Damping: Hysteretic Damping and Models *

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Definition of Hysteretic Damping

Vibrational damping is associated with the dissipation of energy during mechanical vibrations/deformations of an elastic body, usually via the conversion of mechanical energy to thermal energy producing heat in the body which is readily dissipated. Damping is most properly embodied in constitutive laws that relate a body's deformation (strain ε) or displacement to the stress σ or force associated with this deformation. These are most often given in terms of stress/strain laws $\sigma = \mathcal{F}(\varepsilon)$, the most elementary of which is Hooke's law

$$\sigma = E\varepsilon \tag{1}$$

where E is the material dependent Young's modulus or modulus of elasticity. Here and throughout we discuss concepts in a one-dimensional formulation such as occurs for example in the case of elongation of a simple uniform rod. In more general deformations one must use tensor analogues of the stress σ, the strain ε and parameters such as the modulus of elasticity E. We shall also assume small deformations throughout so that infinitesimal strain theory can be used.

Hooke's law (which is an idealization) does not allow for dissipation which occurs to some extent in all materials. The most widely accepted model which does is the Voigt model

$$\sigma = E\varepsilon + C\dot{\varepsilon} \tag{2}$$

where $\dot{\varepsilon} = \frac{d\varepsilon}{dt}$ is the strain rate. To characterize stress-strain laws, it is common to consider material behavior during cyclic loading-unloading and plot the associated stress-strain curves as given in Figure 1 for the Voigt law.

Whenever the material behaves differently under unloading than under loading (as in the case for (2) but not for (1)), the material is said to be *viscoelastic* and to exhibit the phenomenon of *hysteresis*. Hysteresis is present in many materials such as filled rubber and filled rubber-like composites (often referred to as elastomers), shape memory alloys (SMA's) such as Nitinol (a nickel-titanium alloy) and CuZnAl (a cooper zinc aluminum alloy), piezoelectrics (e.g., PZT's or lead zirconate titanates), electrostrictives (e.g., PMN's or lead magnesium niobates) and magnetostrictives (e.g., Terfenol-D, a terbium iron dysprosium alloy). The Voigt law (2) is inadequate in characterizing the hysteresis in such materials under many conditions where the hysteresis loop can be, as depicted in Figures 2 and 3, quite complex.

A number of different modeling approaches and resulting models have been developed to describe quantitatively the various hysteretic dissipation mechanisms such as that manifested in hysteresis loops similar to those shown in Figures 2 and 3.

*Invited Section, *Encyclopedia of Vibration*, Academic Press, London, 2000, to appear.

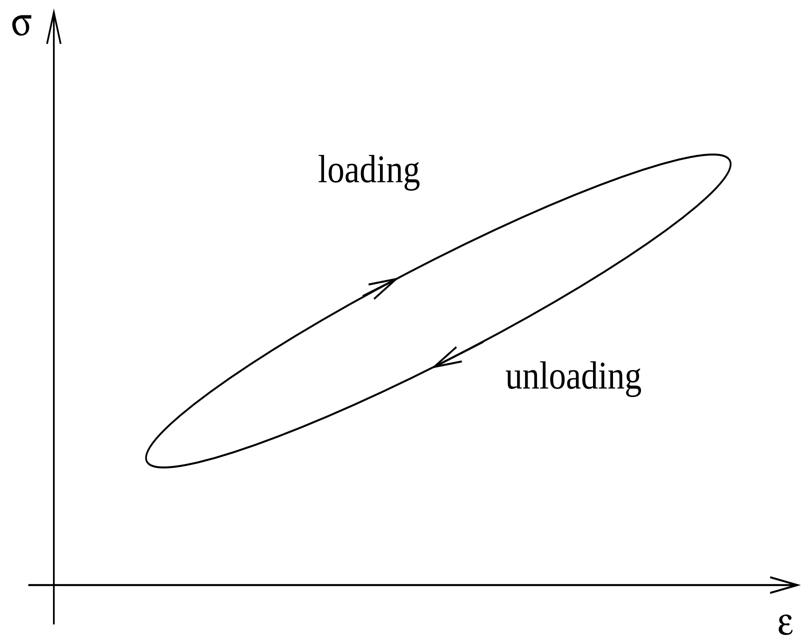

Figure 1: Loading/unloading for a Voigt material

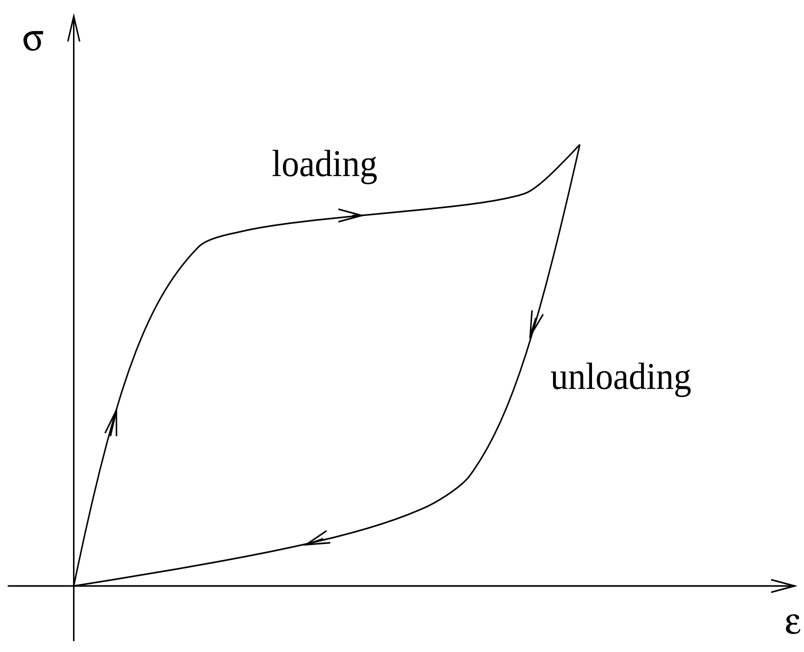

Figure 2: Loading/unloading for an SMA

Models for Hysteresis

The most fundamental model that correctly describes the hysteretic behavior of a number of materials is the *standard linear model* due to Kelvin (and therefore sometimes also referred to as the *Kelvin model*). This model (to be discussed below), along with a number of others, is frequently developed and analyzed in the context of the phenomena of *relaxation* and *creep* which along with *hysteresis* constitute the features of general *viscoelasticity.* If a viscoelastic rod is suddenly stressed (for example, with a unit step force) and held at this level of stress, the rod will continue to deform (as depicted in Figure 4 for the Kelvin model) in a phenomenon called *creep.*
On the other hand, if a rod is suddenly strained (for example, with a unit step displacement) and this strain is then maintained, the resulting stress in the rod will decrease over time as depicted for one model (the Kelvin model) in Figure 5. This phenomenon is called *relaxation.* While the specific shape of the creep and relaxation responses are highly dependent on the particular constitutive law chosen, the general qualitative features represented in Figures 4 and 5 are typical.

The Kelvin or standard linear model

The simplest constitutive model describing the phenomenon of hysteresis which incorporates empirically observed creep and relaxation in materials is the Kelvin or standard linear model given by

$$\sigma + \tau_\varepsilon \dot{\sigma} = E_r(\varepsilon + \tau_\sigma \dot{\varepsilon}) \tag{3}$$

where E_r is called the relaxed modulus of elasticity. This model can be depicted in the context of spring-dashpot analogies as shown in Figure 6.
The creep solution (or "creep function") $\Delta\varepsilon(t)$ of (3) is the solution corresponding to step stress $\Delta\sigma$ at time $t = s$. That is, $\Delta\varepsilon(t)$ is the solution of (3) corresponding to $\sigma_s(t) = H(t-s)\Delta\sigma$, $\dot{\sigma}_s(t) = \delta(t-s)\Delta\sigma$, where H is the Heaviside function and δ is the Dirac delta distribution, and is given by

$$\Delta\varepsilon(t) = \frac{1}{E_r}\left\{1 + (\frac{\tau_\varepsilon}{\tau_\sigma} - 1)e^{-(t-s)/\tau_\sigma}\right\} H(t-s)\Delta\sigma. \tag{4}$$

For obvious reasons, τ_σ is called the constant stress relaxation time.

The relaxation solution (or "relaxation function") $\Delta\sigma(t)$ of (3) is the solution corresponding to a step elongation $\varepsilon_s(t) = H(t-s)\Delta\varepsilon$, $\dot{\varepsilon}_s(t) = \delta(t-s)\Delta\varepsilon$ at time $t = s$ and is given by

$$\Delta\sigma(t) = \frac{1}{E_r}\left\{1 + (\frac{\tau_\sigma}{\tau_\varepsilon} - 1)e^{-(t-s)/\tau_\varepsilon}\right\} H(t-s)\Delta\varepsilon. \tag{5}$$

where τ_ε is called the constant strain relaxation time.

The Boltzmann superposition model

The most direct generalization of the Kelvin or standard linear model is due to Boltzmann and assumes superposition of a continuum of the relaxation solution responses to a continuum of elemental step increments in strain. The stress-strain law (in differential form) is given by

$$d\sigma(t) = E_0 d\varepsilon(t) + k(t-s)d\varepsilon(s) = E_0 d\varepsilon(t) + k(t-s)\dot{\varepsilon}(s)ds \tag{6}$$

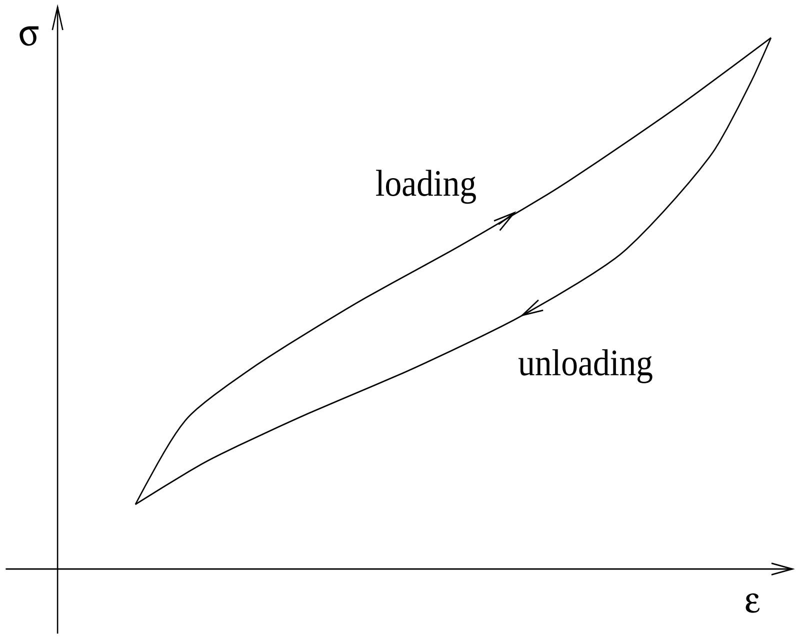

Figure 3: Loading/unloading of filled rubber

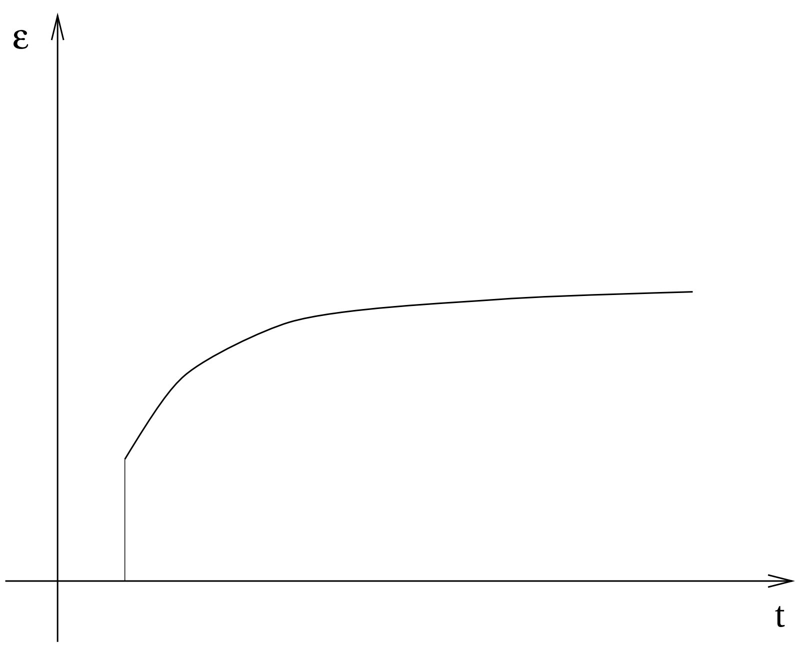

Figure 4: Standard linear model creep response

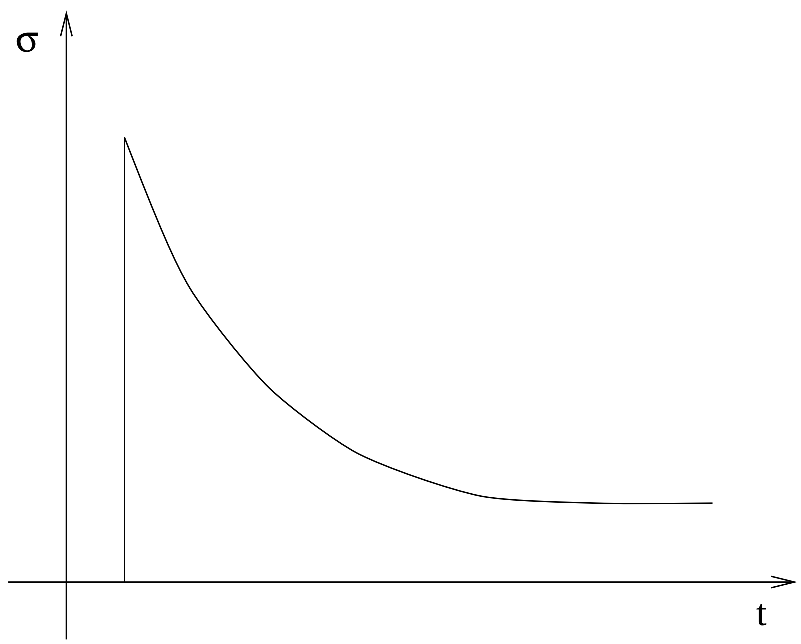

Figure 5: Standard linear model relaxation response

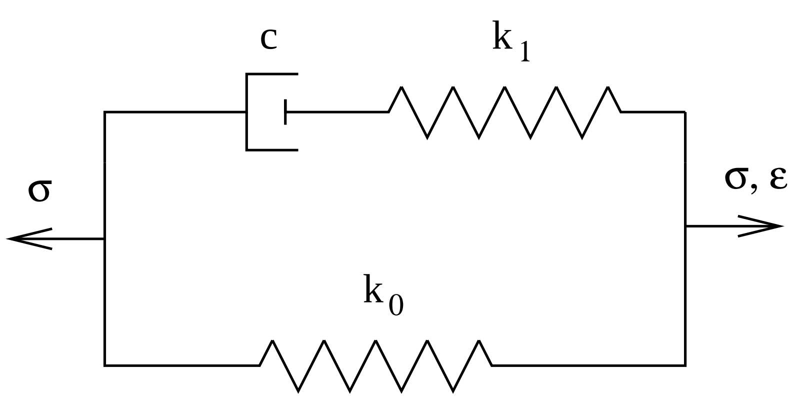

Figure 6: The Kelvin spring-dashpot model

for the incremental stress at time t due to a continuum of elemental elongations $d\varepsilon(s)$ for $s \leq t$. The constant of proportionality E_0 represents the instantaneous response while the proportionality constant $k = k(t-s)$ is dependent on the length of elapsed time $t-s$ and is called the generalized relaxation function. Invoking the linear superposition hypothesis of Boltzmann and summing over the past life of the structure, one obtains (assuming $\sigma(-\infty) = 0, \quad \varepsilon(-\infty) = 0$) the Boltzmann hysteretic constitutive law

$$\sigma(t) = E_0\varepsilon(t) + \int_{-\infty}^{t} k(t-s)\frac{d\varepsilon}{ds}(s)ds \tag{7}$$

where E_0 represents an instantaneous modulus of elasticity and k is the relaxation response kernel. The standard linear model is a special case of the Boltzmann model with the choice $k(t-s) = E_\varepsilon e^{-(t-s)/\tau_\varepsilon}$. More general special cases that are often encountered include a generalization of the single spring-dashpot paradigm of Figure 6 to one with multiple spring-dashpot systems in parallel which results in the relaxation response kernel

$$k(t-s) = \sum_{j=1}^{N} E_j e^{-(t-s)/\tau_j}. \tag{8}$$

We observe that (7), (8) are equivalent to the formulation

$$\sigma(t) = E_0\varepsilon(t) + \sum_{j=1}^{N} E_j\varepsilon_j(t) \tag{9}$$

$$\dot{\varepsilon}_j(t) + \frac{1}{\tau_j}\varepsilon_j(t) = \dot{\varepsilon}(t) \tag{10}$$

$$\varepsilon_j(-\infty) = 0 \qquad j = 1, 2, \ldots, N,$$

since the solution of (10) is given by

$$\varepsilon_j(t) = \int_{-\infty}^{t} e^{-(t-s)/\tau_j}\dot{\varepsilon}(s)ds. \tag{11}$$

The formulation (9)-(10) is sometimes referred to as an *internal variables* model since the variables ε_j can be thought of as *"internal strains"* that are driven by the instantaneous strain according to (10) and that contribute to the total stress via (9). Such models can be viewed as *phenomenological* in nature in that one does not attempt to identify any physical basis for the internal strains ε_j or they can be viewed as *physical* models wherein the strains ε_j are identified with specific mechanisms and/or molecules in a complex composite material with dynamics described by (10).

Generalization of the standard linear/Boltzmann models

The Boltzmann model (7) is one of many generalizations of the standard linear model (3). One such generalization due to Burger (later studied by Golla-Hughes-McTavish) is frequently encountered in the engineering literature wherein one attempts to introduce additional coordinates (essentially internal variables) into state space models to account for hysteresis. The general approach is in the frequency domain (as opposed to the time domain formulations discussed here) and employs complex modulus or loss factor data to fit rational polynomials representing the Laplace transform of hysteresis stress-strain relationships. Specifically, hysteresis is approximated by adjoining a state variable with frequency domain representation

$$h(s) = \frac{\alpha s^2 + \beta s}{s^2 + bs + c}. \tag{12}$$

This is equivalent to an internal dynamics of the form (the Burger model)

$$\ddot{\sigma} + b\dot{\sigma} + c\sigma = \alpha\ddot{\varepsilon} + \beta\dot{\varepsilon} \tag{13}$$

in the time domain, or $\hat{\sigma}\{s^2+bs+c\} = \{\alpha s^2+\beta s\}\hat{\varepsilon}$ in the frequency domain. This can be thought of as a direct generalization of the standard linear model (3) or can be reformulated as a generalization of internal variable/Boltzmann models such as (9)-(10) by defining generalized stress $\tilde{\sigma} = (\sigma, \dot{\sigma})$ and generalized strain $\tilde{\varepsilon} = (\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon})$ vectors and writing (13) as

$$\dot{\tilde{\sigma}} = A\tilde{\sigma} + B\tilde{\varepsilon} \tag{14}$$

so that

$$\tilde{\sigma}(t) = \int_{-\infty}^{t} e^{A(t-\tau)} B\tilde{\varepsilon}(\tau)d\tau \tag{15}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ c & b \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \alpha \end{pmatrix}.$$

Nonlinear hysteresis models

For many materials the linear models for hysteresis discussed above are inadequate to describe experimental data. In particular, many composite materials exhibit nonlinear behavior such as that seen in Figure 2 for SMA's and in Figure 3 for highly filled rubbers. Biological soft tissue is also known to exhibit nonlinear hysteretic behavior.

There are several standard generalizations found in the literature in attempts to treat nonlinear behavior. The most direct one is to allow nonlinear instantaneous strain as well as nonlinear strain rate dependence in (7). This yields models of the form

$$\sigma(t) = f_e(\varepsilon(t)) + \int_{-\infty}^{t} k(t-s) f_v(\dot{\varepsilon}(s))ds \tag{16}$$

and

$$\sigma(t) = f_e(\varepsilon(t)) + \int_{-\infty}^{t} k(t-s)\frac{d}{ds} f_v(\varepsilon(s))ds \tag{17}$$

where f_e represents the instantaneous elastic nonlinear response and f_v represents the viscoelastic nonlinear response function. This can be written in terms of internal strains similar to (9), (10) where (8) defines k and

$$\sigma(t) = f_e(\varepsilon(t)) + \sum_{j=1}^{N} E_j\varepsilon_j(t) \tag{18}$$

whenever we assume *linear* internal dynamics of the form

$$\dot{\varepsilon}_j(t) + \frac{1}{\tau_j}\varepsilon_j(t) = f_j(\dot{\varepsilon}(t)), \;\; j = 1, 2, \ldots, N, \tag{19}$$

or

$$\dot{\varepsilon}_j(t) + \frac{1}{\tau_j}\varepsilon_j(t) = \frac{d}{dt} f_j(\varepsilon(t)), \;\; j = 1, 2, \ldots, N, \tag{20}$$

with $f_j = f_v$ for each j. Models of the form (18), (20) have been successfully used to describe data for highly filled rubber such as that depicted in Figure 3 and data involving nested hysteresis loops as shown in Figure 7 and 8.
A more general nonlinear model results if one assumes different nonlinearities f_j for different j in (19) or (20). If one assumes nonlinear internal dynamics of the form

$$\dot{\varepsilon}_j(t) + g_j(\varepsilon(t)) = f_j(\dot{\varepsilon}(t)), \;\; j = 1, 2, \ldots, N, \tag{21}$$

in place of (19), then the model combining this with (18) *cannot* be written as a Boltzmann generalization in the form (16) and thus constitutes a nontrivial generalization of the Boltzmann superposition approach. A related but somewhat different generalization of the standard linear/Boltzmann approach has been used to model biological soft tissues where the hysteresis loop is independent of strain rate in a finite range of rate variation. In models due to Fung one replaces (7) by

$$\sigma(t) = \int_{-\infty}^{t} G(t - \tau)\dot{\sigma}^e(\tau)d\tau \tag{22}$$

where σ^e is the instantaneous elastic response to a step elongation and G is a *reduced relaxation* kernel defined by

$$G(t) = \{1 + c\,[\mathcal{E}(t/\tau_1) - \mathcal{E}(t/\tau_2)]\}\,/\{1 + c\ln(\tau_2/\tau_1)\} \tag{23}$$

where

$$\mathcal{E}(t) = \int_t^{\infty} \frac{e^{-\tau}}{\tau} d\tau.$$

The generalized kernel G defined in (23) is a decreasing function that has constant slope in the interval (τ_1, τ_2) and produces constant damping in the frequency range $\tau_1 \leq 1/\omega \leq \tau_2$. It effectively replaces the finite spectrum associated with relaxation kernels of the form given in (8) by a continuous spectrum over a finite range. The Fung model is obtained by considering a large number of Kelvin type units as shown in Figure 6 in series where in each unit the springs are nonlinear functions of the elongation while the dashpots are linear functions of the tension in the springs.

The standard linear/Boltzmann models and their generalizations discussed above all entail a physics-based approach in that they usually are derived and used in connection with some (perhaps unknown) internal physics hypothesized for the viscoelastic body. There is however a large literature on purely phenomenological approaches to hysteresis wherein one attempts to approximate directly the hysteresis loops depicted in Figures 2 and 3. The best known of these involve Preisach models and their generalizations due to Krasnosel'skii and Pokrovskii. The approach uses ideal relay operators such as that given in Figure 9 in parallel connections to produce hysteresis loops similar

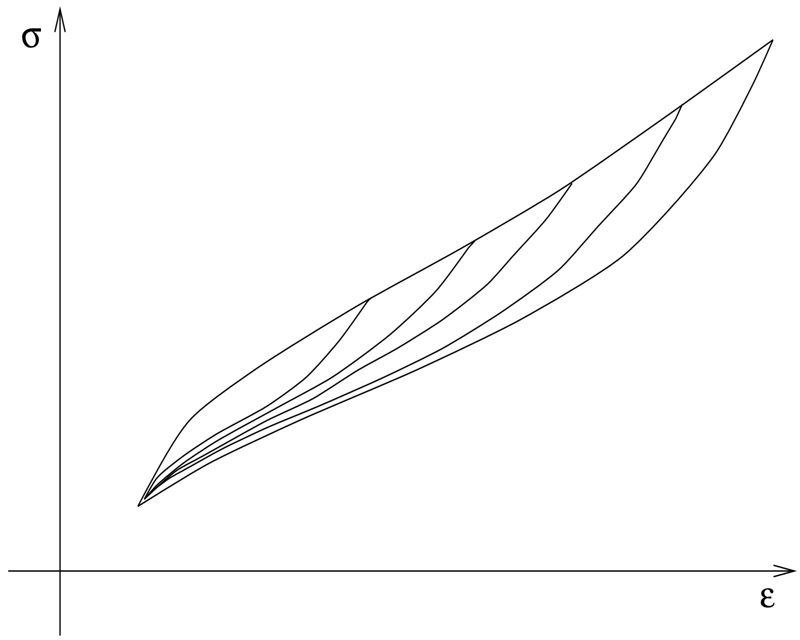

Figure 7: Complete nested hysteresis loops

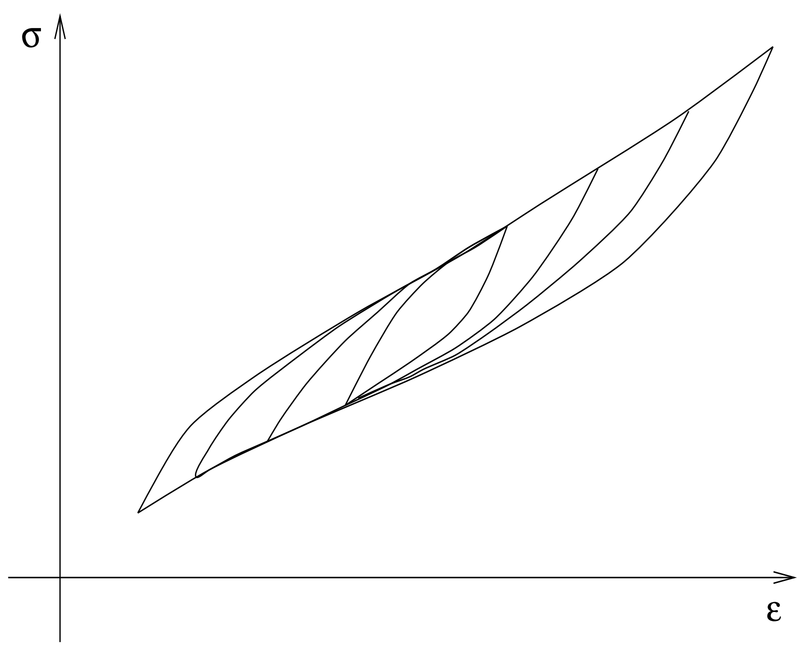

Figure 8: Incomplete nested hysteresis loops

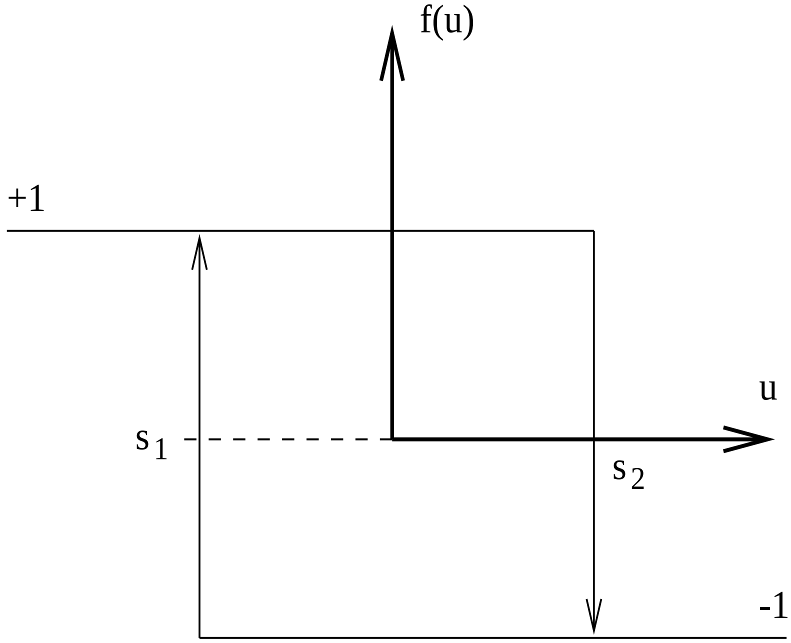

Figure 9: Ideal delayed relay operator

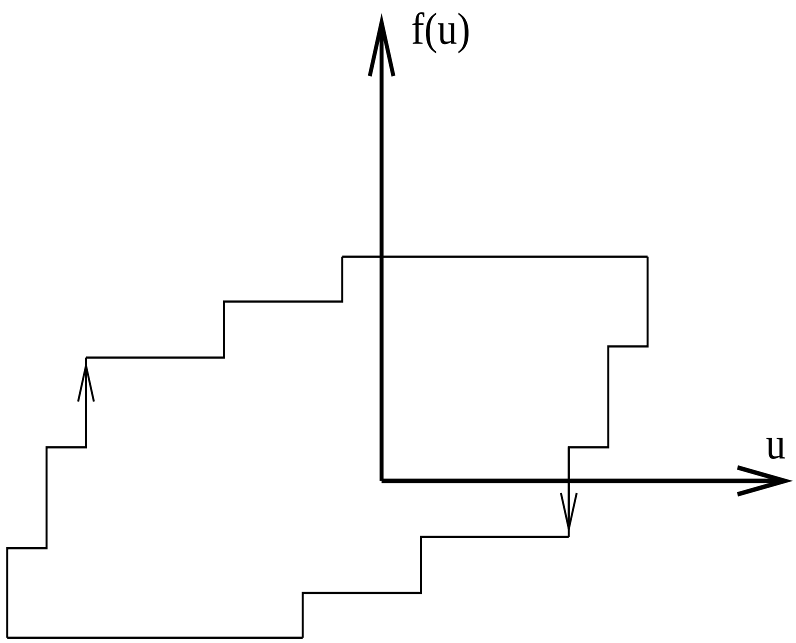

Figure 10: Parallel connection, multiple ideal relays

to that shown in Figure 10. These are then smoothed to produce a smoothed ideal relay as depicted in Figure 11.
Families of these relays (which depend on "switch points" s_1, s_2) can then be used to construct a hysteresis operator of the form

$$\sigma(t) = [P_\mu(\varepsilon, \xi)](t) = \int_{\bar{S}} [k_s(\varepsilon, \xi(s))](t) d\mu(s) \tag{24}$$

where μ is a signed Borel measure on the closed Preisach plane $\bar{S} = \{(s_1, s_2) \,|s_1 \leq s_2\}$ of switches, k_s is a generalized Preisach kernel and ξ contains initial state information. In the classical Preisach formulation, the kernel is defined in terms of linear combinations of relays as depicted in Figure 10 while smoothed relays as in Figure 11 are used in the generalized Krasnosel'skii-Pokrovskii kernels. In addition to the perhaps unsatisfying aspect of being purely phenomenological, this approach has serious difficulties if used as a means to model damping in viscoelastic structures. First, there are a large number of nonphysical parameters (defining the switches and relays) that must be identified. More importantly, the resulting operator in the stress-strain relationship (24) is *rate independent* and hence cannot be expected to yield adequate models to describe hysteretic damping in most viscoelastic structures which usually exhibit strong frequency (and hence rate) dependence.

References

[1] Brokate M and Sprekels J (1996) *Hysteresis and Phase Transitions*, Springer-Verlag, New York.

[2] Christensen RM (1982) *Theory of Viscoelasticity: An Introduction*, 2nd ed., Academic Press, New York.

[3] Drozdov A (1998) *Viscoelastic Structures*, Academic Press, San Diego, CA.

[4] Findley WN, Lai JS and Onaran K (1976) *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North-Holland, Amsterdam.

[5] Fung YC (1965) *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, N.J.

[6] Fung YC (1993) *Biomechanics: Mechanical Properties of Living Tissues*, 2nd ed., Springer-Verlag, New York.

[7] Krasnosel'skii MA and Pokrovskii AV (1989) *Systems with Hysteresis*, Springer-Verlag, Berlin.

[8] Lakes SL (1998) *Viscoelastic Solids*, CRC Press, Boca Raton.

[9] Mayergoyz, ID (1991) *Mathematical Models of Hysteresis*, Springer-Verlag, New York.

[10] Osiński Z (Ed) (1998) *Damping of Vibrations*, A.A. Balkema Publishers, Rotterdam.

[11] Renardy M, Hrusa WJ and Nohel JA (1987) *Mathematical Problems in Viscoelasticity*, Longman Scientific & Technical,Burnt Mill, Harlow, Essex, England; Wiley, New York.

[12] Visintin A (1994) *Differential Models of Hysteresis*, Springer-Verlag, New York.

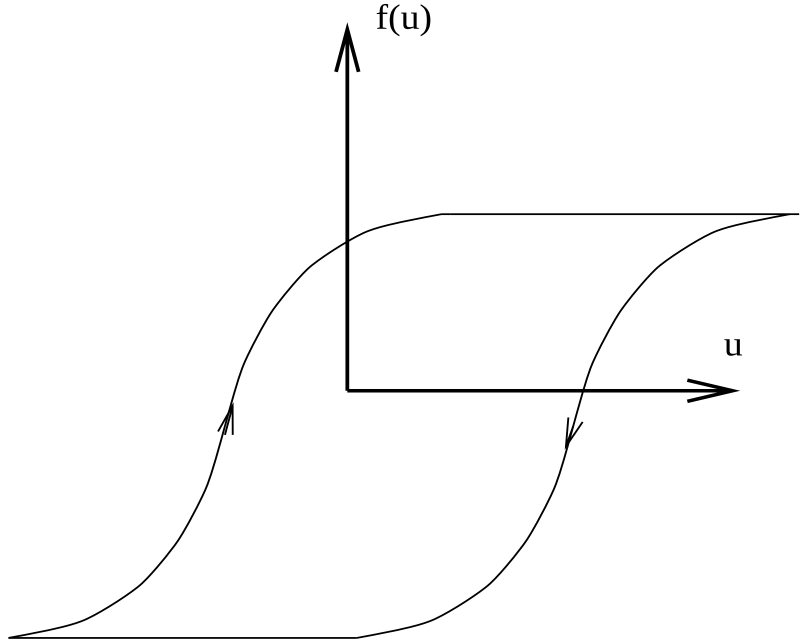

Figure 11: Smoothed ideal relay operator