SOLVING THE VIBRATING BEAM INVERSE PROBLEM

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FORMING AN INVERSE PROBLEM

Mathematical Model:

\[ \ddot{y} + C\dot{y} + Ky = 0 \]

\[ y(0) = y_0, \quad \dot{y}_0 = v_0 \]  \hspace{1cm} (1)

Experimental Data:

- displacement of the vibrating beam
- similar to \( y \) in the model
- denote collected data as \( y_i \) and \( y(t_i) \) in the model as \( y(t_i; C, K) \)

Cost Function: \[ J(C, K) = \frac{1}{2} \sum_{i=1}^{N}(y(t_i; C, K) - y_i)^2 \]

Optimization Problem: minimize \( J(C, K) \) \( \Rightarrow \) \[ \min \frac{1}{2} \sum_{i=1}^{N}(y(t_i; C, K) - y_i)^2 \]
SOLVING AN INVERSE PROBLEM

- Brute Force

  - Construct\( \dot{y}_t \approx \frac{y_t - y_{t-1}}{\Delta t} \)

  - Construct\( \ddot{y}_t \approx \frac{\dot{y}_t - \dot{y}_{t-1}}{\Delta t} \)

  - Minimize

\[
\sum_{i=1}^{N} (\ddot{y}_i - K \dot{y}_i - C y_i)^2
\]

What’s wrong with this approach?
SOLVING AN INVERSE PROBLEM

- Measure the same things
  - Analyze experimental data
  - Adjust either the data or the simulations

- Solve the differential equations from the model

- Choose an optimization method

- Make an educated initial guess on the parameters (C,K)
SOLVING THE MATHEMATICAL MODEL

• Mathematical Model:

\[ \ddot{y} + C\dot{y} + Ky = 0 \tag{2} \]
\[ y(0) = y_0, \quad \dot{y}_0 = v_0 \]

• Constructing a System of 1st ODE:

Let \( z_1 = y, \quad z_2 = \dot{y}, \) and \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \)

\[
\begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -C \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} \tag{3} \]

• Use ode23: \([t,z]=\text{ode23}(@\text{ode\_model},tspan,z_0,[ ],C,K)\)

\[ tspan = \quad z_0 = \]
INVERSE PROBLEMS AND INITIAL GUESSES

- Cost Function: \( \text{cost} = \sum_{i=1}^{N} (y(t_i, q) - y_i)^2 \)

- Minimize Cost Function: \( q = [C, K] \)
  
  \[
  [q, \text{cost}] = \text{fminsearch}(@\text{cost}_\text{beam}, q0, [], \text{time}, y)
  \]

  - How do we choose \( q0 \)?

  Analytic Result: \( y(t) = e^{-\frac{1}{2}Ct} A \sin(\omega t + \phi); \quad \omega = \frac{\sqrt{4K-C^2}}{2} \)

  ◦ \( C \) - related to damping coefficient

  ◦ \( K \) - related to spring constant

- What does a small \( C \) imply? What about a large \( C \)?

- What does a small \( K \) imply? What about a large \( K \)?
• Data does not start at max displacement

• Why?

• How do we fix it?

  Max Displacement: - find max value in the data and start data at the max value

  \[
  \text{maxval} = \max(\text{data}); \quad \% \text{find max value in the data} \\
  \text{index} = \text{find(}\text{data}==\text{maxval}); \quad \% \text{find where max value in the data is located} \\
  y_i = \text{data(index:end)}; \quad \% \text{reset the data so data starts at max value} \\
  \text{time} = \text{time(index:end)}; \quad \% \text{need to reset the time to correspond with data}
  \]

Oscillating Around Zero

\[
\text{y}_i = \text{y}_i - \text{mean}(\text{y}_i); \quad \% \text{(shift data down)}
\]