Electromagnetism as a Fundamental Theory
One theory for all spacial scales

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It is the only interaction that is equally significant in all spacial scales: gigantic cosmic scales when electromagnetic radiation including light propagates through the Universe, macroscopic scales covering our life on Earth and microscopic atomic scales.

The EM interactions are fundamental to the chemistry and consequently to the biology.

The EM interactions is a factual basis of the special relativity principle.
"The most fascinating subject at the time that I was a student was Maxwell’s theory. What made this theory appear revolutionary was the transition from forces at a distance to fields as fundamental variables.

If one views this phase of the development of theory critically, one is struck by the dualism which lies in the fact that the material point in Newton’s sense and the field as continuum are used as elementary concepts side by side.

The electron is a stranger in electrodynamics.

I feel that it is a delusion to think of the electrons and the fields as two physically different, independent entities. Since neither can exist without the other, there is only one reality to be described, which happens to have two different aspects; and the theory ought to recognize this from the start instead of doing things twice."
"Before Maxwell people thought of physical reality-in so far as it represented events in nature-as material points, whose changes consist only in motions which are subject to total differential equations. After Maxwell they thought of physical reality as represented by continuous fields, not mechanically explicable, which are subject to partial differential equations. This change in the conception of reality is the most profound and the most fruitful that physics has experienced since Newton; but it must also be granted that the complete realisation of the programme implied in this idea has not by any means been carried out yet...."
Challenges in constructing a theory of EM Interactions

- It has to be one theory for all spacial scales macroscopic and microscopic.
- Clean separation of elementary and statistical aspects of theory.
- An elementary charge cannot be a mathematical point as it is today in the classical and quantum mechanics, but it has to have a structure.
Maxwell’s (1831–1879) insights on constructing an EM theory

- "As the development of the ideas and methods of pure mathematics has rendered it possible, by forming a mathematical theory of dynamics, to bring to light many truths which could not have been discovered without mathematical training, so, if we are to form dynamical theories of other sciences, we must have our minds imbued with these dynamical truths as well as with mathematical methods.

- We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favorite hypothesis."
Maxwell’s insights on constructing an EM theory

- Maxwell compares his older and last versions of his theory in a letter to his friend Peter Guthrie Tait:
- “The former is built up to show that the phenomena [of electromagnetism] are such as can be explained by mechanism. The nature of the mechanism is to the true mechanism what an orrery is to the Solar System. The latter is built on Lagrange’s Dynamical Equations and is not wise about vortices.”
Maxwell’s insights on constructing an EM theory

- "The aim of Lagrange was to bring dynamics under the power of the calculus.... Certain quantities (expressing the reactions between the parts of the system called into play by its physical connexions) appear in the equations of motion of the component parts of the system, and Lagrange’s investigation, as seen from a mathematical point of view, is a method of eliminating these quantities from the final equations.

- In following the steps of this elimination the mind is exercised in calculation, and should therefore be kept free from the intrusion of dynamical ideas. Our aim, on the other hand, is to cultivate our dynamical ideas. We therefore avail ourselves of the labours of the mathematicians, and retranslate their results from the language of the calculus into the language of dynamics, so that our words may call up the mental image, not of some algebraical process, but of some property of moving bodies."
Some Mathematical Problems in a Neoclassical Theory of Electric Charges

One theory for all spacial scales

"'Give me matter and motion,' he [Descartes] cried,' and I will construct the universe.'"

E. Whittaker.
Our papers on the subject


Some references

Classical Electromagnetic Theory (CEM): Point Charge in EM Field

- Point charge $q$ of the mass $m$ in an external EM field

$$\frac{d}{dt} [mv(t)] = q \left[ \mathbf{E}(r(t), t) + \frac{1}{c} \mathbf{v}(t) \times \mathbf{B}(r(t), t) \right]$$

where $r$ and $\mathbf{v} = \frac{dr}{dt}$ are respectively charge position and velocity, $\mathbf{E}(t, r)$ and $\mathbf{B}(t, r)$ are the electric field and the magnetic induction.

- EM field generated by a moving point charge

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$- \left( \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} \right) = - \frac{4\pi}{c} q \delta (x - r(t)) \mathbf{v}(t),$$

$$\nabla \cdot \mathbf{E} = 4\pi q \delta (x - r(t)).$$
Closed Charge-"EM field" system

- A closed system "charge-EM field" has a problem: divergence of the EM field exactly at the position of the point charge. Even for the relativistic motion equation

\[
\frac{d}{dt} \left[ \gamma m \mathbf{v}(t) \right] = q \left[ \mathbf{E}(\mathbf{r}(t), t) + \frac{1}{c} \mathbf{v}(t) \times \mathbf{B}(\mathbf{r}(t), t) \right],
\]

where \( \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2(t)/c^2}} \) is the Lorentz factor, and the system has a Lagrangian the problem still persists.

- This problem is well known and discussed in detail, for instance, in:
An "indulgence" from Hendric Lorentz (1853-1928)

- An "indulgence" given by the great Hendric Lorentz (1853-1928) in 1927 to future generations of physicists (from Section 42 "Structure of the Electron" of his monograph) reads:

- "Therefore physicists are absolutely free to form any hypotheses on the properties and size of electrons that may best suit them. You can, for instance, choose the old electron (a small sphere with charge uniformly distributed over the surface) or Parson’s ring-shaped electron, endowed with rotation and therefore with a magnetic field; you can also make different hypotheses about the size of the electron. In this connection I may mention that A. H. Compton’s experiments on the scattering of $\gamma$ rays by electrons have led him to ascribe to the electron a size considerably greater than it was formerly supposed to have."
Electrodynamics of balanced charges (BEM)

- The BEM theory is a relativistic Lagrangian theory. It is one theory for all spatial scales: macroscopic and atomic.
- Balanced charge: a new concept for an elementary charge. It is described by a complex-valued wave function as in the Schrödinger wave mechanics.
- A b-charge does not interact with itself electromagnetically.
- Every b-charge has its own elementary EM potential and the corresponding EM field. It is naturally assigned a conserved elementary current via the Lagrangian.
- B-charges interact with each other only through their elementary EM potentials and fields.
- The field equations for the elementary EM fields are exactly the Maxwell equations with the elementary conserved currents.
- Force densities acting upon b-charges are described exactly by the Lorentz formula.
BEM theory relation to the Classical EM and Quantum theories?

- When charges are well separated and move with nonrelativistic velocities the BEM theory can be approximated by point charges governed by the Newton equations with the Lorentz forces.
- Radiative phenomena in the BEM theory are similar to those in the CEM theory in macroscopic scales.
- The Hydrogen Atom spectrum and some other phenomena at atomic scales are described by the BEM theory similarly to the Quantum Mechanics (QM).
However beautiful the strategy,
you should occasionally look at the results.

Sir Winston Churchill.
Electrodynamics of balanced charges (BEM)

- A system of $N$ elementary b-charges $(\psi^\ell, A^\ell\mu)$, $1 \leq \ell \leq N$.
- $\psi^\ell$ is the $\ell$-th b-charge wave function, $A^\ell\mu$ and $F^{\ell\mu\nu} = \partial^\mu A^{\ell\nu} - \partial^\nu A^{\ell\mu}$ are its elementary EM potential and field.
- The action upon the $\ell$-th charge by all other charges is described by the $\ell$-th exterior EM potential and field:

$$
A^\ell\mu \neq = \sum_{\ell' \neq \ell} A^{\ell'\mu}, \quad A^\ell\mu \neq = \left( \phi^\ell \neq, A^\ell \neq \right), \quad F^{\ell\mu\nu} \neq = \sum_{\ell' \neq \ell} F^{\ell'\mu\nu},
$$

$$
F^{\ell\mu\nu} = \partial^\mu A^{\ell\nu} - \partial^\nu A^{\ell\mu}.
$$

- The total EM potential $A^\mu$ and field $F^{\mu\nu}$:

$$
A^\mu = \sum_{1 \leq \ell \leq N} A^{\ell\mu}, \quad F^{\mu\nu} = \sum_{1 \leq \ell \leq N} F^{\ell\mu\nu}.
$$

- The total EM field is just the sum of elementary ones, it has no independent degrees of freedom which can carry EM energy.
Lagrangian for many interacting b-charges

- Lagrangian for the system of $N$ b-charges:

$$ \mathcal{L} \left( \left\{ \psi^\ell, \psi_{;\mu}^\ell \right\}, \left\{ \psi^{\ell*}, \psi_{;\mu}^{\ell*} \right\}, A^{\mu \ell} \right) = \sum_{\ell=1}^{N} \mathcal{L}^\ell \left( \psi^\ell, \psi_{;\mu}^\ell, \psi^{\ell*}, \psi_{;\mu}^{\ell*} \right) + \mathcal{L}_{\text{BEM}}, $$

$$ \mathcal{L}_{\text{BEM}} = \mathcal{L}_{\text{CEM}} - \mathcal{L}_e, \quad \mathcal{L}_{\text{CEM}} = -\frac{F^{\mu \nu} F_{\mu \nu}}{16 \pi}, \quad \mathcal{L}_e = -\sum_{1 \leq \ell \leq N} \frac{F^{\ell \mu \nu} F_{\mu \nu}^{\ell}}{16 \pi}, $$

where $\mathcal{L}^\ell$ is the Lagrangian of the $\ell$-th bare charge, and the covariant derivatives are defined by the following formulas

$$ \psi^\ell_{;\mu} = \tilde{\partial}^{\ell \mu} \psi^\ell, \quad \psi^{\ell*}_{;\mu} = \tilde{\partial}^{\ell \mu*} \psi^{\ell*}, $$

$$ \tilde{\partial}^{\ell \mu} = \partial^{\mu} + \frac{iq^\ell A^{\ell \mu}}{\chi c}, \quad \tilde{\partial}^{\ell \mu*} = \partial^{\mu} - \frac{iq^\ell A^{\ell \mu}}{\chi c}. $$

Covariant differentiation operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{*\mu}$ provide for the coupling between the charge and the EM field.
Lagrangian for many interacting b-charges

- EM part $\mathcal{L}_{\text{BEM}}$ can be obtained by the removal from the classical EM Lagrangian $\mathcal{L}_{\text{CEM}}$ all self-interaction contributions

$$\mathcal{L}_{\text{BEM}} = - \sum_{\{\ell, \ell'\}: \ell' \neq \ell} \frac{F_{\ell \mu \nu} F_{\ell' \mu \nu}}{16\pi} = - \sum_{1 \leq \ell \leq N} \frac{F_{\ell \mu \nu} F_{\ell \mu \nu}}{16\pi}.$$ 

- The "bare" charge Lagrangians $L^\ell$ are

$$L^\ell \left( \psi^\ell, \psi_{;\mu}^\ell, \psi^{\ell*}_-, \psi_{;\mu}^{\ell*}_- \right) = \frac{\chi^2}{2m^\ell} \left\{ \psi_{;\mu}^{\ell*} \psi_{;\mu}^{\ell} - \kappa^2 \psi^{\ell*} \psi^\ell - G^\ell \left( \psi^{\ell*} \psi^\ell \right) \right\},$$

- $G^\ell$ is a nonlinear self-interaction function;
- $m^\ell > 0$ is the charge mass; $q^\ell$ is the value of the charge;
- $\chi > 0$ is a constant similar to the Planck constant $\hbar = \frac{h}{2\pi}$ and

$$\kappa^\ell = \frac{\omega^\ell}{c} = \frac{m^\ell c}{\chi}, \quad \omega^\ell = \frac{m^\ell c^2}{\chi}.$$
Euler-Lagrange field equations

- From the gauge invariance via the Noether’s theorem we get elementary conserved currents,

\[ J^\ell_v = -i \frac{q^\ell}{\chi} \left( \frac{\partial L^\ell}{\partial \psi^\ell_v} \psi^\ell - \frac{\partial L^\ell}{\partial \psi^{*\ell}_v} \psi^{*\ell} \right) = -c \frac{\partial L^\ell}{\partial A^\ell_{\neq v}}, \]

with the conservation law

\[ \partial_v J^\ell_v = 0, \ \partial_t \rho^\ell + \nabla \cdot J^\ell = 0, \ J^\ell_v = \left( \rho^\ell c, J^\ell \right). \]

- Elementary wave equations

\[ \left[ \tilde{\partial}_\mu \tilde{\partial}^{\ell \mu} + \kappa^{\ell 2} + G^{\ell \ell'} \left( |\psi^\ell|^2 \right) \right] \psi^\ell = 0, \ \tilde{\partial}^{\ell \mu} = \partial^\mu + \frac{iq^\ell A^{\ell \mu}_{\neq}}{\chi c}, \]

and similar equations for the conjugate \( \psi^{*\ell} \).
Euler-Lagrange field equations

- Elementary Maxwell equations

\[ \partial_\mu F^{\ell\mu\nu} = \frac{4\pi}{c} J^{\ell\nu}, \]

- or in the familiar vector form

\[ \nabla \cdot \mathbf{E}^\ell = 4\pi q^\ell, \quad \nabla \cdot \mathbf{B}^\ell = 0, \]

\[ \nabla \times \mathbf{E}^\ell + \frac{1}{c} \partial_t \mathbf{B}^\ell = 0, \quad \nabla \times \mathbf{B}^\ell - \frac{1}{c} \partial_t \mathbf{E}^\ell = \frac{4\pi}{c} \mathbf{J}^\ell. \]
Euler-Lagrange field equations

- Elementary currents:

\[ J_{\nu}^l = - \frac{q^l \chi |\psi^l|^2}{m^l} \left( \text{Im} \frac{\partial^\nu \psi^l}{\psi^l} + \frac{q^l A^\ell_{\neq}}{\chi c} \right), \]

- or in the vector form

\[ \rho^l = - \frac{q^l |\psi^l|^2}{m^l c^2} \left( \chi \text{Im} \frac{\partial_t \psi^l}{\psi^l} + q^l \varphi^l_{\neq} \right), \]

\[ J^l = \frac{q^l |\psi^l|^2}{m^l} \left( \chi \text{Im} \frac{\nabla \psi^l}{\psi^l} - \frac{q^l A^\ell_{\neq}}{c} \right). \]
Single relativistic charge and the nonlinearity

- **Lagrangian**

\[ L_0 = \frac{\chi^2}{2m} \left\{ \left| \tilde{\nabla} t \psi \right|^2 - \left| \tilde{\nabla} \psi \right|^2 - \kappa_0^2 |\psi|^2 - G(\psi^*\psi) \right\}. \]

- Without EM self-interaction, \( L_0 \) does not depend on the potentials \( \varphi, A \). Though we can still find the potentials based on the elementary Maxwell equations, they have no role to play and carry no energy.

- **Rest state of the b-charge**

\[
\psi(t, x) = e^{-i\omega_0 t} \hat{\psi}(x), \quad \omega_0 = \frac{mc^2}{\chi} = c\kappa_0,
\]

\[
\varphi(t, x) = \hat{\varphi}(x), \quad A(t, x) = 0,
\]

where \( \hat{\psi}(|x|) \) and \( \hat{\varphi} = \hat{\varphi}(|x|) \) are real-valued radial functions, and we refer to them, respectively, as form factor and form factor potential.

- **Rest charge equations**:

\[
-\nabla^2 \hat{\psi} + G'(\left|\hat{\psi}\right|^2) \hat{\psi} = 0, \quad -\nabla^2 \hat{\varphi} = 4\pi |\hat{\psi}|^2.
\]
Charge equilibrium equation

Charge equilibrium equation for the resting charge:

\[-\nabla^2 \psi + G' \left( |\psi|^2 \right) \psi = 0.\]

It signifies a complete balance of the two forces: (i) internal elastic deformation force \(-\Delta \psi\); (ii) internal nonlinear self-interaction \(G' \left( |\psi|^2 \right) \psi\).

We pick the form factor \(\psi\) considering it as the model parameter and then the nonlinear self interaction function \(G\) is determined based on the charge equilibrium equation.

We integrate the size of the b-charge into the model via size parameter \(a > 0\):

\[G'_a (s) = a^{-2} G'_1 \left( a^3 s \right), \text{ where } G' (s) = \partial_s G (s).\]
Self-interaction nonlinearity

- There is a certain degree of freedom in choosing the form factor and the resulting nonlinearity.
- The proposed below choice is justified by its unique property: the energies and the frequencies of the time-harmonic states of the Hydrogen atom satisfy exactly the Einstein-Planck energy-frequency relation: \( E = \hbar \omega \) (\( E = \chi \omega \)).
- The form factor is Gaussian and defined by

\[
\hat{\psi}(r) = C_g e^{-r^2/2}, \quad C_g = \frac{1}{\pi^{3/4}},
\]

implying

\[
\nabla^2 \hat{\psi}(r) = r^2 - 3 = -\ln \left( \hat{\psi}^2(r) / C_g^2 \right) - 3.
\]

\[
\varphi(x) = q \int_{\mathbb{R}^3} \frac{|\hat{\psi}(y)|^2}{|y - x|} \, dy
\]
Consequently, the nonlinearity reads

\[ G'(s) = - \ln \left( \frac{s}{C_g^2} \right) - 3, \]

implying

\[ G(s) = -s \ln s + s \left( \ln \frac{1}{\pi^{3/2}} - 2 \right). \]

and we call it the **logarithmic nonlinearity**.

The nonlinearity explicit dependence on the size parameter \( a > 0 \) is

\[ G'_a(s) = -a^{-2} \ln \left( \frac{a^3 s}{C_g^2} \right) - 3. \]
Nonrelativistic BEM Lagrangian and its Field Equations

- Non-relativistic Lagrangian

\[ \hat{\mathcal{L}}_0 \left( \{ \psi^\ell \}_{\ell=1}^N, \{ \varphi^\ell \}_{\ell=1}^N \right) = \frac{1}{8\pi} \left| \mathbf{A}^{\ell} \right|^2 + \sum_{\ell} \hat{\mathcal{L}}^{\ell} (\psi^\ell, \psi^{\ell*}, \varphi), \]

\[ \hat{\mathcal{L}}^{\ell} = \frac{\mathbf{k}_1}{2} \left[ \psi^{\ell*} \partial_t \psi^\ell - \psi^\ell \partial_t \psi^{\ell*} \right] - \frac{\mathbf{k}_2}{2m^\ell} \left\{ \left| \mathbf{\nabla}_{\text{ex}} \psi^\ell \right|^2 + G^\ell (\psi^{\ell*} \psi^\ell) \right\} - q^\ell \left( \varphi_{\neq \ell} + \varphi_{\text{ex}} \right) \psi^\ell \psi^{\ell*} - \frac{1}{8\pi} \left| \mathbf{V}^\ell \right|^2, \]

where \( \mathbf{A}_{\text{ex}} (t, \mathbf{x}) \) and \( \varphi_{\text{ex}} (t, \mathbf{x}) \) are potentials of external EM fields.

- The Euler-Lagrange field equations

\[ i\mathbf{k}_1 \partial_t \psi^\ell = -\frac{\mathbf{k}_2}{2m^\ell} \left( \mathbf{\nabla}_{\text{ex}}^\ell \right)^2 \psi^\ell + q^\ell \left( \varphi_{\neq \ell} + \varphi_{\text{ex}} \right) \psi^\ell + \frac{\mathbf{k}_2}{2m^\ell} \left[ G^\ell_{a} \right]' \left( \left| \psi^\ell \right|^2 \right) \]

\[ \nabla^2 \varphi^\ell = -4\pi q^\ell \left| \psi^\ell \right|^2, \quad \ell = 1, \ldots, N. \]

Consequently, potentials \( \varphi^\ell \) can be represented as
Exact wave-corpuscle solution for non-relativistic set up

- Let us assume a purely electric external EM field: \( A_{\text{ex}} = 0 \), \( E_{\text{ex}}(t, x) = -\nabla \varphi_{\text{ex}}(t, x) \).
- We define the wave-corpuscle (soliton) \( \psi, \varphi \) by

\[
\psi(t, x) = e^{iS/\chi \hat{\psi}}, \quad S = m v(t) \cdot (x - r) + s_p(t),
\]

\[
\hat{\psi} = \hat{\psi}(|x - r|), \quad \varphi = \hat{\varphi}(|x - r|), \quad r = r(t).
\]

where \( \hat{\psi} \) is the Gaussian form factor with the corresponding potential \( \hat{\varphi} \);
- \( r(t) \) is determined from the point charge equation

\[
m \frac{d^2 r(t)}{dt^2} = q E_{\text{ex}}(t, r),
\]

and \( v(t), s_p(t) \) are determined by formulas

\[
v(t) = \frac{dr}{dt}, \quad s_p(t) = \int_0^t \left( \frac{mv^2}{2} - q \varphi_{\text{ex}}(t, r(t)) \right) dt'.
\]
Exact wave-corpuscle (soliton) solution

Theorem

Suppose that \( \varphi_{\text{ex}}(t, x) \) is a continuous function which is linear with respect to \( x \). Then the wave-corpuscle wave function and its potential

\[
\psi(t, x) = e^{iS/\chi} \hat{\psi}, \quad S = m\mathbf{v}(t) \cdot (\mathbf{x} - \mathbf{r}) + s_p(t),
\]

\[
\hat{\psi} = \hat{\varphi}(|\mathbf{x} - \mathbf{r}|), \quad \varphi = \hat{\varphi}(|\mathbf{x} - \mathbf{r}|), \quad \mathbf{r} = \mathbf{r}(t).
\]

is the exact solution to the Euler-Lagrange field equation provided \( \mathbf{r}(t) \) satisfies the point charge equation

\[
m \frac{d^2 \mathbf{r}(t)}{dt^2} = q \mathbf{E}_{\text{ex}}(t, \mathbf{r}),
\]

and \( \mathbf{v}(t), s_p(t) \) are determined by formulas

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt}, \quad s_p = \int_0^t \left( \frac{m\mathbf{v}^2}{2} - q\varphi_{\text{ex}}(t, \mathbf{r}(t)) \right) \, dt'.
\]
Multiharmonic solutions for a system of many charges

- Field equations without external fields

\[ i\chi \partial_t \psi^\ell + \frac{\chi^2}{2m^\ell} \nabla^2 \psi^\ell - q^\ell \varphi_{\neq \ell} \psi^\ell = \frac{\chi^2}{2m^\ell} G' \left( |\psi^\ell|^2 \right) \psi^\ell, \quad \ell = 1, \ldots, N, \]

\[ \varphi_{\neq \ell} = \sum_{\ell' \neq \ell} \varphi^{\ell'}, \quad \frac{1}{4\pi} \nabla^2 \varphi^\ell = -q^\ell |\psi^\ell|^2. \]

- The total conserved energy

\[ \mathcal{E} = \sum_{\ell} \mathcal{E}_\ell, \]

\[ \mathcal{E}_\ell = \frac{1}{2} \int q^\ell |\psi^\ell|^2 \varphi_{\neq \ell} \, dx + \int \frac{\chi^2}{2m^\ell} \left\{ |\nabla \psi^\ell|^2 + G^\ell \left( |\psi^\ell|^2 \right) \right\} \, dx. \]
Multiharmonic solutions for a system of many charges

- \( \ell \)-th charge energy in the system’s field

\[
E_{0\ell} = \int q^\ell |\psi^\ell|^2 \varphi \neq \ell \, dx + \int \frac{\chi^2}{2m^\ell} \left\{ |\nabla \psi^\ell|^2 + G^\ell \left( |\psi^\ell|^2 \right) \right\} dx.
\]

- For a single charge \( E_{0\ell} = E_\ell = E \). In the general case \( N \geq 2 \) the total energy \( E \) does not equal the sum of \( E_{0\ell} \), and the difference between the sum of \( E_{0\ell} \) and the total energy \( E \) coincides with the total energy of EM fields.

- We assume

\[
\psi^\ell \in \mathcal{E}, \quad \text{where } \mathcal{E} = \left\{ \psi \in H^1(\mathbb{R}^3) : \|\psi\|^2 = \int |\psi|^2 \, dx = 1 \right\}.
\]
Multiharmonic solutions for a system of many charges

Let us consider now multiharmonic solutions

\[ \psi^\ell (t, x) = e^{-i\omega t} \psi^\ell (x), \quad \phi^\ell (t, x) = \phi^\ell (x). \]

\[ \frac{1}{4\pi} \nabla^2 \phi^\ell = -q^\ell \left| \psi^\ell \right|^2, \quad \text{or} \quad \phi^\ell (t, x) = q^\ell \int_{\mathbb{R}^3} \frac{\left| \psi^\ell \right|^2 (t, y)}{|y - x|} dy. \]

Then \( \psi^\ell (x) \) satisfy the following nonlinear eigenvalue problem,

\[ \chi \omega \psi^\ell + \frac{\chi^2}{2m} \nabla^2 \psi^\ell - q^\ell \psi^\ell - \frac{\chi^2}{2m} G' (|\psi^\ell|^2) \psi^\ell = 0. \]
Multiharmonic solutions for a system of many charges

Theorem

Let \( G_\ell(s), \ell = 1, \ldots, N \) be the logarithmic functions with \( a = a^\ell \). Suppose \( \{\psi_\sigma^\ell\}_{\ell=1}^N \in \Xi^N, \sigma \in \Sigma \) is a set of solutions to the nonlinear eigenvalue problem with the corresponding frequencies \( \{\omega^\sigma_\ell\}_{\ell=1}^N \) and finite energies \( \{E^\sigma_0\ell\}_{\ell=1}^N \). Then

\[
E_0\ell = \chi \omega_\ell + \frac{\chi^2}{2 (a^\ell)^2 m^\ell}.
\]

and any two solutions \( \psi_\ell^\sigma = \psi_\ell \) and \( \psi_\ell^{\sigma_1} = \psi'_\ell \) with \( \sigma, \sigma_1 \in \Sigma \) satisfy the Planck-Einstein relation

\[
\chi (\omega_\ell^\sigma - \omega_\ell^{\sigma_1}) = E^\sigma_0\ell - E^{\sigma_1}_0\ell, \quad \ell = 1, \ldots, N.
\]
Newtonian Mechanics of point charges as an approximation

- We introduce the $\ell$-th charge position $r^\ell(t)$ and velocity $v^\ell(t)$

\[
    r^\ell(t) = r^\ell_a(t) = \int_{\mathbb{R}^3} x \left| \psi^\ell_a(t, x) \right|^2 \, dx, \quad v^\ell(t) = \frac{1}{q^\ell} \int_{\mathbb{R}^3} J^\ell(t, x) \, dx.
\]

- We find then that the positions and velocities are related exactly as in the point charge mechanics:

\[
    \frac{dr^\ell(t)}{dt} = \int_{\mathbb{R}^3} x \partial_t \left| \psi^\ell \right|^2 \, dx = - \frac{1}{q^\ell} \int_{\mathbb{R}^3} x \nabla \cdot J^\ell \, dx = \frac{1}{q^\ell} \int_{\mathbb{R}^3} J^\ell \, dx = v^\ell(t).
\]

- We obtain the kinematic representation for the total momentum which is exactly the same as for the point charges mechanics

\[
    P^\ell(t) = \frac{m^\ell}{q^\ell} \int_{\mathbb{R}^3} J^\ell(t, x) \, dx = m^\ell v^\ell(t),
\]
Newtonian Mechanics of point charges as an approximation

- We obtain the following system of equations of motion for \( N \) charges:

\[
m^\ell \frac{d^2 \mathbf{r}^\ell(t)}{d^2 t} = q^\ell \int_{\mathbb{R}^3} \left[ \left( \sum_{\ell' \neq \ell} \mathbf{E}^{\ell'} + \mathbf{E}_{\text{ex}} \right) \left| \psi^\ell \right|^2 + \frac{1}{c} \mathbf{v}^\ell \times \mathbf{B}_{\text{ex}} \right] \, d\mathbf{x}, \quad \ell = 1, \ldots, N
\]

The derivation is similar to that of the Ehrenfest Theorem in quantum mechanics,

- Suppose that for every \( \ell \)-th charge the densities \( \left| \psi^\ell \right|^2 \) and \( \mathbf{J}^\ell \) are localized in \( a \)-vicinity of the position \( \mathbf{r}^\ell(t) \), and that \( \left| \mathbf{r}^\ell(t) - \mathbf{r}^{\ell'}(t) \right| \geq \gamma > 0 \) with \( \gamma \) independent on \( a \) on time interval \([0, T]\). Then if \( a \to 0 \) we get

\[
\left| \psi^\ell \right|^2 (t, \mathbf{x}) \to \delta \left( \mathbf{x} - \mathbf{r}^\ell(t) \right), \quad \mathbf{v}^\ell (t, \mathbf{x}) = \mathbf{J}^\ell / q^\ell \to \mathbf{v}^\ell (t) \delta \left( \mathbf{x} - \mathbf{r}^\ell(t) \right)
\]
Newtonian Mechanics of point charges as an approximation

- We infer then

\[ \varphi^\ell(t, \mathbf{x}) \rightarrow \varphi_0^\ell(t, \mathbf{x}) = \frac{q^\ell}{|\mathbf{x} - \mathbf{r}^\ell|}, \quad \nabla \mathbf{r} \varphi^\ell(t, \mathbf{x}) \rightarrow \frac{q^\ell (\mathbf{x} - \mathbf{r}^\ell)}{|\mathbf{x} - \mathbf{r}^\ell|^3} \]  

as \( a \rightarrow 0 \).

- Passing to the limit \( a \rightarrow 0 \) we get Newton’s equations of motion for point charges

\[ m^\ell \frac{d^2 \mathbf{r}^\ell}{dt^2} = \mathbf{f}^\ell, \]

where \( \mathbf{f}^\ell \) are the Lorentz forces

\[ \mathbf{f}^\ell = \sum_{\ell' \neq \ell} q^\ell \mathbf{E}_0^{\ell'} + q^\ell \mathbf{E}_{\text{ex}}(\mathbf{r}^\ell) + \frac{1}{c} \mathbf{v}^\ell \times \mathbf{B}_{\text{ex}}(\mathbf{r}^\ell), \quad \ell = 1, \ldots, N, \]
Wave-corpuscle concept for an accelerating charge

- We define the wave-corpuscle (soliton) $\psi, \varphi$ by

$$
\psi(t, x) = e^{iS/\chi \hat{\psi}}, \quad S = m\mathbf{v}(t) \cdot (x - r) + s_p(t), \\
\hat{\psi} = \hat{\psi}(|x - r|), \quad \varphi = \hat{\varphi}(|x - r|), \quad r = r(t).
$$

where $\hat{\psi}$ is the Gaussian form factor with the corresponding potential $\hat{\varphi}$;
- If the potential $\varphi_{\text{ex}}(t, x)$ is linear in $x$ then for any given trajectory $r(t)$

$$
\varphi_{\text{ex}}(t, x) = \varphi_{0,\text{ex}}(t) + \varphi'_{0,\text{ex}}(t) \cdot (x - r(t)),
$$

where

$$
\varphi'_{0,\text{ex}}(t) = \nabla_x \varphi_{\text{ex}}(r(t), t), \quad \varphi_{0,\text{ex}}(t) = \varphi_{\text{ex}}(t, r(t)).
$$
Wave-corpuscle concept for an accelerating charge

Observe then

\[ \partial_t \psi = \exp \left( \frac{i S}{\chi} \right) \left\{ \left[ \frac{im}{\chi} (\partial_t v \cdot (x - r) - v \cdot \partial_t r) + \frac{is_p}{\chi} \right] \hat{\psi} - \partial_t r \cdot \nabla \hat{\psi} \right\}, \]

\[ \nabla^2 \psi = \exp \left( \frac{i S}{\chi} \right) \left[ \left( \frac{imv}{\chi} \right)^2 \hat{\psi} + 2 \frac{im}{\chi} v \cdot \nabla \hat{\psi} + \nabla^2 \hat{\psi} \right]. \]

Substituting above expressions into the field equations we obtain equations for functions \( v, r, s_p \):

\[ \left[ -m \partial_t v \cdot (x - r) - v \cdot \partial_t r - s_p \right] \hat{\psi} - i\chi \partial_t r \cdot \nabla \hat{\psi} \]

\[ - \frac{m}{2} v^2 \hat{\psi} + i\chi v \cdot \nabla \hat{\psi} + \frac{\chi^2}{2m} \nabla^2 \hat{\psi} - q \varphi_{\text{ex}} \hat{\psi} - \frac{\chi^2}{2m} G' \hat{\psi} = 0. \]
Wave-corpuscle concept for an accelerating charge

- Using the charge equilibrium equation we eliminate $G$ and $\nabla^2$:

$$- \left\{ m [\partial_t \mathbf{v} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \partial_t \mathbf{r}] + \frac{m \mathbf{v}^2}{2} + \ddot{s}_p + q \varphi_{\text{ex}} \right\} \hat{\psi}$$

$$-i \chi (\partial_t \mathbf{r} - \mathbf{v}) \nabla \hat{\psi} = 0.$$

- We equate then to zero the coefficients before $\nabla \hat{\psi}$ and $\hat{\psi}$ above:

$$\mathbf{v} = \partial_t \mathbf{r}, \quad m [\partial_t \mathbf{v} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \partial_t \mathbf{r}] + \frac{m \mathbf{v}^2}{2} + \ddot{s}_p + q \varphi_{\text{ex}} = 0,$$

and recast the second equation as

$$m [\partial_t \mathbf{v} \cdot (\mathbf{x} - \mathbf{r}) - \mathbf{v} \cdot \partial_t \mathbf{r}] + \ddot{s}_p + \frac{m \mathbf{v}^2}{2} + q \left[ \varphi_{0,\text{ex}} + \varphi'_{0,\text{ex}} \cdot (\mathbf{x} - \mathbf{r}) \right] = 0.$$

- We equate to zero the coefficient before $(\mathbf{x} - \mathbf{r})$ and the remaining coefficient and obtain

$$m \partial_t \mathbf{v} = -q \varphi'_{0,\text{ex}}(t), \quad \ddot{s}_p - m \mathbf{v} \cdot \partial_t \mathbf{r} + \frac{m \mathbf{v}^2}{2} + q \varphi_{0,\text{ex}}(t) = 0.$$
de Broglie factor for accelerating charge

- For purely electric external field we define

\[ k(t) = \int_{\mathbb{R}^3} \text{Im} \frac{\nabla \psi(t, x)}{\dot{\psi}(t, x)} |\dot{\psi}(t, x)|^2 \, dx. \]

- The Fourier transform \( \mathcal{F} \) of the wave-corpuscle \( \psi(t, x) \):

\[
[\mathcal{F} \psi](t, k) = \exp \left\{ i r(t) k - \frac{i s_p(t)}{\chi} \right\} (\mathcal{F} [\dot{\psi}]) \left( k - \frac{m v(t)}{\chi} \right),
\]

implying \( k(t) = \frac{m v(t)}{\chi}, \quad v(t) = \frac{dr}{dt}(t). \)

- The charge velocity \( v(t) \) equals the group velocity \( \nabla_k \omega(k(t)) \):

\[
\omega(k) = \frac{\chi k^2}{2m}, \quad \nabla_k \omega(k) = \frac{\chi k}{m}, \quad \text{implying} \quad \nabla_k \omega(k(t)) = v(t).
\]
Planck-Einstein energy-frequency relation

- Let us consider now multiharmonic solutions

\[ \psi^\ell (t, x) = e^{-i\omega^\ell t} \psi^\ell (x), \quad \varphi^\ell (t, x) = \varphi^\ell (x), \]

\[ \frac{1}{4\pi} \nabla^2 \varphi^\ell = -q^\ell \left| \psi^\ell \right|^2, \text{ or } \varphi^\ell (t, x) = q^\ell \int_{\mathbb{R}^3} \frac{\left| \psi^\ell \right|^2 (t, y)}{|y - x|} dy. \]

- Then \( \{ \psi^\ell (x) \}_{\ell=1}^N \) satisfy the nonlinear eigenvalue problem

\[ \chi \omega^\ell \psi^\ell + \frac{\chi^2}{2m^\ell} \nabla^2 \psi^\ell - q^\ell \varphi^\ell \neq \ell \psi^\ell - \frac{\chi^2}{2m^\ell} G^\ell \left( \left| \psi^\ell \right|^2 \right) \psi^\ell = 0. \]

- This problem may have many solutions; every solution \( \{ \psi^\ell \}_{\ell=1}^N \) determines a set of frequencies \( \{ \omega^\ell \}_{\ell=1}^N \) and energies \( \{ E_{0\ell} \}_{\ell=1}^N \):

\[ E_{0\ell} = \int q^\ell \left| \psi^\ell \right|^2 \varphi^\ell \neq \ell dx + \int \frac{\chi^2}{2m^\ell} \left\{ \left| \nabla \psi^\ell \right|^2 + G^\ell \left( \left| \psi^\ell \right|^2 \right) \right\} dx. \]
Planck-Einstein energy-frequency relation

- We seek nonlinearities $G^\ell$ such that any two solutions $\{\psi_\ell\}_{\ell=1}^N$, $\{\psi'_\ell\}_{\ell=1}^N$ satisfy Planck-Einstein frequency-energy relation:

$$\chi (\omega_\ell - \omega'_\ell) = E_{0\ell} - E'_{0\ell}, \quad \ell = 1, \ldots, N.$$ 

- Based on the nonlinear eigenvalue equations and the charge normalization condition $\|\psi_\ell\| = 1$ we obtain an integral representation for the frequencies $\omega_\ell$:

$$\chi \omega_\ell = \int \left[ \frac{\chi^2}{2m^\ell} |\nabla \psi_\ell|^2 \, dy + q^\ell \varphi_{\neq \ell} |\psi_\ell|^2 + \frac{\chi^2}{2m^\ell} G'_{\ell} (|\psi_\ell|^2) |\psi_\ell|^2 \right] \, dy.$$ 

- Comparing the above with the integral for $E_{0\ell}$ we see that

$$\chi \omega_\ell - E_{0\ell} = \frac{\chi^2}{2m^\ell} \int \left[ G'_{\ell} (|\psi_\ell|^2) |\psi_\ell|^2 - G_{\ell} (|\psi_\ell|^2) \right] \, dy.$$
Planck-Einstein energy-frequency relation

Consequently, we obtain for any two solutions \( \{ \psi_\ell \}_{\ell=1}^N, \{ \psi'_\ell \}_{\ell=1}^N \):

\[
\chi (\omega_\ell - \omega'_\ell) - (E_{0\ell} - E'_{0\ell}) = \\
= \frac{\chi^2}{2m^\ell} \int G_\ell \left( |\psi'_\ell|^2 \right) - G'_\ell \left( |\psi'_\ell|^2 \right) |\psi'_\ell|^2 \, d\mathbf{y} - \\
- \frac{\chi^2}{2m^\ell} \int G_\ell \left( |\psi_\ell|^2 \right) - G'_\ell \left( |\psi_\ell|^2 \right) |\psi_\ell|^2 \, d\mathbf{y}.
\]

Observe that for the above to hold it is sufficient that for every \( |\psi_\ell|^2 \) with \( \| \psi_\ell \| = 1 \), there exists constants \( C_\ell \) such that

\[
\int \left[ G_\ell \left( |\psi_\ell|^2 \right) - G'_\ell \left( |\psi_\ell|^2 \right) |\psi_\ell|^2 \right] \, d\mathbf{y} = C_\ell.
\]
Planck-Einstein energy-frequency relation

- The previous integral identities in turn will hold if the following differential equations hold for some constants $K_G$: 

$$s \frac{d}{ds} G_\ell (s) - G_\ell (s) = K_G s.$$

- The above together with the normalization $\| \psi_\ell \| = 1$ condition yield 

$$\chi \omega_\ell - E_{0\ell} = \frac{\chi^2}{2m_\ell} \int \left[ G'_\ell \left( |\psi_\ell|^2 \right) |\psi_\ell|^2 - G_\ell \left( |\psi_\ell|^2 \right) \right] dy = -\frac{\chi^2}{2m_\ell} K_G$$

implying the Planck-Einstein energy-frequency relation.

- Solving the above differential equations we obtain the following explicit formula 

$$G_\ell (s) = K_G s \ln s + C_\ell s.$$

yielding $K_G < 0$ exactly the logarithmic nonlinearity which corresponds to the Gaussian factor.
Planck-Einstein energy-frequency relation

- For the proper choice of constants $K_{G_{\ell}}$ we obtain

\[ G_{\ell}(s) = G_{\ell,a}(s) = -\frac{1}{(a^\ell)^2} s \ln s + \frac{1}{(a^\ell)^2} s \left( \ln \frac{1}{\pi^{3/2}} - 2 - 3 \ln a \right) \]

where $a^\ell$ is the size parameter for $\ell$-th charge.

- If $K_G = 0$ then $G_{\ell}\left(|\psi_{\ell}|^2\right)$ is quadratic and the eigenvalue equations turn into the linear Schrödinger equations for which fulfillment of the Planck-Einstein relation is a well-known fundamental property.

- It is remarkable that the logarithmic nonlinearity which is singled out by the fulfillment of the Planck-Einstein relation has a second crucial property: it allows for a Gaussian localized soliton solution.

- The above arguments continue to hold if there is an external time-independent electric field.