tilde{f}$  is isomorphic to  $f$  with  $\|f\|_2 = \|\tilde{f}\|_2$ .  $f$  and  $\tilde{f}$  are usually identified with each other and with this identification  $H_2(Z)$  is a subspace of  $L_2((-\infty, \infty); Z)$ . We denote by  $P_{H_2}$  the orthogonal projection of  $H_2(Z)$  in  $L_2((-\infty, \infty); Z)$ .

The results depend on the *Popov function*  $\Pi$  defined by

$$\Pi(j\omega) = G(j\omega) + G(j\omega)^*. \quad (2.3)$$

**Theorem 2.1** *Suppose that  $A$  is the infinitesimal generator of an exponentially stable  $C_0$ -semigroup on  $X$ ,  $B \in \mathcal{L}(\mathbb{C}^m, X)$  and  $C \in \mathcal{L}(X, \mathbb{C}^m)$ .*

(a) *If  $\Pi(j\omega) \geq 0$  and  $\Pi(j\omega)$  has invertible values for all  $\omega \in \mathbb{R}$  satisfying*

$$\frac{\int_{-\infty}^{\infty} \log^+ \|\Pi(j\omega)\|_{\mathcal{L}(\mathbb{C}^m)} d\omega}{1 + \omega^2} < \infty \quad (2.4)$$

and

$$\frac{\int_{-\infty}^{\infty} \log^+ \|\Pi(j\omega)^{-1}\|_{\mathcal{L}(\mathbb{C}^m)} d\omega}{1 + \omega^2} < \infty, \quad (2.5)$$

then there exists a spectral factor  $W \in H^\infty(\mathcal{L}(\mathbb{C}^m))$  such that

$$\Pi(j\omega) = W(j\omega)^* W(j\omega) \quad \text{for } \omega \in \mathbb{R}. \quad (2.6)$$

(b) *If there exists a  $C_W \in \mathcal{L}(D(A), \mathbb{C}^m)$  such that for  $x \in X$*

$$C_W(sI - A)^{-1}x \in H_2(\mathbb{C}^m) \quad (2.7)$$

and (see (2.16) Lemma 2.7 of Curtain [5])

$$P_{H_2} \left\{ W(s)^\sim C_W(sI - A)^{-1}x \right\} = C(sI - A)^{-1}x \quad (2.8)$$

where  $W(s)^\sim = W(-\bar{s})^*$ , then

$$W(s) = C_W(sI - A)^{-1}B, \quad (2.9)$$

and for  $x \in D(A)$

$$A^*Qx + QAx = -C_W^*C_Wx \quad (2.10)$$

$$B^*Qx = Cx, \quad (2.11)$$

where  $Q \in \mathcal{L}(X)$  is the observability gramian of  $(A, B, C_W)$ .

In general, (2.7) and (2.8) are very difficult to check, but in the case that  $A$  is a Riesz-spectral operator we have the following verifiable conditions.

**Lemma 2.3** *Suppose that  $A$  has compact resolvent with eigenvalues  $\{\lambda_n, n \in \mathbb{N}\}$  and suppose that the eigenvectors  $\{e_n, n \in \mathbb{N}\}$  form a Riesz basis for  $X$  and  $\sup_{n \in \mathbb{N}} \{Re \lambda_n\} < 0$ . The following are sufficient conditions for the constrained Lyapunov equations (2.10), (2.11) to have solutions  $Q \in \mathcal{L}(X)$  and  $C_W \in \mathcal{L}(D(A), X)$ .*

(i)

$$C_W^n = C_W e_n = (W(-\bar{\lambda}_n)^{-1})^* C e_n; \quad (2.12)$$

(ii) *Either  $A$  generates a holomorphic semigroup or there exist numbers  $0 < a \leq b$  and  $\alpha \geq 0$  such that*

$$a|Im \lambda_k|^\alpha \leq -Re \lambda_k \leq b|Im \lambda_k|^\alpha; \quad (2.13)$$

(iii) *there exists  $M \geq 0$  such that for  $h + i\omega \in \mathcal{C}^+$*

$$\left\| \sum_{-\lambda_n \in R(h, \omega)} C_W^n (C_W^n)^* \right\|_{\mathcal{L}(\mathcal{C}^m)} \leq Mh, \quad (2.14)$$

where  $R(h, \omega) = \{z \in \mathcal{C} \mid 0 < Re z \leq h, \omega - h \leq Im z < \omega + h\}$ .

Conditions (ii) and (iii) ensure that  $C_W$  is an infinite-time admissible observation operator for  $T(t)$ , in the terminology of “well-posed” linear system (see Hansen and Weiss [11]). Lemma 2.3 covers a large class of partial differential systems. We apply it to our previous Examples 2.A and 2.B, both of which have the same structure. The transfer functions are of the form

$$g(s) = \sum_n \frac{\langle c, \phi_n \rangle \langle b, \phi_n \rangle}{s + \lambda_n},$$

where  $b_n = \langle b, \phi_n \rangle$  and  $c_n = \langle c, \phi_n \rangle$  are both of order  $\frac{1}{n}$  for large  $n$  and  $\lambda_n \sim -n^2$  in both examples. Consequently, the analysis and the conclusions are the same.

Now

$$\Pi(j\omega) = \sum_{n=1}^{\infty} \frac{2c_n b_n \lambda_n}{\omega^2 + \lambda_n^2},$$

and since  $c_n$  and  $b_n$  are of the order of  $\frac{1}{n}$  for large  $n$ , we can estimate  $\Pi(j\omega) \sim \frac{1}{\omega^{\frac{3}{2}}}$  and it is not difficult to verify that (2.4) holds and  $\Pi$  has a spectral factorization ((2.5) follows from (2.4) since  $\Pi(j\omega)$  takes scalar values). In Curtain [5] the estimate  $W(n^2) \geq \frac{\text{const}}{n^{\frac{3}{4}}}$  was obtained and so  $|C_W^n| \leq \text{const } n^{-\frac{1}{4}}$ . Now for our example,  $C_W$  will be bounded if it is of the order  $n^{\frac{1}{2}+\varepsilon}$  for some positive  $\varepsilon$  and the admissibility condition (2.14) will be

satisfied if  $|C_W^n| \leq \text{const}\sqrt{n}$ . So  $C_W$  is unbounded, but admissible. Hence we have satisfied all the conditions of Lemma 2.3 and there exist  $Q$  and  $C_W \in \mathcal{L}(D(A), \mathbb{C}^m)$  satisfying the constrained Lyapunov equations.

We remark that, using Lemma 2.1 or Lemma 2.2 on Example 2.A, we obtain two different  $Q_1$ 's  $\in \mathcal{L}(X)$  satisfying the constrained Lyapunov equation (2.10), (2.11). Using Lemma 2.2 on Example 2.B we also obtain a different  $Q_2 \in \mathcal{L}(X)$  and  $L_2 \in \mathcal{L}(D(A), X)$  satisfying the constrained Lyapunov equation (2.10), (2.11). In fact, for these two examples  $Q_1$  and  $Q_2$  are boundedly invertible and in section 3 we shall see that they are actually a better choice for our adaptive observer application.

As already noted, Lemmas 2.1 and 2.2 only apply to special classes of SISO systems. Lemma 2.3 applies to a much wider class of partial differential systems. Unfortunately, Lemma 2.3 is not applicable to delay systems. In the following example we show how (2.7) and (2.8) can be verified directly for a delay system.

**Example 2.C** Consider the delay system

$$\dot{x}(t) = -ax(t) - bx(t-1) + u(t); \quad a, b > 0 \quad (2.15)$$

$$y(t) = x(t) \quad (2.16)$$

with the transfer function

$$g(s) = \frac{1}{s + a + be^{-s}} \in H_\infty.$$

Now

$$\begin{aligned} \Pi(j\omega) &= g(j\omega) + g(j\omega)^* \\ &= \frac{2(a + b \cos(\omega))}{(a + b \cos(\omega))^2 + (\omega - b \sin(\omega))^2} \\ &\geq 0 \quad \text{if } a \geq |b|. \end{aligned}$$

In this case, it is easy to find the spectral factor

$$W(s) = \frac{\alpha + \beta e^{-s}}{s + a + be^{-s}}, \quad \alpha^2 + \beta^2 = 2a, \quad \alpha\beta = b. \quad (2.17)$$

$W \in H_\infty$  and the candidate for  $C_W$  is

$$(C_W x)(t) = \alpha x(t) + \beta x(t-1). \quad (2.18)$$

The delay system (2.15), (2.16) can be formulated on the state-space  $X = \mathbb{C} \oplus L_2(-1, 0)$  with generating operators defined by

$$Bu = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad C \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = r, \quad (2.19)$$

$$\begin{aligned} D(A) &= \left\{ \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in X \mid f \text{ is absolutely continuous,} \right. \\ &\quad \left. \frac{df}{d\theta}(\cdot) \in L_2(-1, 0) \text{ and } f(0) = r \right\}, \\ A \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} &= \begin{pmatrix} -ar - bf(-1) \\ \frac{df}{d\theta} \end{pmatrix} \end{aligned} \quad (2.20)$$



(see Curtain and Zwart [8], chapter 2.4). We recall from Infante and Walker [12] that  $A$  generates an exponentially stable semigroup if  $a - |b| \geq \mu > 0$  for some positive constant  $\mu$ . While  $C$  and  $B$  are bounded operators,  $C_W$  is not. It is known that it is admissible (Salamon, [20]), which is equivalent to satisfying (2.7), but we shall verify (2.7) directly. The resolvent is given by (Curtain and Zwart, [8], Lemma 2.4.5)

$$(sI - A)^{-1} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix}, \quad (2.21)$$

where

$$\begin{aligned} g(\theta) &= e^{s\theta} g(0) - \int_0^\theta e^{s(\theta-\mu)} f(\mu) d\mu \\ g(0) &= \frac{1}{\Delta(s)} \left( r - b \int_{-1}^0 e^{-s(\mu+1)} f(\mu) d\mu \right). \end{aligned}$$

So

$$\begin{aligned} C_W(sI - A)^{-1} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} &= C_W \begin{pmatrix} g(0) \\ g(\cdot) \end{pmatrix} = (\alpha g(0) + \beta g(-1)) \\ &= \frac{\alpha + \beta e^{-s}}{\Delta(s)} r + \left( 1 - \frac{b(\alpha + \beta e^{-s})}{\Delta(s)} \right) e^{-s} \int_{-1}^0 e^{-s\theta} f(\theta) d\theta \\ &= W(s)r + (1 - bW(s)) e^{-s} \int_{-1}^0 e^{-s\theta} f(\theta) d\theta. \end{aligned}$$

Now  $W \in H_\infty$  and  $e^{-s} \in H_\infty$  and so  $C_W(sI - A)^{-1} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in H_2$  if  $\int_{-1}^0 e^{-s\theta} f(\theta) d\theta \in H_2$  and this is a consequence of the Paley-Wiener theorem:  $\int_{-1}^0 e^{-j\omega\theta} f(\theta) d\theta$  is the Fourier transform of  $f(\theta)1_{[-1,\theta]}$ .

Next we verify (2.8). Since the eigenvectors of  $A$  span  $X$  (see Curtain and Zwart [8] Theorem 2.5.10), it suffices to verify (2.8) for each eigenvector

$$\phi_n = \begin{pmatrix} 1 \\ e^{\lambda_n \cdot} \end{pmatrix} \quad (2.22)$$

corresponding to the eigenvalue  $\lambda_n$ :

$$\Delta(\lambda_n) = (\lambda_n + a + be^{-\lambda_n}) = 0 \quad (2.23)$$

(see Curtain and Zwart [8], Theorem 2.4.6). Now

$$(sI - A)^{-1} \phi_n = \frac{1}{s - \lambda_n} \phi_n \quad \text{and} \quad C(sI - A)^{-1} \phi_n = \frac{1}{s - \lambda_n}.$$

Next

$$\begin{aligned} W(s) \sim C_W(sI - A)^{-1} \phi_n &= \frac{\alpha + \beta e^s}{\Delta(-s)} C_W \begin{pmatrix} \frac{1}{s - \lambda_n} \\ \frac{e^{\lambda_n s}}{s - \lambda_n} \end{pmatrix} \\ &= \frac{(a + \beta e^s)(\alpha + \beta e^{-\lambda_n})}{\Delta(-s)} \frac{1}{s - \lambda_n}. \end{aligned}$$

Now the only pole of  $W(s) \sim C_W(sI - A)^{-1}\phi_n$  in  $\text{Re } s < 0$  is  $s = \lambda_n$  and so

$$\begin{aligned}
P_{H_2} \{W(s) \sim C_W(sI - A)^{-1}\phi_n\} &= \frac{(\alpha + \beta e^{\lambda_n})(\alpha + \beta e^{-\lambda_n})}{\Delta(-\lambda_n)} \frac{1}{s - \lambda_n} \\
&= \frac{\alpha^2 + \beta^2 + \alpha\beta(e^{\lambda_n} + e^{-\lambda_n})}{-\lambda_n + a + \beta e^{\lambda_n}} \frac{1}{s - \lambda_n} \\
&= \frac{2a + b(e^{\lambda_n} + e^{-\lambda_n})}{2a + b(e^{\lambda_n} + e^{-\lambda_n})} \frac{1}{s - \lambda_n} \\
&\quad \text{from (2.17) and (2.23)} \\
&= C(sI - A)^{-1}\phi_n \quad \text{as desired.}
\end{aligned}$$

In fact, it is easily verified that the solution to the constrained Lyapunov equation (2.10), (2.11) is

$$Q = \begin{pmatrix} I & 0 \\ 0 & \beta^2 I \end{pmatrix}.$$

For the general retarded system with vector-valued inputs and outputs, see Curtain, [5].

### 3 An Adaptive Observer: main results

The proposed state estimator is

$$\dot{\hat{x}}(t) = A_0 \hat{x}(t) + Bu(t) + B\hat{\Gamma}(t)y(t) + f(t), \quad \hat{x}(0) = \hat{x}_0, \quad (3.1)$$

where  $\hat{x}(t)$  is the state estimate at time  $t$  and  $\hat{\Gamma}(t)$  is the adaptive estimate of the unknown gain. In order to extract the adaptation rule for  $\hat{\Gamma}(t)$ , we use the Lyapunov redesign method [14, 15] that has proved successful for finite-dimensional systems. In this section, we show that the same adaptive observer that was proposed in Curtain, Demetriou and Ito [6] for scalar (SISO) systems can be extended to the larger class of multivariable (MIMO) systems considered in this paper.

**Theorem 3.1** *Consider the structurally perturbed system (1.1)-(1.3), where  $A_0$  is the generator of an exponentially stable semigroup on  $X$ ,  $B \in \mathcal{L}(\mathcal{C}^m, X)$ ,  $C \in \mathcal{L}(X, \mathcal{C}^m)$ ,  $f(t)$  is a known exogenous signal which is locally Bochner integrable, and  $\Gamma$  is an unknown matrix feedback gain. If there exist  $Q \in \mathcal{L}(X)$  and  $L \in \mathcal{L}(D(A), X)$  or  $\mathcal{L}(D(A), \mathcal{C}^m)$  satisfying the constrained Lyapunov equation for  $x \in D(A)$*

$$(A_0 + \mu I)^* Qx + Q(A_0 + \mu I)x = -L^* Lx \quad (3.2)$$

$$B^* Qx = Cx, \quad (3.3)$$

then the state estimator defined by (3.1) and the adaptation rule with adaptation matrix gain  $G = G^T > 0$  given by

$$\begin{aligned}
\dot{\hat{\Gamma}}(t) &= GCe(t)y^T(t) \\
\hat{\Gamma}(0) &= \hat{\Gamma}_0
\end{aligned} \quad (3.4)$$

have the following properties:

- (i) If  $u, y \in L_\infty(0, \infty; \mathbb{C}^m)$ , then  $\hat{\Gamma}(t)$ ,  $\hat{x}(t)$  and the estimation error  $e(t) = x(t) - \hat{x}(t)$  are bounded in norm for  $t \geq 0$  and  $\|e^{\frac{\mu}{2}t} Q^{\frac{1}{2}}(x(t) - \hat{x}(t))\|_X \rightarrow 0$  as  $t \rightarrow \infty$  ;
- (ii) If  $y \in L_\infty(0, \infty; \mathbb{C}^m) \cap L_2(0, \infty; \mathbb{C}^m)$  then  $\|e^{\frac{\mu}{2}t}(\hat{x}(t) - x(t))\|_X \rightarrow 0$  as  $t \rightarrow \infty$  ;
- (iii) If the conditions in (ii) hold and, in addition, the plant is persistently exciting, i.e., there exist  $T_0, \delta_0$  and  $\epsilon_0$  such that for each sufficiently large  $t > 0$  there exists  $\tilde{t} \in [t, t + T_0]$  such that

$$\left| \int_{\tilde{t}}^{\tilde{t} + \delta_0} y^T(\tau) \mathbf{w} d\tau \right| \geq \epsilon_0$$

for every unit vector  $\mathbf{w} \in \mathbb{R}^m$ , then we can achieve parameter convergence, i.e.,

$$\hat{\Gamma}(t) \rightarrow \Gamma, \quad \text{as } t \rightarrow \infty.$$

Proof (i) Consider the dynamics of the state error

$$\begin{aligned} \dot{e}(t) &= A_0 e(t) + B\Gamma y(t) - B\hat{\Gamma}(t)y(t) \\ &= A_0 e(t) + B\tilde{\Gamma}(t)y(t) \end{aligned} \tag{3.5}$$

$$e(0) = x_0 - \hat{x}_0 = e_0. \tag{3.6}$$

We propose the following dynamics for the parameter error  $\tilde{\Gamma}(t) = \Gamma - \hat{\Gamma}(t)$

$$\begin{aligned} \dot{\tilde{\Gamma}}(t) &= -GCe(t)y^T(t), \\ \tilde{\Gamma}(0) &= \Gamma - \hat{\Gamma}_0 = \tilde{\Gamma}_0. \end{aligned} \tag{3.7}$$

First we need to examine the well-posedness of the coupled system (3.5), (3.7) which is, in fact, a linear time-dependent system

$$\frac{d}{dt} \begin{bmatrix} e(t) \\ \tilde{\Gamma}(t) \end{bmatrix} = \begin{bmatrix} A_0 & B[\cdot]y(t) \\ -GC[\cdot]y^T(t) & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \tilde{\Gamma}(t) \end{bmatrix}. \tag{3.8}$$

The perturbation term  $D(t) : X \oplus \mathbb{C}^{m \times m} \rightarrow X \oplus \mathbb{C}^{m \times m}$  is given by

$$D(t) = \begin{bmatrix} 0 & B[\cdot]y(t) \\ -GC[\cdot]y^T(t) & 0 \end{bmatrix}. \tag{3.9}$$

So, if  $y \in L_\infty(0, t_1; \mathbb{C}^m)$ , (3.8) has a unique solution given by

$$\begin{bmatrix} e(t) \\ \tilde{\Gamma}(t) \end{bmatrix} = U(t, 0) \begin{bmatrix} e_0 \\ \tilde{\Gamma}_0 \end{bmatrix}, \tag{3.10}$$

where  $U(t, s)$  is a mild evolution operator [7] defined for  $0 \leq s \leq t \leq t_1$ . In fact,  $y(t)$  defined by (1.1)-(1.5) will always be in  $C(0, t_1; \mathbb{C}^m)$  for  $u, f \in L_p(0, t_1; \mathbb{C}^m)$ ,  $p = 1, 2$  or  $\infty$ . In general,  $e(t)$  will not be in  $D(A_0)$ , even if  $e_0 \in D(A_0)$ . Sufficient conditions for  $e(t) \in D(A_0)$  are that  $y(t) \in C^1(0, t_1; \mathbb{C}^m)$ , which are very strong. However, we assume this initially to

facilitate the Lyapunov argument. We examine the asymptotic properties of (3.10) using the following Lyapunov functional for  $\begin{pmatrix} e \\ \tilde{\Gamma} \end{pmatrix}$

$$V(e, \tilde{\Gamma}) = \langle e, Qe \rangle + \text{Tr} \left\{ \tilde{\Gamma}^T G^{-1} \tilde{\Gamma} \right\}, \quad (3.11)$$

where  $Q$  is the solution to (3.2). Since  $e(t) \in D(A_0)$  we may differentiate  $V$  along solutions of (3.8) for  $0 \leq t \leq t_1$  to obtain

$$\begin{aligned} \dot{V}(e, \tilde{\Gamma}) &= \langle A_0 e + B \tilde{\Gamma} y, Qe \rangle + \langle Qe, A_0 e + B \tilde{\Gamma} y \rangle + 2 \text{Tr} \left\{ \dot{\tilde{\Gamma}}^T G^{-1} \tilde{\Gamma} \right\} \\ &= -\|Le\|^2 - 2\mu \langle e, Qe \rangle - 2(Ce)^T \tilde{\Gamma} y + 2 \text{Tr} \left\{ y(Ce)^T \tilde{\Gamma} \right\} \\ &\quad \text{using Lemma 2.1 (3.2), (3.3) and (3.7)} \\ &= -\|Le\|^2 - 2\mu \langle e, Qe \rangle \text{ using (2.9) and } b^T a = \text{Tr}(ab^T). \end{aligned} \quad (3.12)$$

We now integrate (3.12) from  $t = 0$  to  $t = t_1$  to obtain

$$\begin{aligned} \langle e(t_1), Qe(t_1) \rangle + \text{Tr} \left\{ \tilde{\Gamma}^T(t_1) G^{-1} \tilde{\Gamma}(t_1) \right\} + \int_0^{t_1} \|Le(t)\|^2 dt + 2\mu \int_0^{t_1} \langle Qe(t), e(t) \rangle dt \\ = \langle e_0, Qe_0 \rangle + \text{Tr} \left\{ \tilde{\Gamma}_0^T G^{-1} \tilde{\Gamma}_0 \right\}. \end{aligned} \quad (3.13)$$

Notice that although we have assumed that  $e_0 \in D(A_0)$  and  $y \in C^1(0, t_1; \mathbb{C}^m)$  to derive (3.13), all terms make perfectly good sense for  $e_0 \in X$  and  $y \in C(0, t_1; \mathbb{C}^m)$ . Moreover, (3.10) and the facts that  $\sup_{0 \leq s \leq t \leq t_1} \|U(t, s)\| < \infty$ , and that  $D(A_0)$  is dense in  $X$ , show that (3.13) can be extended to all  $e_0 \in X$ . We now extend (3.13) to all  $y \in L_\infty(0, t; \mathbb{C}^m)$  by appealing to Lemma A.1 in the appendix, which shows that if we approximate  $y$  by a sequence  $y_n \in C^1(0, t_1; \mathbb{C}^m)$  satisfying

$$\int_0^{t_1} \|y(s) - y_n(s)\|^2 ds \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then there holds

$$\sup_{0 \leq s \leq t \leq t_1} \|U(t, s) - U_n(t, s)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.14)$$

So the respective solutions to (3.8) satisfy

$$\sup_{0 \leq s \leq t \leq t_1} \left( \|e(t) - e_n(t)\|_X + \|\tilde{\Gamma}(t) - \tilde{\Gamma}_n(t)\| \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.15)$$

and this suffices to show that (3.13) holds for any  $y \in L_\infty(0, t_1; \mathbb{C}^m)$  and  $e_0 \in X$ . This implies that  $\tilde{\Gamma} \in L_\infty(0, \infty; \mathbb{C}^{m \times m})$ .

Next, we define  $q(t) := \|Q^{\frac{1}{2}} e(t)\|^2$  and deduce the following from (3.13)

$$q(t_1) + 2\mu \int_0^{t_1} f(s) ds \leq f(0) + \text{Tr} \left\{ \tilde{\Gamma}_0^T G^{-1} \tilde{\Gamma}_0 \right\}. \quad (3.16)$$

Equation (3.16) implies that  $q(t_1) \leq e^{-2\mu t_1} V(0)$  or equivalently

$$\|e^{\mu t_1} Q^{\frac{1}{2}} e(t_1)\|^2 \leq \|Q^{\frac{1}{2}} e_0\|^2. \quad (3.17)$$

Now  $t_1$  can be chosen arbitrarily large and so  $\|e^{\frac{\mu}{2}t}Q^{\frac{1}{2}}e(t)\|^2 \rightarrow 0$  as  $t \rightarrow \infty$ .

Finally, since  $\tilde{\Gamma}(t)$  and  $y(t)$  are uniformly bounded in norm for  $t \geq 0$  and  $A_0$  generates an exponentially stable semigroup, (3.5) shows that  $e(t)$  is uniformly bounded in norm for  $t \geq 0$ . Similarly, (2.3) shows that  $\hat{x}(t)$  is uniformly bounded in norm for  $u \in L_\infty(0, \infty; \mathbb{C}^m)$  and hence  $x(t) = e(t) + \hat{x}(t)$  has the same property.

(ii) If  $y \in L_2(0, \infty; \mathbb{C}^m)$ , then from (i), the forcing term  $B\tilde{\Gamma}(t)y(t)$  in (3.5) is in  $L_2(0, \infty; X)$ . This, together with the fact that  $A_0$  generates an exponentially stable semigroup implies that  $e(t) \rightarrow 0$  as  $t \rightarrow \infty$  (see Lemma 12 in Curtain and Oostveen [16]).

(iii) The parameter convergence is proven by applying the results in Baumeister *et al* in [2] to equation (3.8). We discuss briefly the minor subtleties required to apply Definitions 2.1 and 3.3 and Theorem 3.4 of [2]. The definition of the (admissible) plant requires that  $|\langle B\Gamma y(t), \psi \rangle_X| \leq \gamma(y)\|\tilde{\Gamma}\|\|\psi\|_X$ . Since the operator  $B$  is bounded and the output  $y \in L_\infty(0, \infty; \mathbb{C}^m)$  we have that indeed the plant satisfies Definition 2.1 of [2] with  $\gamma(y) = \gamma|y|_{L_\infty}$ . The persistence of excitation condition in Definition 3.3 of [2] takes the form of

$$\sup_{\|\psi\| \leq 1} \left| \int_{\tilde{t}}^{\tilde{t}+\delta_0} \langle B\Gamma y(\tau), \psi \rangle d\tau \right| \geq \epsilon_0,$$

for all  $\Gamma \in \mathbb{R}^{m \times m}$  with  $\|\Gamma\| = 1$ , which simplifies to

$$\sup_{\|\psi\| \leq 1} \left| \int_{\tilde{t}}^{\tilde{t}+\delta_0} y^T(\tau)\Gamma^T (B^*\psi) d\tau \right| \geq \epsilon_0.$$

This finally becomes

$$\left| \int_{\tilde{t}}^{\tilde{t}+\delta_0} y^T(\tau)\mathbf{w} d\tau \right| \geq \epsilon_0.$$

for some  $\mathbf{w} \in \mathbb{R}^m$  with unit norm. Then a direct application of Theorem 3.4 in [2] yields the required result. This completes the proof of Theorem 3.1.  $\square$

## 4 Adaptive Compensators

In this section, we propose an adaptive compensator for the perturbed plant (1.1)-(1.3) where  $f(t)$  is a known exogenous signal. We obtain results for  $f(t)$  a periodic signal and for  $f \in L_2(0, \infty; X)$ . First we apply output injection to obtain a modified control problem:

$$u(t) = u_2 - \hat{\Gamma}(t)y(t). \quad (4.1)$$

This has the advantage of producing the new estimator dynamics

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0\hat{x}(t) + Bu_2(t) + f(t) \\ \hat{x}(0) &= \hat{x}_0 \end{aligned} \quad (4.2)$$

and the same error dynamics (3.8) for  $\begin{pmatrix} e(t) \\ \hat{\Gamma}(t) \end{pmatrix}$  as before.

So it remains to design a controller  $u_2(t)$  for the system (4.2).

We use the LQR control design from Da Prato and Ichikawa [19]. Suppose that  $(A_0, B, C_2)$  is exponentially stabilizable and detectable and  $R = R^T > 0 \in \mathcal{L}(\mathbb{C}^m)$ . We seek to minimize the average cost

$$J(u) = \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left( \|C_2 \hat{x}(t)\|^2 + \|R^{-\frac{1}{2}} u(t)\|^2 \right) dt \quad (4.3)$$

over all controls satisfying  $\overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|u(t)\|^2 dt < \infty$  and for which the corresponding closed loop trajectory is bounded on  $t \geq 0$ . They showed that if  $f(t)$  is periodic, the minimizing control law is given by

$$u_2(t) = -R^{-1} B^* (P \hat{x}(t) + r(t)), \quad (4.4)$$

where  $P = P^* \in \mathcal{L}(X)$  is the solution to the Riccati equation for  $x \in D(A_0)$

$$A_0^* P x + P A_0 x - P B R^{-1} B^* P x + C_2^* C_2 x = 0 \quad (4.5)$$

and  $r(t)$  is the solution to

$$\dot{r}(t) = (A_0^* - P B R^{-1} B^*) r(t) - P f(t), \quad r(t) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (4.6)$$

Equation (4.6) has the solution

$$r(t) = \int_t^\infty T_P^*(s-t) P f(s) ds, \quad (4.7)$$

where  $T_P(t)$  is the exponentially stable  $C_0$ -semigroup generated by  $A - B R^{-1} B^* P$ .

The closed loop trajectory converges exponentially fast to the periodic solution

$$p(t) = \int_{-\infty}^t T_P(t-s) \left( f(s) - B R^{-1} B^* r(s) \right) ds, \quad (4.8)$$

i.e.,

$$\lim_{t \rightarrow \infty} e^{\nu t} \|\hat{x}(t, t_0) - p(t)\|_X = 0, \quad (4.9)$$

where  $\nu$  is the decay rate of  $T_P(t)$ .

In the case of a constant exogenous signal,  $f(t) = f_0$ ,

$$p(t) = - \left( A_0 - B R^{-1} B^* P \right)^{-1} A_0^* \left( A_0 - B R^{-1} B^* P \right) f_0.$$

We note that if  $f \in L_2(0, \infty; X)$ , then the feedback control law

$$u_2(t) = -R^{-1} B^* P \hat{x}(t)$$

has the effect of ensuring that  $e^{\mu t} x(t) \rightarrow 0$  as  $t \rightarrow \infty$  (see Lemma 12 in Curtain and Oostveen [16]).

We propose the following adaptive compensator for the case of a known periodic exogenous input

$$\begin{aligned} \dot{\hat{x}}(t) &= \left( A_0 - B R^{-1} B^* P \right) \hat{x}(t) - B R^{-1} B^* r(t) + f(t); \quad \hat{x}(0) = \hat{x}_0 \\ \dot{r}(t) &= \left( A_0^* - P B R^{-1} B^* \right) r(t) - P f(t), \quad r(t) \rightarrow 0 \text{ as } t \rightarrow \infty \\ u(t) &= -R^{-1} B^* (P \hat{x}(t) + r(t)) - \hat{\Gamma}(t) y(t) \\ \dot{\hat{\Gamma}}(t) &= G C e(t) y^T(t), \quad \hat{\Gamma}(0) = \hat{\Gamma}_0 \end{aligned}$$

for our structurally perturbed plant (1.1)-(1.3).

In section 3 we showed that

$$\|e^{\frac{\mu}{2}t} Q^{\frac{1}{2}} (x(t) - \hat{x}(t))\|_X \rightarrow 0 \text{ as } t \rightarrow \infty$$

independently of the choice of the control. So combining this with the results in this section, we conclude that for the case of a known periodic input  $f(t)$

$$\|e^{\beta t} Q^{\frac{1}{2}} (x(t) - \hat{x}(t))\|_X \rightarrow 0 \text{ as } t \rightarrow \infty,$$

where  $\beta = \min(\nu, \frac{\mu}{2})$  and  $p(t)$  is given by (4.8). If  $f(t)$  is in  $L_2(0, \infty; X)$ , then

$$\|e^{\frac{\mu}{2}t} Q^{\frac{1}{2}} (x(t) - \hat{x}(t))\|_X \rightarrow 0 \text{ as } t \rightarrow \infty.$$

## 5 Examples and Numerical Results

We present some numerical results for the three examples considered in section 2. For each of these examples, there exists a solution  $Q \in \mathcal{L}(X)$  satisfying (3.2) for a certain  $\mu > 0$ , and in all three cases  $Q$  is invertible. Consequently, we can conclude that for the adaptive observer (3.1) with adaptation rule (3.4)

$$\|e^{\frac{\mu}{2}t} (x(t) - \hat{x}(t))\|_X \rightarrow 0$$

and with the adaptive compensator of section 4

$$\|e^{\beta t} (x(t) - p(t))\|_X \rightarrow 0.$$

All the computations described below were carried out on a Digital Personal Workstation 433 *au-Series* in the Mechanical Engineering Department at Worcester Polytechnic Institute. A finite element Galerkin approximation scheme based on spline elements was used for the spatial discretization of the two PDE's similar to the one developed in [2]. The resulting finite dimensional ODE systems were integrated in time using a Fehlberg fourth-fifth Runge-Kutta method. The delay system in Example 2.C was discretized using the method presented in the paper by Ito and Kappel, [13]. The resulting evolution (finite dimensional) system was similarly integrated using the Runge-Kutta code `rkf45.f`.

**Example 2.A** As was already mentioned in section 2, we can choose in this case  $\mu = \pi^2 - \epsilon$  in (2.1) and define the operators  $Q$  and  $L$  via (2.2). Alternatively, when Lemma 2.1 is used, we have  $Q = I$  and  $L = 0$  with the same  $\mu$ . The input operator  $b(x)$  was chosen as

$$b(x) = \begin{cases} 1 & \text{on } [0, 1) \\ 0 & \text{elsewhere} \end{cases}$$

The unknown gain was chose as  $\Gamma = 1$  and as initial conditions we chose  $z(x, 0) = \sin(\pi x)$  and  $\hat{z}(x, 0) = \cos(2\pi x) - 1$ . The exogenous input  $f(x, t) = 50\chi_{[0,1]}(x) \sin(2\pi t)$ . The initial guess for  $\hat{\Gamma}(0) = 0$  with an adaptive gain of  $G = 20$ . Figure 1(a) depicts the time evolution of the output state error  $Ce(t) = \int_0^1 (z(x, t) - \hat{z}(x, t)) dx$ . The convergence to zero is achieved within 0.5 seconds. The parameter estimate  $\hat{\Gamma}(t)$  (dashed) and the actual value of  $\Gamma = 1$  are depicted in Figure 1(b). Parameter convergence is achieved in 4 seconds.

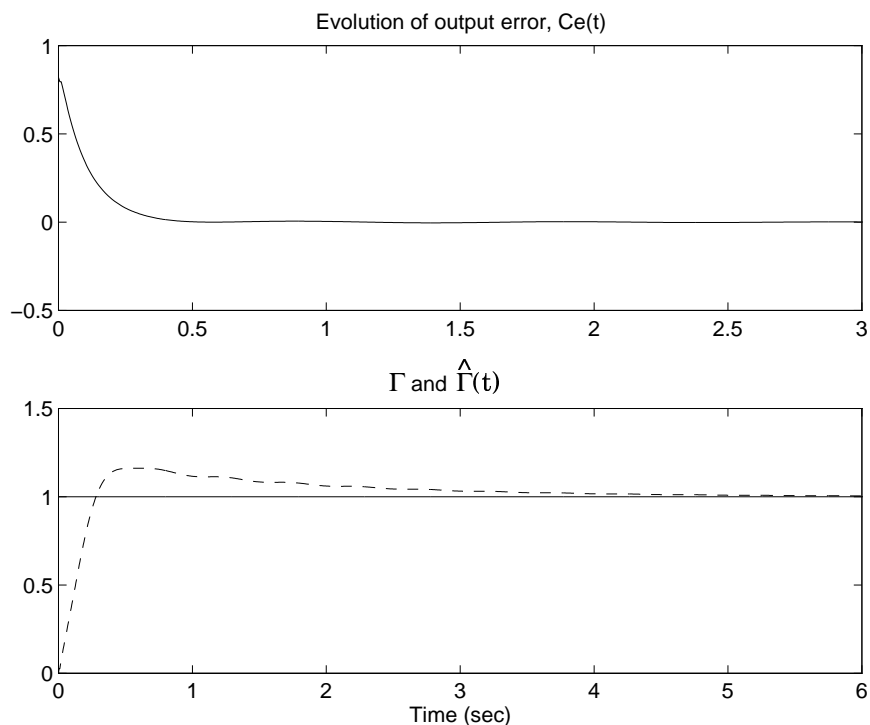


Figure 1: Evolution of (a) output error and (b) parameter estimate  $\hat{\Gamma}(t)$  (dashed) – actual parameter  $\Gamma$  (solid).

**Example 2.B** Equations (2.1) and (2.2) can be satisfied with  $\mu = \frac{\alpha^2}{4}\pi^2 - \epsilon$  where the parameter  $\alpha = 0.2$ . Initial conditions were set as  $z(x, 0) = \sin(\pi x)$  and  $\hat{z}(x, 0) = -0.25 \sin(\pi x)$ . A constant in space and time exogenous function is implemented as  $f(x, t) = 50\chi_{[0,1]}(x)$  and  $\hat{\Gamma}(0) = 0$  with  $G = 2$ . With these values of initial conditions, it is observed in Figure 2(a) that the the output state error converges to zero in 3.5 seconds. Furthermore, the parameter  $\hat{\Gamma}(t)$  converges to the actual value  $\Gamma = 1$  in 4 seconds. as shown is Figure 2(b).

**Example 2.C** The plant parameters were chosen as  $a = 3$ ,  $b = 1$ . In this case the solution to the constrained Lyapunov equations (2.10), (2.11) is  $Q = \begin{bmatrix} I & 0 \\ 0 & (3 \pm \sqrt{8})I \end{bmatrix}$ .

The actual value of the parameter was  $\Gamma = 0.4$  with initial condition for its estimate chosen as  $\hat{\Gamma}(0) = 0.2$ . The initial state was set at  $x(t-1) = \sin(4t-1) - \sin(-1)$  and the state estimate as  $\hat{x}(t-1) = 0.5 \sin(4t-1)$ ; thus  $x(0) = \sin(3) + \sin(1)$  and  $\hat{x}(0) = 0.5 \sin(3)$ . Here we had  $f(t) = 0$  for the exogenous signal and chose an adaptive gain of  $G = 500$ . It is observed from Figure 3(a) that the state estimate converges to the plant state in about 2 seconds. Parameter convergence is also achieved at about 5 seconds. For numerical results for a multivariable example the reader is directed to [9].

## References

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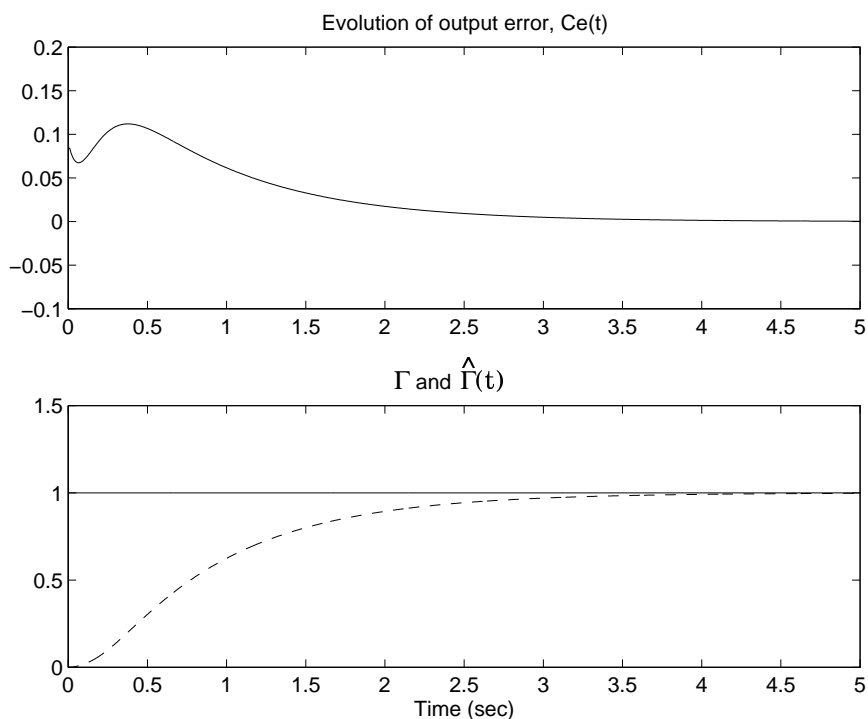


Figure 2: Evolution of (a) output error and (b) parameter estimate  $\hat{\Gamma}(t)$  (dashed) – actual parameter  $\Gamma$  (solid).

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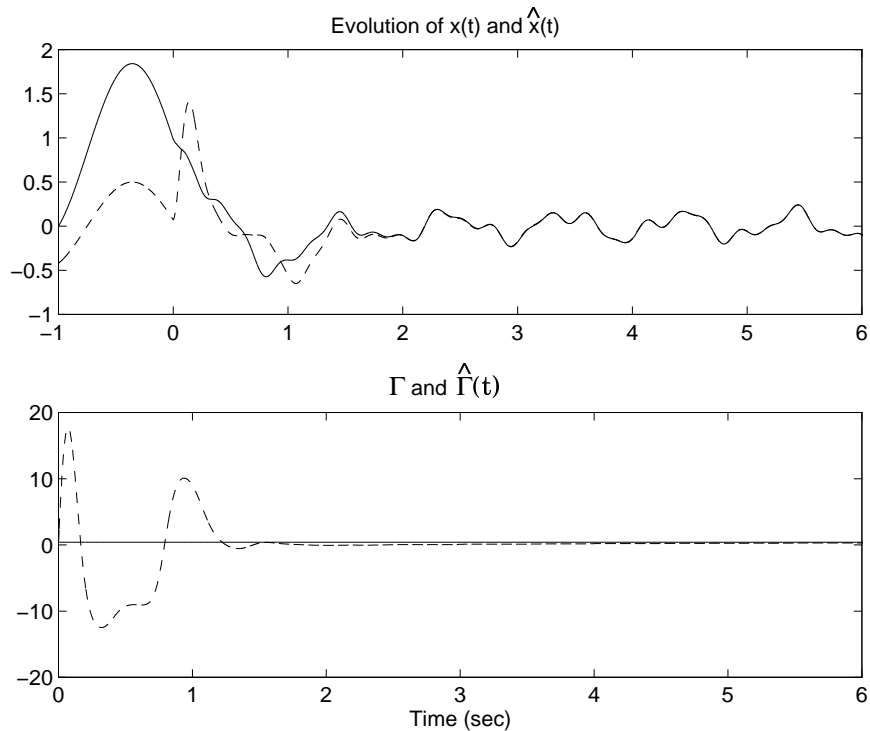


Figure 3: Evolution of (a) plant state  $x(t)$  (solid) and state estimate  $\hat{x}(t)$  (dashed); (b) parameter estimate  $\hat{\Gamma}(t)$  (dashed) – actual parameter  $\Gamma$  (solid).

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## Appendix

**Lemma A.1** *Suppose that  $A$  generates a  $C_0$ -semigroup on the Hilbert space  $X$  and consider the mild evolution operator  $U(t, s)$  generated by  $A + \sum_{i=1}^k D_i y_i(t)$ ,  $D_i \in \mathcal{L}(X)$  and  $y_i \in L_\infty(0, t_1)$ . Let  $U_n(t, s)$  be the evolution operator generated by  $A + \sum_i^k D_i y_i^n(t)$ , where for each  $i$   $y_i^n(t)$  is a sequence of functions in  $C^1(0, t_1)$  satisfying*

$$\int_0^{t_1} \|y_i^n(t) - y_i(t)\|^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

There holds

$$\sup_{0 \leq s \leq t \leq t_1} \|U(t, s) - U_n(t, s)\|_{\mathcal{L}(X)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Proof We only give a detailed proof for  $k = 1$ , since the arguments extend readily to any finite  $k$ . We recall from Curtain and Zwart [8] the defining equations for  $U(t, s)$  and  $U_n(t, s)$ :

$$U(t, s)x = T(t - s) + \int_s^t T(t - \alpha) D y(\alpha) U(\alpha, s) x d\alpha \quad (\text{A.1})$$

$$U_n(t, s)x = T(t - s) + \int_s^t T(t - \alpha) D y_n(\alpha) U_n(\alpha, s) x d\alpha \quad (\text{A.2})$$

and the estimate

$$\|U(\alpha, s)\| \leq M e^{(\alpha-s)(\omega+\mu)}, \quad (\text{A.3})$$

$$\text{where } \|T(t)\| \leq M e^{\omega t}, \quad t \geq 0, \quad (\text{A.4})$$

$$\text{and } \mu = M \|D\| \|y\|_{L_\infty(0, t_1)} > 0.$$

Consider the following estimates obtained using (A.1) and (A.2)

$$\begin{aligned} \|U(t, s) - U_n(t, s)\| &\leq \int_s^t \|T(t - \alpha) D\| \|y(\alpha) - y_n(\alpha)\| \|U(\alpha, s)\| d\alpha \\ &\quad + \int_s^t \|T(t - \alpha) D\| \|y_n\|_{L_\infty} \|U(\alpha, s) - U_n(\alpha, s)\| d\alpha \\ &\leq \|D\| \int_s^t M e^{\omega(t-\alpha)} M e^{(\omega+\mu)(\alpha-s)} |y(\alpha) - y_n(\alpha)| d\alpha \\ &\quad + \|D\| \|y_n\|_{L_\infty} \int_s^t M e^{\omega(t-\alpha)} \|U(\alpha, s) - U_n(\alpha, s)\| d\alpha. \end{aligned}$$

Defining  $f_n(t, s) = e^{-\omega(t-s)} \|U(t, s) - U_n(t, s)\|$  we obtain

$$\begin{aligned}
f_n(t, s) &\leq M^2 \|D\| \int_s^t e^{\mu\alpha} e^{-\mu s} |y(\alpha) - y_n(\alpha)| d\alpha \\
&\quad + \|D\| \|y_n\|_{L_\infty} \int_s^t f_n(\alpha, s) d\alpha \\
&\leq M^2 \|D\| \left( \int_s^t |y(\alpha) - y_n(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_s^t e^{2\mu(\alpha-s)} d\alpha \right)^{\frac{1}{2}} \\
&\quad + \|D\| \|y_n\|_{L_\infty} \int_s^t f_n(\alpha, s) d\alpha \\
&= C_1 |e^{2\mu(t-s)} - 1|^{\frac{1}{2}} \left( \int_s^{t_1} |y(\alpha) - y_n(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \\
&\quad + C_2 \int_s^t f_n(\alpha, s) d\alpha
\end{aligned}$$

where  $C_1$  and  $C_2$  only depend on  $t_1$ .

Thus

$$f_n(t, s) \leq 2C_1 e^{\mu(t-s)} \|y - y_n\|_{L_2(0, t_1)} + C_2 \int_s^t f_n(\alpha, s) ds$$

and differentiating with respect to  $t$  for fixed  $s$  yields

$$\frac{df_n}{dt}(t, s) \leq 2C_1 \mu e^{\mu(t-s)} \|y - y_n\|_{L_2(0, t_1)} + C_2 f_n(t, s)$$

and

$$\frac{d}{dt} \left( e^{-C_2 t} f_n(t, s) \right) \leq 2C_2 \mu e^{\mu(t-s)} e^{-C_2 t} \|y - y_n\|_{L_2(0, t_1)}.$$

We integrate from  $t$  to  $s$  noting that  $f_n(s, s) = 0$  to obtain

$$\begin{aligned}
e^{-C_2 t} f_n(t, s) &\leq 2C_1 \mu e^{-\mu s} \int_s^t e^{(\mu-C_2)\beta} d\beta \|y - y_n\|_{L_2(0, t_1)} \\
&= \frac{2C_1 \mu}{\mu - C_2} e^{-\mu s} \left( e^{(\mu-C_2)t} - e^{(\mu-C_2)s} \right) \|y - y_n\|_{L_2(0, t_1)}
\end{aligned}$$

and

$$f_n(t, s) \leq \frac{2C_1 \mu}{\mu - C_2} \left( e^{(\mu-C_2)t} - e^{(\mu-C_2)s} \right) \|y - y_n\|_{L_2(0, t_1)}$$

and

$$\|U(t, s) - U_n(t, s)\| \leq \frac{2C_1 \mu}{\mu - C_2} \left[ e^{(\omega+\mu)(t-s)} - e^{(\omega+C_2)(t-s)} \right] \|y - y_n\|_{L_2(0, t_1)}$$

which proves our claim.