Supporting Students' Development of an Understanding of Functions Using Nonstandard, Dynamic Representations

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Though gains have been made in recent years in mathematics achievement on the National Assessment of Educational Progress, only 34% of eighth grade students were considered at or above proficient in 2009 (National Center for Education Statistics, 2009). This is particularly troubling when one considers that roughly one-third of the items on the eighth grade NAEP are from the algebra strand (National Center for Education Statistics). A focus on algebra has persisted as a mainstay in mathematics education reform. For the National Council of Teachers of Mathematics (NCTM), “The ideas included in the Algebra Standard constitute a major component of the school mathematics curriculum and help to unify it... All students should learn algebra” (2000, p. 36).

This attention to algebra is to be understood, not as a focus solely on procedures and computations, but as a development of algebraic thinking. Algebraic thinking has been described as developing habits of mind that include the ability to do a particular action and undo it (reversibility), to abstract general properties of numbers from computations (structure), and to build models that represent functions (Driscoll, 1999). Children can begin their mathematical study of reversibility when learning to do arithmetic operations and explore fact families. They have the opportunity to learn mathematical structure as they discover and appropriate properties of equality related to these operations through elementary and middle grades.

One approach for introducing algebraic thinking in elementary and middle school is to engage students in pattern exploration. Ideas relating to patterns are part of students' informal knowledge of algebra (Lannin, Barker, & Townsend, 2006). Pattern finding tasks, especially those which encourage students to make generalizations, promote the development of algebraic reasoning in students. In early grades, students study repeating and growth patterns. Some mathematics educators advocate that these early explorations can transition to the study of functions in late elementary and middle grades. In a review of the research literature on patterns, algebra, and functions, Smith (2003) dichotomizes the study of these topics in school mathematics as either one of stasis or as one of change, with a static view of mathematics underlying most of school mathematics at the expense of an examination of the “mathematics of change”. Whereas a static view of pattern and functions entail attention to a fixed relationship between elements, to the objects and the invariants in them, a perspective examining change focuses on the ways that patterns and functions can be described, translated, and extended.

For example, consider the pattern in Figure 1. A static view may focus on the repetition of 3 two-triangle units. Alternatively, one may observe change by focusing on characteristics that vary within the pattern, such as the accumulation of the total number of triangles, the fact that the number of total sides increases by three from term to term, or a 60° or 180° rotation from term to term. Smith (2003) suggests that through coordination of both the static and dynamic characteristics of patterns, students build a more robust foundation for a formal study of functions and calculus.
These possibilities described by Smith, as well as the historical utility of functions for unifying different areas of mathematics (Dreyfus & Eisenberg, 1982), led us to investigate the ways that students come to understand functions and to design learning activities to support this learning. A review of research literature indicated that the stasis/change dichotomy manifests in two different perspectives on functions. A static view of functions is referred to as correspondence perspective, where one focuses on the relationship between inputs and outputs. A dynamic view is referred to as covariational perspective, where one describes the variation in the two variables and coordinates that change (Saldanha & Thompson, 1998; Confrey & Smith, 1991).

As an example of these two perspectives, consider the pattern in Figure 2 to be a square train. One could investigate its perimeter in relation to the number of cars in the train. By explicitly introducing the number of squares (i.e., cars) to the pattern, there is now a relationship between two varying quantities, the perimeter and the number of cars in the train. A static or correspondence perspective of this function considers each instantiation, associating a specific number of cars with a specific perimeter of the train, such as a two-car train has a perimeter of six units. Conversely, a covariational perspective to this function considers that as the number of cars increases by one, the perimeter increases by two, and the coordination of this change results in twice the number of sides with an adjustment for the adjacent sides. This can be generalized as $f(x) = 2x + 2$, where $x$ represents the number of cars.

The literature also suggests that students' learning of functions is supported by experiences with a variety of representations, both those that are conventional, such as, graphs, tables, symbols, words, and contexts, and those that are nonstandard, like diagrams, gestures, models, and metaphors. Friedlander and Tabach (2001) suggest the use of conventional representations can potentially create meaning for students as they develop algebraic concepts by assisting students in making sense of mathematical ideas. In many cases, the conventional representations mentioned above serve to communicate information about a particular function. Such uses are supported by advances in technology tools which permit new ways of representing mathematical ideas that may deepen understandings of those ideas. Students also communicate their ideas about functional relationships using nonstandard representations. Nonstandard representations may support connecting the context of the pattern with students’ algebraic reasoning and assist students in communicating how they make sense of these relationships. Many times, even when students have had previous exposure to graphing linear functions and relations, students make sense of patterns by representing relationships using nonstandard representations (e.g. numeric, both verbal and written words, diagrams, and gestures).
In the case of pattern finding tasks, these varied uses of multiple representations lead students, over time, to more robust understandings among the relationships found in patterns and ideas relating to functions. In a study of 25 sixth graders engaging in patterning tasks, Lannin (2005) describes the stages through which students progressed as they connected experiences with linear growth patterns to functions. Initially, students simply created each instance of a pattern and counted the attribute of attention. In the next stage, they noticed a recursive relationship between the attribute of interest. In the case of the Square Train in Figure 2, they would notice that the perimeter increased by two each time. In the next stage, students began to make the index of the pattern explicit and drew upon their developing understandings of proportional reasoning and made predictions about the attribute of interest. For the square trains, such reasoning would be that since 5 cars has a perimeter of 12, then a train with 10 cars would have a perimeter of 24 because 10 is two times as large as 5. He notes this strategy may or may not yield a correct answer depending on the patterns initial conditions. At the fourth stage, students created symbolic generalizations through guessing and checking their results with collected instances of the pattern. At the highest stage, they made connections explicitly among their representations. For the Square Train, this understanding might be that the coefficient of two corresponds to the additional two sides on the top and bottom of each added square in the train, whereas the constant of two corresponds to the “front” side of the “engine” and the “end” of the “caboose.”

Task Design

From this review of the literature, we sought to design a series of tasks that would assist students in making the transition from patterning tasks to functions by incorporating multiple and dynamic representations of a family of linear functions. Building on students’ experiences with pattern explorations in elementary grades, the activities intend to provide students with the opportunities to describe growth patterns as a sequence, to describe that sequence in relation to its index of growth, and to coordinate these sequences for an explicit description of the growth in context. These descriptions and observations, related to their use with students, are a part of a larger research study focused on algebraic reasoning which has been reported elsewhere (Berenson, Wilson, Mojica, Lambertus, & Smith, 2007; Mojica et al., 2007; Wilson, Wiebe, Berenson, & Mojica, 2007). Therefore, rather than formally present findings from this research, our intention with this article is to describe for practitioners the series of tasks and to provide examples of how students may react to them based on our observations.

We began instruction with a task where students would create “pattern block trains” and predict perimeters of varying length trains using triangles, squares, and hexagons (see Figure 3). These activities would allow students to use their prior experiences with patterning activities and manipulatives to generate a linear growth sequence for the perimeter. By asking students to predict the perimeters for trains composed of varying numbers of cars, our goals were two-fold. First, we hoped to create a problematic situation where students would realize that recursive understandings were inadequate to complete the task. Secondly, we wanted to encourage students to move toward an explicit description. Additionally, the use of the manipulatives provided opportunities for students to create nonstandard representations, such as models of varying train lengths with pattern blocks, to represent their understanding of the relationship between the length of the train and the train’s perimeter. The manipulatives also served as a medium to unify students’ understanding of the functional relationships that were expressed in these nonstandard representations with the context of the tasks and students numeric (and ultimately symbolic) representations. The manipulatives not only offered a means of collecting and verifying data but also a means of generating deeper meaning of the relationship they would describe.
Next in the series of tasks, students investigated patterns in the perimeter of trains created with polygons for which there were no manipulatives available, such as pentagons and octagons, and ultimately generalizing a relationship between a number of arbitrary polygons in a train and its perimeter. For this task, we anticipated students would use their numeric and symbolic representations from the first task to interpolate (in the case of pentagons) or extrapolate (in the case of octagons) descriptions of the functions. From these new functions, we conjectured students would notice an invariance across the constant term, a variation among the coefficient terms for their generalizations, and begin to conceive of these as a family of linear functions. As a final task, we proposed to challenge students to generalize and describe the linear function family for all polygon perimeter trains.

For a final capstone task, we adapted a nonstandard representation of a function created in a dynamic geometry environment to draw students’ attention to the rates of change among the various generalizations (c.f. http://www.dynamicgeometry.com). Our goal was to give students a new representation of these functions that would provide a sense of motion and support a covariational view of the functions. Rather than use a conventional graph of a function, which is based on perpendicular number lines, dynagraphs (Key Curriculum Press, 2009) orient the number lines such that the axes are parallel. On the independent axis, the user may slide an indicator along the number line and view the changes in a corresponding indicator on the dependent axis dynamically. In the context of the pattern trains tasks, the dynagraphs juxtaposed the dependent perimeter value and the independent the number of cars in the train (see Figure 4). As they explored the different dynagraphs, we conjectured that our students would use their previous understandings of changes in perimeters, in the number of cars, and the resulting coordination to enrich their existing connections among representations and to strengthen both their correspondence and covariation understandings of functions.
Outcomes and Refinements

After progressing through the perimeter trains tasks and reaching common understandings among the different functions and their representations, we asked the students from our teaching experiment to explore the dynagraphs from Figure 4 and to decide which function corresponded with which representation. Groups of students investigated the dynamic sketches, discussed their observations and insights, and shared their conclusions in whole class discussion. From this discussion, they used two strategies to identify the functions.

In one approach, the students compared the relative rates of motion in each representation. By dragging the number of blocks slider on each dynagraph and noting the associated speeds of the perimeter indicator, the students performed pair-wise comparisons and ranked them according to this rate. Then, they connected these ranked dynagraphs to their previous representations of the functions to assimilate the new representation. By controlling the variation in the independent variable, observing the variation in dependent variable, and coordinating those variations between these variables and then among the four representations, the students were approaching the functions from a perspective of covariation. A reenactment of this approach is provided in the article published in Meridian volume 13, issue 1, 2010.

The other approach students used involved aligning the slider representing the number of blocks and then observing and comparing the associated perimeters across the representations. Initially, some students reported they aligned all of the independent sliders to zero but noticed that all of the corresponding outputs were aligned at two. This observation led them to connect this to the constant of 2 from their symbolic representations. From this insight, they chose to align the independent sliders at 1 which corresponded to the perimeter of a train of only 1 block. After ranking these four perimeters, the students quickly incorporated each representation into their previous conceptions of the functions. Whereas the previous approach focused on the variations among the different variables, students using this
approach focused on equating the value of the independent variable and then comparing their matching outputs and were approaching the functions from a perspective of correspondence. As with the previous approach, a reenactment of this strategy is provided in the article published in Meridian volume 13, issue 1, 2010.

There was one unanticipated outcome of the activity, however. When we adapted the dynagraphs to represent the perimeter train of the functions, we did not remove the scale on the axes. In retrospect, the ability to assign numeric values may have privileged the correspondence approach that some groups of students reported. Removing these scales would have encouraged students to attend to the rates of change and thus supported a covariational approach.

**Conclusion**

Overall, the design of the instructional sequence supported students in transitioning from ideas of linear growth patterns to linear functions. Through an approach that encouraged multiple representations and supported both covariational and correspondence perspectives on functions, students were able to move from recursive notions of patterns to explicit and contextual understandings of linear functions. The use of the nonstandard, dynamic representations afforded students the opportunities focus on Smith’s (2003) notions of mathematical change and to incorporate these ideas with their static understandings of mathematics. Here, we have described how a creative appropriation of technology tools can overcome possible limitations of conventional treatments of functions.

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