Convergence of Approximations in Feedback Control of Structures[1]

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Abstract

Convergence of linear quadratic regulator (LQR) problems in structures is discussed. The abstract formulation of the system using a variational framework based on sesquilinear forms is considered. Since convergence theorems require uniform stabilizability of the finite dimensional approximating system, we present a detailed proof of a fundamental lemma due to Banks and Ito which can be used to easily verify this condition for many applications. Existing results for the well posedness of the infinite dimensional system and convergence of Galerkin approximations are summarized.

Keywords

feedback control, approximation, LQR, control convergence

1 Introduction

In this paper we discuss in detail the proof of Lemma 6.2 in the electronic and CRSC technical report versions of [1] (stated as Lemma 7.13 in [2]) which allows verification of uniform stabilizability of a family of finite dimensional approximating systems arising in feedback control formulations. This uniform stabilizability condition is sufficient for the desired convergence theorems for gains, controls and trajectories. Since structural applications are of interest to us, the partial differential equations we consider are second order in time and are motivated by a simple example of a cantilever beam. We discuss details of this proof since it has appeared only in the electronic version of [1] and the proof appearing there was only a sketch of the arguments. We believe the ideas behind this proof can be extended to treat a much larger class of examples than those indicated below.

The system we consider entails Kelvin-Voigt or strong damping and the lemma requires a strong assumption regarding the relationship between the stiffness, Kelvin-Voigt damping and air damping coefficients. This assumption is reasonable for homogeneous structures such as beams, plates or shells with actuators (such as piezoceramic patches) embedded in a manner so that models with material properties which do not vary across the region of the actuators are good approximations.

The theoretical control results we present are valid for systems with bounded observation operators. Since current sensing devices yield observation operators that are discontinuous in nature, we refer the reader to extended results regarding unbounded observation operators in [1,

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Section 6.2]. We also assume no persistent exogenous force is acting on the system and thus we are interested in applications in which the structure or actuators start with a deformation and vibrates to a steady state. This is also useful in applications where the system is subjected to an impulsive force. For systems with exogenous disturbance, the theory is less complete but a number of computational investigations have been carried out on systems with periodic exogenous forces. We refer the reader to [2] for a summary of results for infinite and finite dimensional control of systems with periodic exogenous disturbance. Numerical confirmation involving LQR control of thin cylindrical shells with transient and periodic exogenous disturbances can be found in [3, 4, 5]. For experimental and numerical results of control of plate systems, see [6].

In Section 2 we discuss the abstract system and introduce the motivating example which is control of transverse vibrations of a beam as presented in [2]. We then summarize results for infinite dimensional control in Section 3. Approximation, LQR control and the detailed proof of Lemma 6.2 of [1] are given in Section 4.

2 Abstract System

Consider the abstract second-order (in time) variational system

$$\begin{aligned} &\langle \ddot{w}(t), \psi \rangle_{V^*,V} + \sigma_2(\dot{w}(t), \psi) + \sigma_1(w(t), \psi) = \left\langle \tilde{\mathcal{B}} u(t), \psi \right\rangle_{V^*,V} \\ &w(0) = w_0, \qquad \dot{w}(0) = w_1 \ , \end{aligned} \tag{2.1}$$

where σ_1 and σ_2 are sesquilinear forms from $V \times V$ to $\mathbb{C}$. Let V and H be complex Hilbert spaces with V continuously embedded in H (i.e., $||\phi||_H \leq k||\phi||_V$), forming a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ (for details regarding Gelfand triples, see [7, 8]). Here V^* and H^* are the dual spaces to V and H, respectively and H is identified with H^* through the Riesz map. We take the duality product $\langle \cdot, \cdot \rangle_{V^* \times V}$ on $V^* \times V$ to be the unique extension by continuity of the scalar product $\langle \cdot, \cdot \rangle_H$ of H defined on $H \times V$. Thus, the elements $v^* \in V^*$ have the representation $v^*(v) = \langle v^*, v \rangle_{V^*,V}$. Furthermore, assume that the embedding i from V into H is compact, and that the stiffness sesquilinear form σ_1 is V-continuous, positive and symmetric, i.e.,

(H1) $|\sigma_1(\phi, \psi)| \leq c_1 \, |\phi|_V \, |\psi|_V$, for some $c_1 \in \mathbb{R}$ (V-continuous)

(H2) $Re\sigma_1(\phi, \phi) \geq c_2 \, |\phi|_V^2$, for some $c_2 > 0$ (V-elliptic)

(H3) $\sigma_1(\phi, \psi) = \overline{\sigma_1(\psi, \phi)}$ (symmetric) .

Assume further that the damping form σ_2 has the properties

(H4) $|\sigma_2(\phi, \psi)| \leq c_3 \, |\phi|_V \, |\psi|_V$, for some $c_3 \in \mathbb{R}$ (V-continuous)

(H5) $Re\sigma_2(\phi, \phi) \geq c_4 \, |\phi|_V^2$, for some $c_4 > 0$ (V-elliptic) .

From the continuity properties (H1) and (H4), we obtain operators $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{L}(V, V^*)$ defined by

$$\begin{aligned} \langle \mathcal{A}_1 \phi, \psi \rangle_{V^*,V} &= \sigma_1(\phi, \psi) \ , \quad \forall \phi, \psi \in V \\ \langle \mathcal{A}_2 \phi, \psi \rangle_{V^*,V} &= \sigma_2(\phi, \psi) \ , \quad \forall \phi, \psi \in V \ , \end{aligned} \tag{2.2}$$

and hence an equivalent formulation of (2.1) is given by

$$\begin{aligned} &\ddot{w}(t) + \mathcal{A}_2 \dot{w}(t) + \mathcal{A}_1 w(t) = \tilde{\mathcal{B}} u(t) \qquad \text{in } V^* \\ &w(0) = w_0, \qquad \dot{w}(0) = w_1 \ . \end{aligned} \tag{2.3}$$

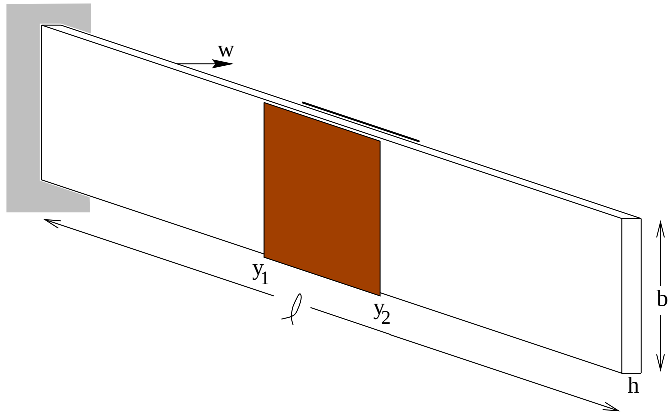

Figure 1: Cantilever beam with piezoceramic patch pair.

The control operator $\tilde{\mathcal{B}} : \mathcal{U} \to V^*$ where $\mathcal{U}$ is the control input space is typically unbounded due to the discontinuous geometry of actuators. The input $u(t) \in \mathcal{U}$ usually models voltage input to the actuators in smart material structure applications.

The system (2.1) or equivalently (2.3) arises in the abstract formulation of partial differential equations governing smart material structures. To illustrate, consider the transverse vibrations of a homogeneous beam with length ℓ, thickness h, width b, linear mass density ρ, Young's modulus E and Kelvin Voigt damping c_D. Assume that a pair of identical piezoceramic patches are bonded to opposite sides of the beam covering the region $y_1 \leq y \leq y_2$ (see Figure 1). We denote the Young's modulus, linear mass density, thickness and damping coefficient of the patches by $E_{pe}, \rho_{pe}, h_{pe}$ and $c_{D_{pe}}$, respectively. Cantilever end conditions are assumed with the fixed end at $y = 0$ and free end at $y = \ell$. Transverse displacements of the beam will be denoted by w and air damping coefficient by c_a. Force and moment balancing yields the strong form of the equation (see [2] for details in the derivation) which when written in weak form is given by

$$\int_0^\ell \left\{ \widetilde{\rho} \frac{\partial^2 w}{\partial t^2} \phi + \widetilde{EI} \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \phi}{\partial y^2} + \widetilde{c_D I} \frac{\partial^3 w}{\partial y^2 \partial t} \frac{\partial^2 \phi}{\partial y^2} + c_a \frac{\partial w}{\partial t} \phi - (bM_y)_{pe} \frac{\partial^2 \phi}{\partial y^2} \right\} dy = 0 \ , \qquad (2.4)$$

for all $\phi \in H_L^2(0,\ell)$, where

$$H_L^2(0,\ell) = \left\{ \phi \in H^2(0,\ell) | \phi(0) = \phi'(0) = 0 \right\} \ . \qquad (2.5)$$

Due to the presence of the patches, the linear mass density $\widetilde{\rho}(y) = \rho h b + 2b\rho_{pe} h_{pe} \chi_{pe}(y)$ is piecewise constant with the characteristic function $\chi_{pe}(y)$ used to isolate patch contributions. Here

$$\begin{aligned} \widetilde{EI}(y) &= E\frac{h^3 b}{12} + \frac{2b}{3} E_{pe} a_3 \chi_{pe}(y) \\ \widetilde{c_D I}(y) &= c_D \frac{h^3 b}{12} + \frac{2b}{3} c_{D_{pe}} a_3 \chi_{pe}(y) \end{aligned} \qquad (2.6)$$

and $a_3 = (h/2 + h_{pe})^3 - h^3/8$.

The external moment $(bM_y)_{pe}$ depends on the voltages supplied to the two patches. Denoting the outer and inner patch voltages by $V_1(t)$ and $V_2(t)$, respectively, the external moment is given by

$$(bM_y)_{pe} = -\mathcal{K}^B \chi_{pe}(y) \left[V_1(t) - V_2(t) \right] \ , \qquad (2.7)$$

where $\mathcal{K}^B = -\frac{1}{2}E_{pe}bd_{31}(h+h_{pe})$ depends on the piezoceramic material properties. Coupled to the system are the cantilever boundary conditions

$$w(t,0) = \frac{\partial w}{\partial y}(t,0) = 0, \quad M_y(t,\ell) = \frac{\partial}{\partial y}M_y(t,\ell) = 0 \tag{2.8}$$

and initial conditions

$$w(0,y) = w_0(y), \quad \frac{\partial w}{\partial y}(0,y) = w_1(y) \; . \tag{2.9}$$

Here M_y is the internal moment resultant with expression

$$M_y = -\frac{1}{b}\left[\widetilde{EI}\frac{\partial^2 w}{\partial y^2} + \widetilde{c_D I}\frac{\partial^2 w}{\partial y^2 \partial t}\right] \; .$$

To abstractly formulate the weak form (2.4), let $V = H_L^2(0,\ell)$ and $H = L^2(0,\ell)$. It readily follows from standard Sobolev theory that V is continuously, densely and compactly embedded in H and hence with H forms a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$. For $\phi,\psi \in H$, define the H inner product to be

$$\langle \phi,\psi\rangle_H = \int_0^\ell \rho h b \phi\psi dy \; ,$$

and for $\phi,\psi \in V$, the stiffness and damping sesquilinear forms are defined by

$$\begin{aligned} \sigma_1(\phi,\psi) &= \int_0^\ell \widetilde{EI}(y)\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\psi}{\partial y^2}dy \\ \sigma_2(\phi,\psi) &= \int_0^\ell \widetilde{c_D I}(y)\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\psi}{\partial y^2}dy + c_a\int_0^\ell \phi\psi dy \; . \end{aligned} \tag{2.10}$$

The control operator $\tilde{\mathcal{B}} : \mathcal{U} \to V^*$ is given by

$$\left\langle \tilde{\mathcal{B}}u(t),\psi\right\rangle_{V^*,V} = \int_0^\ell (bM_y)_{pe}\frac{\partial^2\psi}{\partial y^2}dy \; . \tag{2.11}$$

Here $\mathcal{U} = \mathbb{R}$ and the vector $u(t) \in \mathcal{U}$ represents the time varying voltage to the inner and outer patches. Finally, we define the H_L^2-equivalent inner product on V using the stiffness sesquilinear form, i.e., $\langle\cdot,\cdot\rangle_V = \sigma_1(\cdot,\cdot)$.

It can be easily shown that the sesquilinear forms (2.10) satisfy (H1)-(H5) and thus the weak form (2.4) with corresponding boundary (2.8) and initial conditions (2.9) can be abstractly formulated using (2.1) (equivalently (2.14)).

To obtain a first order formulation of the system (2.1) (or equivalently (2.3)) amenable to semigroup analysis and control methodologies, we define the product spaces $\mathcal{H} = V \times H$ and $\mathcal{V} = V \times V$ with norms

$$\begin{aligned} \|(\phi_1,\phi_2)\|_{\mathcal{H}}^2 &= \|\phi_1\|_V^2 + \|\phi_2\|_H^2 \\ \|(\phi_1,\phi_2)\|_{\mathcal{V}}^2 &= \|\phi_1\|_V^2 + \|\phi_2\|_V^2 \; . \end{aligned}$$

It can be readily verified that these product spaces also form a Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$, where $\mathcal{V}^* = V \times V^*$. The control operator is then reformulated as

$$\mathcal{B}u(t) = \begin{bmatrix} 0 \\ \tilde{\mathcal{B}}u(t) \end{bmatrix} ,$$

and we define a sesquilinear form $\sigma : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ by

$$\sigma(\Phi,\Psi) = -\langle \phi_2,\psi_1\rangle_V + \sigma_1(\phi_1,\psi_2) + \sigma_2(\phi_2,\psi_2)$$

for $\Phi = (\phi_1, \phi_2), \Psi = (\psi_1, \psi_2) \in \mathcal{V}$. Thus, for $z(t) = (w(t), \dot{w}(t))$, the second-order system (2.1) formulated in first-order form is

$$\begin{aligned} &\langle \dot{z}(t), \Psi \rangle_{\mathcal{V}^*,\mathcal{V}} + \sigma(z(t), \Psi) = \langle \mathcal{B}u(t), \Psi \rangle_{\mathcal{V}^*,\mathcal{V}} \ , \qquad \forall \Psi \in \mathcal{V} \\ &z(0) = z_0 = (w_0, w_1) \ . \end{aligned} \tag{2.12}$$

The $\mathcal{V}$-continuity of σ and $\mathcal{V}$-ellipticity of $\sigma(\cdot,\cdot) + \lambda \langle \cdot, \cdot \rangle_{\mathcal{H}}$ for $\lambda > 0$ was detailed in [2, p.109]. This guarantees the existence of the operator $\tilde{\mathcal{A}} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ given by

$$\sigma(\Phi, \Psi) = \left\langle \tilde{\mathcal{A}}\Phi, \Psi \right\rangle_{\mathcal{V}^*,\mathcal{V}} .$$

To write (2.12) in an equivalent strong form, we restrict $\tilde{\mathcal{A}}$ to the system operator

$$\begin{aligned} &\text{dom}\mathcal{A} = \{(\phi_1, \phi_2) \in \mathcal{H} | \phi_2 \in V, \mathcal{A}_1\phi_1 + \mathcal{A}_2\phi_2 \in H\} \\ &\mathcal{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_1 & -\mathcal{A}_2 \end{bmatrix} , \end{aligned} \tag{2.13}$$

where $\mathcal{A}_1, \mathcal{A}_2$ are defined in (2.2). It should be noted that $\mathcal{A}$ is the negative of the restriction to dom$\mathcal{A}$ of the operator $\tilde{\mathcal{A}}$ so that $\sigma(\Phi, \Psi) = \langle -\mathcal{A}\Phi, \Psi \rangle_{\mathcal{H}}$ for $\Phi \in \text{dom}\mathcal{A}, \Psi \in \mathcal{V}$. A strong form of the abstract system model (2.12) is given by

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}z(t) + \mathcal{B}u(t) \quad \text{in } \mathcal{V}^* \\ z(0) &= z_0 \ . \end{aligned} \tag{2.14}$$

Existence, uniqueness and continuous dependence on data of the solution to (2.1) was first proven in [9] and can also be found in [2, Chapter 4]. For the first order form of the system, existence and uniqueness of the solution together with its equivalence to the second order weak solution are presented in [2, 9].

3 Infinite Dimensional Control

The abstract model (2.12) (equivalently (2.14)) is useful in applications where the structure undergoes initial deformations and vibrates to a steady state. Control methods for this case are designed to attenuate only the transient state responses.

The output observations in the observation space $\mathcal{Y}$ are given by $z_{ob} = \mathcal{C}z(t)$ where $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ is bounded. As already noted, the operator $\mathcal{C}$ is often unbounded in applications but we only consider the bounded case here. We also make the simplifying assumption that the full state $z = (w, \dot{w})$ is available for the computation of the feedback control $u(t)$. In many practical applications, current measuring devices can only deliver partial state measurements hence compensators must be included in the control design. The reader is referred to [2, Chapters 7.5 and 8] for discussions regarding compensators and [1] for a summary of results on unbounded observation operators $\mathcal{C}$.

The quadratic functional we minimize in order to determine the optimal control $\bar{u}$ for the infinite horizon control problem is

$$J(u, z_0) = \int_0^\infty \left\{ ||\mathcal{C}z(t)||^2_{\mathcal{Y}} + \left\| \mathcal{R}^{1/2}u(t) \right\|^2_{\mathcal{U}} \right\} dt \tag{3.15}$$

subject to

$$\begin{aligned}\dot{z}(t) &= \mathcal{A}z(t) + \mathcal{B}u(t) \\ z(0) &= z_0 \,.\end{aligned}$$

Here, the positive, self-adjoint operator $\mathcal{R} = (\mathcal{R}^{1/2})^2 \in \mathcal{L}(\mathcal{U},\mathcal{U})$ is used to soft constrain the control input. We do not state results for the finite horizon problem but the reader is referred to discussions in [1, Theorem 3.1] and [2, Chapter 7.2.1].

We first give the definitions for the pair $(\mathcal{A},\mathcal{B})$ to be stabilizable and $(\mathcal{A},\mathcal{C})$ to be detectable before stating a theorem which which uses these conditions to guarantee the existence of optimal controls minimizing (3.15).

Definition 3.1 *The pair $(\mathcal{A},\mathcal{B})$ is said to be stabilizable if there exists an operator $\mathcal{K} \in \mathcal{L}(\mathcal{V}^*,\mathcal{U})$ such that $\mathcal{A}-\mathcal{B}\mathcal{K}$ generates an exponentially stable semigroup on $\mathcal{V}^*$ (i.e., there exists $M \geq 1, \omega > 0$ such that $\left\|e^{t(\mathcal{A}-\mathcal{B}\mathcal{K})}\right\|_{\mathcal{L}(\mathcal{V}^*)} \leq Me^{-\omega t}$).*

Definition 3.2 *The pair $(\mathcal{A},\mathcal{C})$ is said to be detectable if there exists an operator $\mathcal{F} \in \mathcal{L}(\mathcal{Y},\mathcal{V}^*)$ such that $\mathcal{A}-\mathcal{F}\mathcal{C}$ generates an exponentially stable semigroup on $\mathcal{V}^*$.*

Theorem 3.1 *If $(\mathcal{A},\mathcal{B})$ is stabilizable and $(\mathcal{A},\mathcal{C})$ is detectable, then the algebraic Riccati equation*

$$\left(\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi + \mathcal{C}^*\mathcal{C}\right) z = 0 \qquad \forall z \in \mathcal{V} \tag{3.16}$$

has a unique non-negative solution $\Pi \in \mathcal{L}(\mathcal{V}^*,\mathcal{V})$, $\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi$ generates an exponentially stable closed loop semigroup $\mathcal{S}(t)$ on $\mathcal{H},\mathcal{V},\mathcal{V}^*$, and the optimal control that minimizes (3.15) is given by

$$\bar{u}(t) = -\mathcal{R}^{-1}\mathcal{B}^*\Pi\bar{z}(t)$$

where $\bar{z}(t) = \mathcal{S}(t)z_0$ for $z_0 \in \mathcal{V}^*$.

Note: Theorem 3.1 is Theorem 7.5 in [2] and its proof can be found in [1].

4 Approximation and Finite Dimensional Control

The solutions to (2.1) or (2.14) together with the optimal controls given by Theorem 3.1 are infinite dimensional. For numerical applications, we use Galerkin approximations to obtain solutions in finite dimensional subspaces $\mathcal{V}^{\mathcal{N}} \subset \mathcal{V} \subset \mathcal{H}$. The bases for these subspaces can consist of modes, splines, polynomials, finite elements, or reduced basis elements.

Since $\mathcal{V} = V \times V$, we use the superscript $\mathcal{N}$ to denote $\mathcal{N}$-dimensional subspaces of V and use the superscript $2\mathcal{N}$ to denote the $2\mathcal{N}$-dimensional subspace $\mathcal{V}^{2\mathcal{N}} = V^{\mathcal{N}} \times V^{\mathcal{N}}$. The following approximation condition is necessary for the convergence of the finite dimensional solutions to the infinite dimensional solutions to (2.1) or (2.14)

(H1N) Let $\mathbf{V}$ be a Hilbert space. For any $z \in \mathbf{V}$, there exists a sequence $\tilde{z}^{\mathcal{N}} \in \mathbf{V}^N$ such that $\left|z - \tilde{z}^{\mathcal{N}}\right|_{\mathbf{V}} \to 0$ as $N \to \infty$.

We now define the operators $\mathcal{A}_1^{\mathcal{N}} : V^{\mathcal{N}} \to V^{\mathcal{N}}$ and $\mathcal{A}_2^{\mathcal{N}} : V^{\mathcal{N}} \to V^{\mathcal{N}}$ which approximate $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, by restricting the corresponding sesquilinear forms to $V^{\mathcal{N}} \times V^{\mathcal{N}}$, i.e.,

$$\begin{aligned}\left\langle \mathcal{A}_1^{\mathcal{N}}\phi,\psi\right\rangle_H &= \sigma_1(\phi,\psi) \quad \forall \phi,\psi \in V^{\mathcal{N}} \\ \left\langle \mathcal{A}_2^{\mathcal{N}}\phi,\psi\right\rangle_H &= \sigma_2(\phi,\psi) \quad \forall \phi,\psi \in V^{\mathcal{N}} \,.\end{aligned} \tag{4.17}$$

Similarly, the operator $\mathcal{A}^{2\mathcal{N}} : \mathcal{V}^{2\mathcal{N}} \to \mathcal{V}^{2\mathcal{N}}$ is defined by the restriction of σ on $\mathcal{V}^{2\mathcal{N}} \times \mathcal{V}^{2\mathcal{N}}$ with definition

$$\left\langle -\mathcal{A}^{2\mathcal{N}}\Phi, \Psi \right\rangle_{\mathcal{H}} = \sigma(\Phi, \Psi) \quad \forall \Phi, \Psi \in \mathcal{V}^{2\mathcal{N}} \ . \tag{4.18}$$

It readily follows that

$$\mathcal{A}^{2\mathcal{N}} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_1^{\mathcal{N}} & -\mathcal{A}_2^{\mathcal{N}} \end{bmatrix} . \tag{4.19}$$

The usual projection operators from H onto $V^{\mathcal{N}}$ and from $\mathcal{H}$ onto $\mathcal{V}^{2\mathcal{N}}$ are denoted by $\mathcal{P}^{\mathcal{N}}$ and $\mathcal{P}^{2\mathcal{N}}$, respectively, and are defined by

$$\begin{aligned} &\mathcal{P}^{\mathcal{N}}\phi \in V^{\mathcal{N}} \text{ and } \left\langle \mathcal{P}^{\mathcal{N}}\phi - \phi, \psi \right\rangle_{H} = 0 \quad \forall \psi \in V^{\mathcal{N}} \\ &\mathcal{P}^{2\mathcal{N}}\Phi \in \mathcal{V}^{2\mathcal{N}} \text{ and } \left\langle \mathcal{P}^{2\mathcal{N}}\Phi - \Phi, \Psi \right\rangle_{\mathcal{H}} = 0 \quad \forall \Psi \in \mathcal{V}^{2\mathcal{N}} \ . \end{aligned} \tag{4.20}$$

The control operator $\mathcal{B}$ is approximated by $\mathcal{B}^{2\mathcal{N}}$ by restricting it to the finite dimensional subspace using its adjoint

$$\left\langle \mathcal{B}^{2\mathcal{N}}u, \Psi \right\rangle_{\mathcal{H}} = \langle u, \mathcal{B}^*\Psi \rangle_{\mathcal{U}} \ , \quad \forall \Psi \in \mathcal{V}^{2\mathcal{N}} \ ,$$

and $\mathcal{C}^{2\mathcal{N}}$ is the restriction of $\mathcal{C}$ to $\mathcal{V}^{2\mathcal{N}}$. Thus, the finite dimensional analogue to (2.14) is given by

$$\begin{aligned} \dot{z}^{2\mathcal{N}}(t) &= \mathcal{A}^{2\mathcal{N}}z(t) + \mathcal{B}^{2\mathcal{N}}u(t) \quad \text{in } \mathcal{V}^{2\mathcal{N}} \\ z^{2\mathcal{N}}(0) &= \mathcal{P}^{2\mathcal{N}}z_0 \ . \end{aligned} \tag{4.21}$$

The following is Lemma 4.1 in [1] which guarantees the convergence of the finite dimensional Galerkin solutions.

Theorem 4.1 *(Lemma 4.1 in [1]) Suppose (H1N) is satisfied and let* $\mathcal{B}u \in L^2(0,T;\mathcal{V}^*)$ *and* $z_0 \in \mathcal{H}$. *If* $z(t) \in \mathcal{V}$ *is the solution to (2.14) and* $z^{2\mathcal{N}}(t) \in \mathcal{V}^{2\mathcal{N}}, t \geq 0$ *satisfies*

$$\begin{gathered} \frac{d}{dt}\left\langle z^{2\mathcal{N}}(t), \Psi \right\rangle_{\mathcal{H}} + \sigma(z^{2\mathcal{N}}(t), \Psi) = \left\langle \mathcal{B}^{2\mathcal{N}}u(t), \psi \right\rangle_{\mathcal{V}^*,\mathcal{V}} \ , \quad \forall \Psi \in \mathcal{V}^{2\mathcal{N}} \\ z^{2\mathcal{N}}(0) = \mathcal{P}^{2\mathcal{N}}z_0 \ , \end{gathered} \tag{4.22}$$

then the error function $e^{2\mathcal{N}}(t) = z^{2\mathcal{N}}(t) - z(t)$ *satisfies*

$$\left\| e^{2\mathcal{N}}(t) \right\|_{\mathcal{H}} \to 0$$

and

$$\int_0^t \left\| e^{2\mathcal{N}}(s) \right\|_{\mathcal{V}}^2 ds \to 0 \quad \text{as } \mathcal{N} \to \infty$$

uniformly in $t \in [0,T]$.

The approximate infinite horizon control problem for the system involves finding the control $u \in L^2(0,\infty;\mathcal{U})$ which minimizes

$$J^{2\mathcal{N}}(u, z_0^{2\mathcal{N}}) = \int_0^\infty \left\{ \left\| \mathcal{C}^{2\mathcal{N}}z^{2\mathcal{N}}(t) \right\|_{\mathcal{Y}}^2 + \left\| \mathcal{R}^{1/2}u(t) \right\|_{\mathcal{U}}^2 \right\} dt \tag{4.23}$$

subject to $z^{2\mathcal{N}}$ satisfying

$$\begin{aligned} \dot{z}^{2\mathcal{N}}(t) &= \mathcal{A}^{2\mathcal{N}}z^{2\mathcal{N}} + \mathcal{B}^{2\mathcal{N}}u(t) \ , \quad t > 0 \\ z^{2\mathcal{N}}(0) &= \mathcal{P}^{2\mathcal{N}}z_0 = z_0^{2\mathcal{N}} \ . \end{aligned} \tag{4.24}$$

The analogous definition of stabilizable and detectable matrix pairs for finite dimensional systems will be stated before we present the convergence theorem.

Definition 4.1 *The pair* $(\mathcal{A}^{2\mathcal{N}}, \mathcal{B}^{2\mathcal{N}})$ *is said to be* *uniformly stabilizable* *if there exist constants* $M_1 \geq 1, \omega_1 > 0$ *independent of* $\mathcal{N}$ *and a sequence of operators* $\mathcal{K}^{2\mathcal{N}} \in \mathcal{L}(\mathcal{V}^{2\mathcal{N}}, \mathcal{U})$ *such that* $\sup_{\mathcal{N}} \left|\mathcal{K}^{2\mathcal{N}}\right| < \infty$ *and*

$$\left\| e^{t\left(\mathcal{A}^{2\mathcal{N}} - \mathcal{B}^{2\mathcal{N}}\mathcal{K}^{2\mathcal{N}}\right)}\mathcal{P}^{2\mathcal{N}} z \right\|_{\mathcal{H}} \leq M_1 e^{-\omega_1 t} \left\| z \right\|_{\mathcal{H}}$$

for $z \in \mathcal{H}$.

Definition 4.2 *The pair* $(\mathcal{A}^{2\mathcal{N}}, \mathcal{C}^{2\mathcal{N}})$ *is said to be* *uniformly detectable* *if there exist constants* $M_2 \geq 1, \omega_2 > 0$ *independent of* $\mathcal{N}$ *and a sequence of operators* $\mathcal{F}^{2\mathcal{N}} \in \mathcal{L}(\mathcal{Y}, \mathcal{V}^{2\mathcal{N}})$ *such that* $\sup_{\mathcal{N}} \left|\mathcal{F}^{2\mathcal{N}}\right| < \infty$ *and*

$$\left\| e^{t\left(\mathcal{A}^{2\mathcal{N}} - \mathcal{F}^{2\mathcal{N}}\mathcal{C}^{2\mathcal{N}}\right)}\mathcal{P}^{2\mathcal{N}} z \right\|_{\mathcal{H}} \leq M_2 e^{-\omega_2 t} \left\| z \right\|_{\mathcal{H}}$$

for $z \in \mathcal{H}$.

Theorem 4.2 *Suppose* $V \overset{i}{\hookrightarrow} H$, *where the embedding* i *is compact. Let the sesquilinear form* σ *associated with the first-order system (2.14) be continuous and* $\mathcal{V}$*-elliptic. Assume that the operators* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *are such that* $(\mathcal{A}, \mathcal{B})$ *is stabilizable and* $(\mathcal{A}, \mathcal{C})$ *is detectable where* $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{V}^*)$ *is unbounded and* $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ *is bounded. Consider an approximation method which satisfies* $(H1N)$. *Finally, suppose that for fixed* $\mathcal{N}_0$ *and* $\mathcal{N} > \mathcal{N}_0$, *the pair* $(\mathcal{A}^{2\mathcal{N}}, \mathcal{B}^{2\mathcal{N}})$ *is uniformly stabilizable and* $(\mathcal{A}^{2\mathcal{N}}, \mathcal{C}^{2\mathcal{N}})$ *is uniformly detectable.*

Then for $\mathcal{N}$ *sufficiently large, there exists a unique nonnegative self-adjoint solution* $\Pi^{2\mathcal{N}} \in \mathcal{L}(\mathcal{V}^*, \mathcal{V})$ *to the* $\mathcal{N}^{th}$ *approximate algebraic Riccati equation (3.16) in* $\mathcal{V}^{2\mathcal{N}}$ *with* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *replaced by* $\mathcal{A}^{2\mathcal{N}}, \mathcal{B}^{2\mathcal{N}}, \mathcal{C}^{2\mathcal{N}}$, *respectively. There also exist constants* $M_3 \geq 1$ *and* $\omega_3 > 0$ *independent of* $\mathcal{N}$ *such that* $\mathcal{S}^{2\mathcal{N}}(t) = e^{\left(\mathcal{A}^{2\mathcal{N}} - \mathcal{B}^{2\mathcal{N}}\mathcal{R}^{-1}\mathcal{B}^{2\mathcal{N}^*}\Pi^{2\mathcal{N}}\right)t}$ *satisfies*

$$\left\| \mathcal{S}^{2\mathcal{N}}(t) \right\|_{\mathcal{V}^{2\mathcal{N}}} \leq M_3 e^{-\omega_3 t} \ , t > 0$$

or equivalently

$$\left\| e^{t\left(\mathcal{A}^{2\mathcal{N}} - \mathcal{B}^{2\mathcal{N}}\mathcal{R}^{-1}\mathcal{B}^{2\mathcal{N}^*}\right)}\mathcal{P}^{2\mathcal{N}} z_0 \right\|_{\mathcal{H}} \leq M_3 e^{-\omega_3 t} \left\| z_0 \right\|_{\mathcal{H}} \ , t > 0, z_0 \in \mathcal{H} \ .$$

Additionally, the convergences

$$\Pi^{2\mathcal{N}}\mathcal{P}^{2\mathcal{N}} z \overset{s}{\to} \Pi z \text{ in } \mathcal{V} \text{ for every } z \in \mathcal{V}^*$$

$$\left\| \mathcal{B}^{2\mathcal{N}^*}\Pi^{2\mathcal{N}}\mathcal{P}^{2\mathcal{N}} - \mathcal{B}^*\Pi \right\|_{\mathcal{L}(\mathcal{H},\mathcal{U})} \to 0 \ ,$$

as $\mathcal{N} \to \infty$, *of the Riccati and control operators are obtained. Moreover, the feedback system operator* $\mathcal{A} - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^{2\mathcal{N}^*}\Pi^{2\mathcal{N}}$ *generates an exponentially stable analytic semigroup on* $\mathcal{H}$ *and for every* $z_0 \in \mathcal{H}$,

$$J^{2\mathcal{N}}\left(-\mathcal{B}^{2\mathcal{N}^*}\Pi^{2\mathcal{N}} z(\cdot), z_0\right) - J(\bar{u}, z_0) \leq \varepsilon(\mathcal{N}) \left\| z_0 \right\|_{\mathcal{H}}^2$$

where $\varepsilon(\mathcal{N}) \to$ *as* $\mathcal{N} \to \infty$.

Theorem 4.2 is Theorem 7.10 in [2] in which the assumptions of uniform stabilizability of $(\mathcal{A}^{2\mathcal{N}}, \mathcal{B}^{2\mathcal{N}})$ and uniform detectability of $(\mathcal{A}^{2\mathcal{N}}, \mathcal{C}^{2\mathcal{N}})$ can be difficult to directly verify. Hence we need additional results based on readily confirmed assumptions. For general first-order systems, Lemma 4.7 in [1] (which is restated as Lemma 7.12 in [2]) guarantees that $(\mathcal{A}^{2\mathcal{N}}, \mathcal{B}^{2\mathcal{N}})$ is stabilizable provided $(\mathcal{A}, \mathcal{B})$ is stabilizable and the injection $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact. For the second-order system rewritten as a first-order system, the definition of the product spaces $\mathcal{V} = V \times V$ and $\mathcal{H} = H \times V$ precludes the possibility of compactness of $i : \mathcal{V} \hookrightarrow \mathcal{H}$, even when $V \hookrightarrow H$ is compact. Hence we require the following lemma to obtain uniform exponential bounds on the approximating semigroups $\mathcal{S}^{2\mathcal{N}}(t)$.

Lemma 4.1 *(Lemma 6.2 in [1]) Suppose $V \overset{i}{\hookrightarrow} H$, where i is compact. Moreover, suppose that the damping sesquilinear form σ_2 can be decomposed as $\sigma_2 = \gamma\sigma_1 + \tilde{\sigma}_2$, for some $\gamma > 0$, where the continuous sesquilinear form $\tilde{\sigma}_2$ satisfies for some $\mu \in \mathbb{R}$*

$$Re\tilde{\sigma}_2(\phi, \phi) \geq -\frac{c_2\gamma}{2}\|\phi\|_V^2 - \mu\|\phi\|_H^2 \text{ for all } \phi \in V .$$

Finally, suppose that the operator $\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2$, where $\tilde{\mathcal{A}}_2 \in \mathcal{L}(V, V^*)$ is defined by

$$\langle \tilde{\mathcal{A}}_2\phi, \eta \rangle_{V^*,V} = \tilde{\sigma}_2(\phi, \eta) \tag{4.25}$$

is compact on V.

Let $\mathcal{T}$ denote the open loop semigroup generated by the product space operator $\mathcal{A}$ and let $\mathcal{T}^{2\mathcal{N}}$ be generated by $\mathcal{A}^{2\mathcal{N}}$. If for some $\nu \geq \mu$ and $M \geq 1$

$$\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\nu t} , \quad t \geq 0 , \tag{4.26}$$

then for any $\varepsilon > 0$ there exists an integer $\mathcal{N}_\varepsilon$ such that for $\mathcal{N} \geq \mathcal{N}_\varepsilon$

$$\left\|\mathcal{T}^{2\mathcal{N}}(t)\mathcal{P}^{2\mathcal{N}}\right\|_{\mathcal{L}(\mathcal{H})} \leq \tilde{M}e^{(\nu+\varepsilon)t} , \quad t \geq 0 ,$$

for some constant $\tilde{M} > 0$ independent of $\mathcal{N}$.

Since the proof of Lemma 6.2 in [1] is given only in sketchy form, we give a detailed proof here. We first state and prove the following lemmas which will be used in proving Lemma 4.1 below.

Lemma 4.2 $\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1} \equiv P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}$ *on $V^{\mathcal{N}}$, where $P_{V_1}^{\mathcal{N}} : V \to V^{\mathcal{N}}$ is defined by*

$$\sigma_1\left(P_{V_1}^{\mathcal{N}}\phi - \phi, \psi^{\mathcal{N}}\right) = 0 \quad \text{for all } \psi^{\mathcal{N}} \in V^{\mathcal{N}} , \phi \in V . \tag{4.27}$$

<u>Proof</u>:

- Since $\sigma_1(\cdot,\cdot)$ gives rise to an inner product on V whose induced norm is equivalent to the usual norm in V, then a closer inspection of (4.27) reveals that $P_{V_1}^{\mathcal{N}}$ is the projection operator from V into $V^{\mathcal{N}}$ under the $\sigma_1(\cdot,\cdot)$ inner product, i.e., the projection of V_1 onto $V^{\mathcal{N}}$ where V_1 is V with the σ_1 inner product. It follows immediately that $P_{V_1}^{\mathcal{N}}$ from V onto $V^{\mathcal{N}}$ is well defined and linear. Now let $\phi^{\mathcal{N}}$ be any arbitrary element of $V^{\mathcal{N}}$. Then for any $\psi^{\mathcal{N}} \in V^{\mathcal{N}}$,

$$\begin{aligned}
\sigma_1\left(\mathcal{A}_1^{-1}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) &= \langle \phi^{\mathcal{N}}, \psi^{\mathcal{N}} \rangle_{V^*,V} , \quad \text{(from (2.2))} \\
&= \langle \phi^{\mathcal{N}}, \psi^{\mathcal{N}} \rangle_H , \quad (\text{since } \phi^{\mathcal{N}} \in V) \\
&= \left\langle \mathcal{A}_1^{\mathcal{N}}\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\phi^{\mathcal{N}}, \psi^{\mathcal{N}} \right\rangle_H \\
&= \sigma_1\left(\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) , \quad \text{(from (4.17))} .
\end{aligned}$$

We note that $\mathcal{A}_1^{\mathcal{N}}$ is invertible in $V^{\mathcal{N}}$ follows from (4.17) and the V-ellipticity of σ_1. Thus,

$$\sigma_1\left(\mathcal{A}_1^{-1}\phi^{\mathcal{N}} - \left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) = 0$$

for every $\psi^{\mathcal{N}} \in V^{\mathcal{N}}$. But by the defining the relationships (4.27) we have

$$\sigma_1\left(P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\phi^{\mathcal{N}} - \mathcal{A}_1^{-1}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) = 0 \text{ for all } \phi^{\mathcal{N}}, \psi^{\mathcal{N}} \in V^{\mathcal{N}} .$$

It follows that $P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\phi^{\mathcal{N}} = \left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\phi^{\mathcal{N}}$ and this completes the proof. ∎

Lemma 4.3 $\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}} \equiv P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2$ *on* $V^{\mathcal{N}}$, *where* $\tilde{\mathcal{A}}_2^{\mathcal{N}}$ *is defined by the restriction of* $\tilde{\sigma}_2$ *on* $V^{\mathcal{N}}$, *i.e.,* $\tilde{\mathcal{A}}_2^{\mathcal{N}} : V^{\mathcal{N}} \to V^{\mathcal{N}}$ *is defined by*

$$\left\langle \tilde{\mathcal{A}}_2^{\mathcal{N}}\phi, \psi\right\rangle_H = \tilde{\sigma}_2(\phi,\psi) \quad \forall \phi,\psi \in V^{\mathcal{N}} . \tag{4.28}$$

Proof:

- Let $\phi^{\mathcal{N}} \in V^{\mathcal{N}}$. Then for all $\psi^{\mathcal{N}} \in V^{\mathcal{N}}$

$$\begin{aligned}
\sigma_1\left(\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) &= \left\langle \tilde{\mathcal{A}}_2^{\mathcal{N}}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right\rangle_{V^*,V} \text{ (from Lemma 4.2,}\\
&\quad \text{(4.27) and (2.2))}\\
&= \left\langle \tilde{\mathcal{A}}_2^{\mathcal{N}}\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right\rangle_H \text{ (since } \tilde{\mathcal{A}}_2^{\mathcal{N}}\phi^{\mathcal{N}} \in V^{\mathcal{N}})\\
&= \tilde{\sigma}_2(\phi^{\mathcal{N}}, \psi^{\mathcal{N}}) \text{ (from (4.28))}\\
&= \left\langle \tilde{\mathcal{A}}_2\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right\rangle_{V^*,V} \text{ (from (4.25))}\\
&= \left\langle \mathcal{A}_1\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right\rangle_{V^*,V}\\
&= \sigma_1\left(\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) \text{ (from (2.2))} .
\end{aligned}$$

  Therefore, $\sigma_1\left(\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\phi^{\mathcal{N}} - \mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2\phi^{\mathcal{N}}, \psi^{\mathcal{N}}\right) = 0$,$\forall \psi^{\mathcal{N}} \in V^{\mathcal{N}}$. We then conclude from (4.27) that $\left(\mathcal{A}_1^{\mathcal{N}}\right)^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\phi^{\mathcal{N}} = P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2\phi^{\mathcal{N}}$ $\forall \phi^{\mathcal{N}} \in V^{\mathcal{N}}$. ∎

Lemma 4.4 $\left\|\mathcal{A}_1^{-1} - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\right\|_{\mathcal{L}(V,V)} \to 0$
Proof:

- First we consider $\mathcal{A}_1^{-1}$ as a compact operator from V to V by using the compact injections $i : V \hookrightarrow H$ and $i^* : H^* \hookrightarrow V^*$ and setting $\mathcal{A}_1^{-1} = \mathcal{A}_1^{-1}i^*i$. Next, note that the convergence $P_{V_1}^{\mathcal{N}}w \to Iw$,$\forall w \in V$ is evident from (H1N) where $P_{V_1}^{\mathcal{N}}$ is, as introduced above, the projection operator from V_1 onto $V^{\mathcal{N}}$ (again V_1 is V with the the norm induced by the $\sigma_1(\cdot,\cdot)$ inner product). Thus convergence follows from $\left|P_{V_1}^{\mathcal{N}}z - z\right|_{V_1} \le \left|\tilde{z}^{\mathcal{N}} - z\right|_{V_1} \le k\left|\tilde{z}^{\mathcal{N}} - z\right|_V$. Then

$$\begin{aligned}
\left\|\mathcal{A}_1^{-1} - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\right\|_{\mathcal{L}(V,V)} &= \sup_{\|v\|_V \le 1}\left\|\left(\mathcal{A}_1^{-1} - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\right)v\right\|_V\\
&= \sup_{\|v\|_V \le 1}\left\|(I - P_{V_1}^{\mathcal{N}})\mathcal{A}_1^{-1}v\right\|_V\\
&= \sup_{w \in U}\left\|(I - P_{V_1}^{\mathcal{N}})w\right\|_V
\end{aligned}$$

where $U = \mathcal{A}_1^{-1}\left(\{\|v\|_V \leq 1\}\right)$ is a relatively compact subset in V since $\mathcal{A}_1^{-1}$ is a compact operator from V to V. Thus, by Chatelin's Theorem[4],

$$\sup_{w\in U} \left\|(I - P_{V_1}^{\mathcal{N}})w\right\|_V \to 0 \ .$$

This gives us the desired result

$$\left\|\mathcal{A}_1^{-1} - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\right\|_{\mathcal{L}(V,V)} \to 0 \ .$$

■

Lemma 4.5 $\left\|\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2 - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2\right\|_{\mathcal{L}(V,V)} \to 0$ *where* $\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2$ *is compact on* V.
Proof:

- The proof of this lemma is similar to the proof given for the previous lemma. ■

We now give the proof of Lemma 4.1.
Proof of Lemma 4.1:

- We first express the sesquilinear form σ in terms of components,

$$\sigma(\Phi,\Psi) = -\left\langle \phi_2, \psi_1\right\rangle_{V_1} + \sigma_1(\phi_1,\psi_2) + \sigma_2(\phi_2,\psi_2) \ ,$$

then

$$\begin{aligned}
Re\sigma(\Phi,\Phi) &= Re\left\{-\left\langle \phi_2,\phi_1\right\rangle_{V_1} + \sigma_1(\phi_1,\phi_2) + \sigma_2(\phi_2,\phi_2)\right\} \\
&= Re\sigma_2(\phi_2,\phi_2) \ , \quad \text{since } \langle\cdot,\cdot\rangle_{V_1} \text{ is equivalent to } \sigma_1(\cdot,\cdot) \\
&= \gamma Re\sigma_1(\phi_2,\phi_2) + Re\tilde{\sigma}_2(\phi_2,\phi_2) \ , \quad \text{since} \\
&\qquad\qquad \sigma_2(\psi,\psi) = \gamma\sigma_1(\psi,\psi) + \tilde{\sigma}_2(\psi,\psi) \text{ for some } \gamma > 0 \\
&\geq \gamma c_2\|\phi_2\|_V^2 - \frac{c_2}{2}\gamma\|\phi_2\|_V^2 - \mu\|\phi_2\|_H^2 \ , \quad \text{since} \\
&\qquad\qquad Re\sigma_1(\phi,\phi) \geq c_2\|\phi\|_V^2 \text{ (see (H2)), and} \\
&\qquad\qquad Re\tilde{\sigma}_2(\psi,\psi) \geq -\frac{c_2\gamma}{2}\|\psi\|_V^2 - \mu\|\psi\|_H^2 \\
&= \frac{c_2}{2}\gamma\|\phi_2\|_V^2 - \mu\|\phi_2\|_H^2 \\
&= \frac{c_2}{2}\gamma\|\phi_2\|_V^2 + \mu\|\phi_1\|_H^2 - \mu\left\{\|\phi_2\|_H^2 + \|\phi_1\|_H^2\right\} \\
&\geq \min\left\{\frac{\gamma c_2}{2},\mu\right\}\|\Phi\|_V^2 - \mu\,\|\Phi\|_H^2 \ .
\end{aligned}$$

It follows that the linear operator $\mathcal{A}$ associated with σ (see (2.13)) is the infinitesimal generator of a C_0 semigroup $\mathcal{T}(t)$ satisfying

$$\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\mu t} \ , \tag{4.29}$$

[4]Theorem 3.2 in [10]: Suppose that $T_n x \to Tx, x \in X$. Then for any relatively compact set U, $\sup_{x\in U} \|(T_n - T)x\| \to 0$.

and thus (4.26) holds with $\nu = \mu$. Now suppose (4.26) holds for any $\nu \geq \mu$, i.e., suppose $\|\mathcal{T}(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\nu t}$ is satisfied. To prove the lemma, we show $\forall \varepsilon > 0$, there exists N_ε such that for $N \geq N_\varepsilon$

$$\left\|\mathcal{T}^{2\mathcal{N}}(t)\mathcal{P}^{2\mathcal{N}}\right\|_{\mathcal{L}(\mathcal{H})} \leq \tilde{M}e^{(\nu+\varepsilon)t} \ , \quad t \geq 0 \ . \tag{4.30}$$

The next step involves writing the semigroup $\mathcal{T}(t)$ in terms of the generator $\mathcal{A}$ using the inverse of the Laplace transform. From the result above that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup satisfying (4.26), it follows that $\mathcal{A} - \nu I$ generates a C_0 semigroup of contractions $\mathcal{S}(t)$ satisfying $\|\mathcal{S}(t)\|_{\mathcal{L}(\mathcal{H})} \leq M$, where $\mathcal{T}(t) = \mathcal{S}(t)e^{\nu t}$. Theorem 6.A in [7] guarantees the existence of δ such that $0 < \delta < \pi/4$ and

$$\rho(\mathcal{A} - \nu I) \supset \Sigma_\delta = \{z \in \mathbb{C} : |\arg(z)| < \pi/2 + \delta\} \ .$$

By Theorem 7.7 in [11], we can express $\mathcal{S}(t)$ as

$$\begin{aligned} \mathcal{S}(t) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - (\mathcal{A} - \nu I))^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}((\lambda + \nu)I - \mathcal{A})^{-1} d\lambda \ , \end{aligned}$$

where Γ is a smooth curve in $\Sigma_\delta \cup \{0\}$ running from $\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$ for $\pi/2 < \vartheta < \pi/2 + \delta$. Since $\mathcal{T}(t) = e^{\nu t}\mathcal{S}(t)$, we have

$$\mathcal{T}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\nu t} e^{\lambda t}((\lambda + \nu)I - \mathcal{A})^{-1} d\lambda \ .$$

Shifting the path of integration Γ by ν and denoting it by Γ', we have

$$\mathcal{T}(t) = \frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda t}(\lambda I - \mathcal{A})^{-1} d\lambda \ . \tag{4.31}$$

The finite dimensional semigroup can similarly be written as

$$\mathcal{T}^{2\mathcal{N}}(t) = \frac{1}{2\pi i}\int_{\Gamma'} e^{\lambda t}\left(\lambda I - \mathcal{A}^{2\mathcal{N}}\right)^{-1} d\lambda \ .$$

Multiplying by $\mathcal{P}^{2\mathcal{N}}$ and obtaining the norm of both sides, we have the estimate

$$\left\|\mathcal{T}^{2\mathcal{N}}(t)\mathcal{P}^{2\mathcal{N}}\right\|_{\mathcal{L}(\mathcal{H})} \leq \left|\frac{1}{2\pi i}\right| \int_{\Gamma'} \left|e^{\lambda t}\right| \left\|\left(\lambda I - \mathcal{A}^{2\mathcal{N}}\right)^{-1}\mathcal{P}^{2\mathcal{N}}\right\|_{\mathcal{L}(\mathcal{H})} |d\lambda| \ .$$

Thus, if we find M_0 and N_ε such that whenever $Re\lambda \geq \nu + \varepsilon$ and $N \geq N_\varepsilon$,

$$\left\|\left(\lambda I - \mathcal{A}^{2\mathcal{N}}\right)^{-1}\mathcal{P}^{\mathcal{N}}\right\|_{\mathcal{L}(\mathcal{H})} \leq M_0 \ ,$$

then we establish (4.30). This uniform boundedness will be shown by decomposing the resolvent $\left(\lambda I - \mathcal{A}^{2\mathcal{N}}\right)^{-1}$ into its components. To this end, consider the resolvent equation

$$\left(\lambda I - \mathcal{A}^{2\mathcal{N}}\right)^{-1}\mathcal{P}^{2\mathcal{N}}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} z_1^{\mathcal{N}} \\ z_2^{\mathcal{N}} \end{pmatrix} ,$$

where $(f, g) \in \mathcal{H}$ and $(z_1^{\mathcal{N}}, z_2^{\mathcal{N}}) \in \mathcal{V}^{2\mathcal{N}}$. Using the components of $\mathcal{A}$ given in (4.19), and letting $\mathcal{P}^{2\mathcal{N}}(f, g) = (f^{\mathcal{N}}, g^{\mathcal{N}})$, we obtain

$$\begin{aligned} f^{\mathcal{N}} &= \lambda z_1^{\mathcal{N}} - z_2^{\mathcal{N}} \\ g^{\mathcal{N}} &= \lambda z_2^{\mathcal{N}} + \mathcal{A}_2^{\mathcal{N}} z_2^{\mathcal{N}} + \mathcal{A}_1^{\mathcal{N}} z_1^{\mathcal{N}} . \end{aligned}$$

Eliminating $z_2^{\mathcal{N}}$ in the system of equations above and using the assumption $\mathcal{A}_2 = \gamma\mathcal{A}_1 + \tilde{\mathcal{A}}_2$ we have

$$\begin{gathered}\left(I + \frac{\lambda^2}{\lambda\gamma+1}(\mathcal{A}_1^{\mathcal{N}})^{-1} + \frac{\lambda}{\lambda\gamma+1}(\mathcal{A}_1^{\mathcal{N}})^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\right) z_1^{\mathcal{N}} = \\ \frac{\gamma f^{\mathcal{N}}}{\lambda\gamma+1} + \frac{(\mathcal{A}_1^{\mathcal{N}})^{-1}}{\lambda\gamma+1}\left(g^{\mathcal{N}} + \lambda f^{\mathcal{N}} + \tilde{\mathcal{A}}_2^{\mathcal{N}} f^{\mathcal{N}}\right) .\end{gathered} \tag{4.32}$$

To simplify notation, let

$$G = I + \frac{\lambda^2}{\lambda\gamma+1}\mathcal{A}_1^{-1} + \frac{\lambda}{\lambda\gamma+1}\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2 \in \mathcal{L}(V,V) ,$$

and

$$\xi = \frac{\gamma f}{\lambda\gamma+1} + \frac{(\mathcal{A}_1)^{-1}}{\lambda\gamma+1}\left(g + \lambda f + \tilde{\mathcal{A}}_2 f\right) \in V ,$$

and denote the corresponding finite dimensional expressions by $G^{\mathcal{N}}$ and $\xi^{\mathcal{N}}$, respectively. If we show that $\|z_1^{\mathcal{N}}\|_V$ is bounded, then (4.30) is satisfied where $z_1^{\mathcal{N}}$ is given by $G^{\mathcal{N}} z_1^{\mathcal{N}} = \xi^{\mathcal{N}}$, for $\|(f,g)\|_{\mathcal{H}} \le 1$, $Re\lambda \ge \nu + \varepsilon$ and $\mathcal{N} \ge \mathcal{N}_\varepsilon$. The next step is then to show that the inverse of G exists and is bounded in $\mathcal{L}(V,V)$ whenever $Re\lambda > \nu$. Since $(\lambda I - \mathcal{A})^{-1}$ exists and is bounded in $\mathcal{L}(\mathcal{H},\mathcal{H})$ whenever $Re\lambda > \nu$, then for every $(f,g) \in \mathcal{H}$, we can solve for $(z_1, z_2) \in \mathcal{H}$ such that

$$(\lambda I - \mathcal{A})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} .$$

Solving for z_1 yields

$$z_1 = \left(I + \frac{\lambda}{\lambda\gamma+1}\mathcal{A}_1^{-1}(\lambda + \tilde{\mathcal{A}}_2)\right)^{-1}\left(\frac{\gamma f}{\lambda\gamma+1} + \frac{\mathcal{A}_1^{-1}}{\lambda\gamma+1}\left(g + \lambda f + \tilde{\mathcal{A}}_2 f\right)\right)$$

and $\|z_1\|_V \le M_1$ whenever $\|(f,g)\|_{\mathcal{H}} \le 1$. Thus, G^{-1} exists and is bounded in $\mathcal{L}(V,V)$ whenever $Re\lambda > \nu$. Now, we consider the finite dimensional operators and show boundedness of $\|z_1^{\mathcal{N}}\|_V$. Note first that $G^{\mathcal{N}} z_1^{\mathcal{N}}$ can be expressed as

$$\begin{aligned} G^{\mathcal{N}} z_1^{\mathcal{N}} &= \left(I + \frac{\lambda^2}{\lambda\gamma+1}(\mathcal{A}_1^{\mathcal{N}})^{-1} + \frac{\lambda}{\lambda\gamma+1}(\mathcal{A}_1^{\mathcal{N}})^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\right) z_1^{\mathcal{N}} \\ &= \left(I + \frac{\lambda^2}{\lambda\gamma+1}\mathcal{A}_1^{-1} + \frac{\lambda}{\lambda\gamma+1}\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2\right) z_1^{\mathcal{N}} \\ &\quad -\frac{\lambda^2}{\lambda\gamma+1}\left(\mathcal{A}_1^{-1} - (\mathcal{A}_1^{\mathcal{N}})^{-1}\right) z_1^{\mathcal{N}} \\ &\quad -\frac{\lambda}{\lambda\gamma+1}\left(\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2 - (\mathcal{A}_1^{\mathcal{N}})^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\right) z_1^{\mathcal{N}} . \end{aligned}$$

Since $G^{\mathcal{N}} z_1^{\mathcal{N}} = \xi^{\mathcal{N}}$, we have

$$\xi^{\mathcal{N}} = G z_1^{\mathcal{N}} - \frac{\lambda^2}{\lambda\gamma+1}\left(\mathcal{A}_1^{-1} - (\mathcal{A}_1^{\mathcal{N}})^{-1}\right) z_1^{\mathcal{N}} - \frac{\lambda}{\lambda\gamma+1}\left(\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2 - (\mathcal{A}_1^{\mathcal{N}})^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\right) z_1^{\mathcal{N}} .$$

Equivalently,

$$\begin{aligned} z_1^{\mathcal{N}} &= G^{-1}\left\{\xi^{\mathcal{N}} + \frac{\lambda^2}{\lambda\gamma+1}\left(\mathcal{A}_1^{-1} - (\mathcal{A}_1^{\mathcal{N}})^{-1}\right) z_1^{\mathcal{N}}\right. \\ &+ \left.\frac{\lambda}{\lambda\gamma+1}\left(\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2 - (\mathcal{A}_1^{\mathcal{N}})^{-1}\tilde{\mathcal{A}}_2^{\mathcal{N}}\right) z_1^{\mathcal{N}}\right\} . \end{aligned}$$

Taking the V norm of both sides and using Lemma 4.2 and Lemma 4.3, we obtain

$$\|z_1^{\mathcal{N}}\|_V \;\leq\; \|G^{-1}\|_{\mathcal{L}(V,V)} \left(\|\xi^{\mathcal{N}}\|_V + \left|\frac{\lambda^2}{\lambda\gamma+1}\right| \left\|\mathcal{A}_1^{-1} - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\right\|_{\mathcal{L}(V,V)} \left\|z_1^{\mathcal{N}}\right\|_V \right.$$
$$\left. + \left|\frac{\lambda}{\lambda\gamma+1}\right| \left\|(\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2 - P_{V_1}^{\mathcal{N}}\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2)\right\|_{\mathcal{L}(V,V)} \left\|z_1^{\mathcal{N}}\right\|_V \right) .$$

From the convergences in Lemma 4.4 and Lemma 4.5, it can be seen that $\|z_1^{\mathcal{N}}\|_V$ is bounded and this completes the proof. ∎

Remarks:

1. For the motivating example (2.4), the assumption $\sigma_2 = \gamma\sigma_1 + \tilde{\sigma}_2$ in Lemma 4.1 is satisfied if we ignore passive patch contributions to the system and assume that the density, Young's modulus, air damping and Kelvin-Voigt damping are constant. In this case, $\sigma_1(\phi,\psi) = \int_0^\ell EI\phi''\psi''dy$, $\sigma_2(\phi,\psi) = \int_0^\ell c_D I\phi''\psi''dy + c_a\int_0^\ell \phi\psi dy$ and thus $\gamma = c_D/E$ and $\tilde{\sigma}_2 = c_a\int_0^\ell \phi\psi dy$, where $\tilde{\sigma}_2$ satisfies

   $$Re\tilde{\sigma}_2(\phi,\phi) = \frac{c_a}{\rho hb}\|\phi\|_H^2 \geq -\frac{\gamma}{2}\|\phi\|_V^2 .$$

   The operator $\tilde{\mathcal{A}}_2$ generated by $\tilde{\sigma}_2$ is $\tilde{\mathcal{A}}_2 = c_a I$ which is clearly bounded. We then write $\mathcal{A}_1^{-1}$ as a compact operator on V by $\mathcal{A}_1^{-1} = \mathcal{A}_1^{-1}i^*i$, since $i: V \hookrightarrow H$ and $i^*: H \hookrightarrow V^*$ are compact and $\mathcal{A}_1^{-1}$ is bounded (due to (H2)). Since $\tilde{\mathcal{A}}_2$ is bounded, it follows that $\mathcal{A}_1^{-1}\tilde{\mathcal{A}}_2$ is compact on V.

2. If passive contributions are incorporated in the model, i.e., σ_1 and σ_2 are of the form (2.10), Lemma 4.1 is applicable only if the actuators are employed in such a manner that the sum of the structure and actuator stiffness coefficients is a multiple of the sum of the structure and actuator Kelvin-Voigt damping (this is necessary to satisfy the assumption $\sigma_2 = \gamma\sigma_1 + \tilde{\sigma}_2$). One possible way to achieve this is to embed the actuators so that material properties of the composite structure remain constant over the region covered by the patches.

3. The assumption of exponential stability of $(\mathcal{A}, \mathcal{B})$ in Theorem 4.2 is guaranteed by Theorem 7.15 in [2].

5 Concluding Remarks

We presented and proved Lemma 4.1 which can be used to establish uniform stabilizability of approximate finite dimensional systems arising in structural systems. In the context presented, uniform stabilizability is a sufficient condition for the convergence of the Riccati solution and control gains. As demonstrated in the remarks above and in [2, Chapter 7], the conditions in the lemma are easily verifiable for the motivating example (2.4) in which passive patch contributions are ignored. We point out that these ideas can be readily extended to systems involving shells or plates.

For models incorporating general passive patch contributions, the assumption $\sigma_2 = \gamma\sigma_1 + \tilde{\sigma}_2$ with $\tilde{\sigma}_2$ satisfying the hypothesis in Lemma 4.1 may not be satisfied and the authors are currently extending the lemma to include this more general case. Such an extension would complete the theory for convergence of Riccati and optimal control solutions for systems with no exogenous force and bounded observation operator $\mathcal{C}$. Numerical results demonstrating convergence of suboptimal controls for thin shell structures incorporating passive piezoceramic patch contributions have been demonstrated in [12, 4, 5].

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