LPV Control Theory and
Its Aerospace Applications

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Gain-scheduling Control

Overview

Idea:

Perform linearization-based control designs at several different operating conditions, then interpolate the local designs to yield global nonlinear controller.

- Ad-hoc engineering practice guided by rules of thumb such as
  - the scheduling variable should capture the plant’s nonlinearities
  - the scheduling variable should vary slowly,

- Lack of theoretical foundation, design often needs to be verified through extensive simulation,

- Various LTI control design techniques applicable to solve nonlinear problem.

Since then, systematic gain-scheduling control design technique has been developed within the linear parameter-varying framework.
Motivated by gain-scheduling control design methodology.

Linear Parameter Varying (LPV) system $P_\theta$

\[
\begin{bmatrix}
\dot{x}(t) \\
e(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A(\theta(t)) & B_1(\theta(t)) & B_2(\theta(t)) \\
C_1(\theta(t)) & D_{11}(\theta(t)) & D_{12}(\theta(t)) \\
C_2(\theta(t)) & D_{21}(\theta(t)) & D_{22}(\theta(t))
\end{bmatrix}
\begin{bmatrix}
x(t) \\
d(t) \\
u(t)
\end{bmatrix}
\]

Parameter $\theta$:

- is time-varying,
- takes on values in a known, compact set, $\mathcal{P} = \{\theta : \ h_i(\theta) \geq 0, \ i = 1, \ldots, n_h\}$, and known bounds on $\underline{\theta} \leq \dot{\theta} \leq \bar{\theta}$,
- parameter values $\theta(t)$ are not known in advance, but measurable in real-time.
For control,

- controller is also parameter-dependent, using the available real-time information of the parameter variation,
- controller $K_\theta$ depends on
  - $y$ linearly
  - $\theta$ nonlinearly,
- closed-loop performance should be “optimized” with respect to time-varying parameters.
Parameter-dependent System

Description (cont’d)

Class of LPV controller $K_\theta$

\[
\begin{bmatrix}
\dot{x}_k(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
A_k(\theta(t), \dot{\theta}(t)) & B_k(\theta(t), \dot{\theta}(t)) \\
C_k(\theta(t), \dot{\theta}(t)) & D_k(\theta(t), \dot{\theta}(t))
\end{bmatrix}
\begin{bmatrix}
x_k(t) \\
y(t)
\end{bmatrix}
\]

LPV system performance index measures “worst-case” (over all allowable parameter trajectories) induced $\mathcal{L}_2$ gain from $d \to e$

\[||G_\theta||_{i,2} = \min_{K_\theta \text{ allowable } \theta \text{ and } d} \max ||\mathcal{F}_\ell(P_\theta, K_\theta)d||_2 \frac{||\mathcal{F}_\ell(P_\theta, K_\theta)d||_2}{||d||_2}\]

where $\mathcal{F}_\ell(P_\theta, K_\theta)$ is the interconnection of plant and controller.
The stability of LPV system can be determined by finding a parameter-dependent Lyapunov function \( V(x, \theta) = x^T P(\rho)x > 0 \) such that

\[
\frac{\partial V}{\partial x} \dot{x} = x^T \left( A^T(\rho)P(\rho) + P(\rho)A(\rho) \right) x < 0
\]

Moreover, using \( S \)-procedure, the performance of the system is guaranteed if

\[
\frac{\partial V}{\partial x} \dot{x} + \frac{1}{\gamma} e^T e - \gamma d^T d < 0
\]
The closed-loop LPV system is stabilized exponentially by an LPV controller and 
\[ \|e\|_2 < \gamma \|d\|_2 \]

if

there exist matrix functions \( R(\theta), S(\theta) > 0 \), such that for all \( \theta \in \mathcal{P} \),

\[
\begin{align*}
N_R^T(\theta) &= \begin{bmatrix}
\begin{bmatrix}
A(\theta)R(\theta) + R(\theta)A^T(\theta) - \{\nu, \bar{\nu}\} \frac{\partial R}{\partial \theta} & R(\theta)C_1^T(\theta) & B_1(\theta) \\
C_1(\theta)R(\theta) & -I & D_{11}(\theta) \\
B_1^T(\theta) & D_{11}(\theta) & -I
\end{bmatrix}
\end{bmatrix} N_R(\theta) < 0 \\
N_S^T(\theta) &= \begin{bmatrix}
\begin{bmatrix}
A^T(\theta)S(\theta) + S(\theta)A(\theta) + \{\nu, \bar{\nu}\} \frac{\partial S}{\partial \theta} & S(\theta)B_1(\theta) & C_1^T(\theta) \\
B_1^T(\theta)S(\theta) & -I & D_{11}(\theta) \\
C_1(\theta) & D_{11}(\theta) & -I
\end{bmatrix}
\end{bmatrix} N_S(\theta) < 0
\end{align*}
\]

where

\[
N_R(\theta) = \text{Ker} \begin{bmatrix} B_2^T(\theta) & D_{12}^T(\theta) & 0 \end{bmatrix}, \quad N_S(\theta) = \text{Ker} \begin{bmatrix} C_2(\theta) & D_{21}(\theta) & 0 \end{bmatrix}.
\]
• The bound on $\dot{\theta}$ is captured by the term $\{\nu, \tilde{\nu}\} \frac{\partial(\cdot)}{\partial \theta}$,

• Look for matrix functions $R(\theta), S(\theta)$, needs to be solved for all $\theta \in \mathcal{P}$,

• Convex constraints on the functions $R(\theta)$ and $S(\theta)$, called linear matrix inequalities (LMIs) or semidefinite programming,

• If all the matrices are constant, this set of LMIs is a generalization of the well-known $\mathcal{H}_\infty$ control condition for LTI systems.
How do we solve this parameter-dependent LMIs?

- Pick basis for $R(\theta)$ and $S(\theta)$, so $R(\theta) = \sum_{i=1}^{n_f} f_i(\theta) R_i$, $S(\theta) = \sum_{i=1}^{n_g} g_i(\theta) S_i$,
- Grid the set $\mathcal{P}$,
- Solve the remaining convex optimization problem in the matrix variables $R_i, S_i$, subject to the constraints at the grid points.

Ad-hoc approach of function parameterization.

Computationally intensive.

Need to validate the solution over the entire set $\mathcal{P}$.

Other approaches include different convex relaxations of LPV problems, but generally conservative.
**Sum-of-Squares (SOS)**

A multivariate polynomial $f(x_1, \cdots, x_n)$ is an SOS, if there exist polynomials $f_1(x), \cdots, f_m(x)$ such that $f(x) = \sum_{i=1}^{m} f_i^2(x)$.

The SOS decomposition can be solved using semidefinite programming.

This could help

- to provide coherent methodology of synthesizing Lyapunov functions for nonlinear systems,
- to provide tractable relaxations for many difficult optimization problems (better than $S$-procedure).

The application of powerful SOS tool enables us to tackle polynomial nonlinear systems and improve system performance by searching more sophisticated Lyapunov functions.
New SOS Approach for LPV Control  Sufficient Condition

There exist polynomial matrix functions \( R(\theta), S(\theta) > 0 \) and SOS multipliers \( m_{1i}(z, \theta, \nu), m_{2i}(z, \theta, \nu), m_{3i}(z, \theta), n_{1}(z, \theta), n_{2}(z, \theta) \) such that

\[
-z_1^T \mathcal{N}_R^T(\rho) \left[ A(\theta)R(\theta) + R(\theta)A^T(\theta) - \{\nu, \bar{\nu}\} \frac{\partial R}{\partial \theta} \begin{bmatrix} RC_1^T(\theta) & B_1(\theta) \\ C_1(\theta)R & -\gamma I \end{bmatrix} \begin{bmatrix} RC_1^T(\theta) & B_1(\theta) \\ C_1(\theta)R & -\gamma I \end{bmatrix} \right] \mathcal{N}_R(\rho) z_1
\]

\[
- \sum_{i=1}^{r_f} m_{1i}(z_1, \theta, \nu)h_i(\theta) - n_1(z_1, \theta)(\nu - \bar{\nu})(\bar{\nu} - \nu) - \epsilon_1 z_1^T z_1 \text{ is SOS}
\]

\[
-z_2^T \mathcal{N}_S^T(\rho) \left[ A^T(\theta)S(\theta) + S(\theta)A(\theta) + \{\nu, \bar{\nu}\} \frac{\partial S}{\partial \theta} \begin{bmatrix} SB_1(\theta) & C_1^T(\theta) \\ B_1^T(\theta)S & -\gamma I \end{bmatrix} \begin{bmatrix} SB_1(\theta) & C_1^T(\theta) \\ B_1^T(\theta)S & -\gamma I \end{bmatrix} \right] \mathcal{N}_S(\rho) z_2
\]

\[
- \sum_{i=1}^{r_f} m_{2i}(z_2, \theta, \nu)h_i(\theta) - n_2(z_2, \theta)(\nu - \bar{\nu})(\bar{\nu} - \nu) - \epsilon_2 z_2^T z_2 \text{ is SOS}
\]

\[
z_3^T \begin{bmatrix} R(\theta) & I \\ I & S(\theta) \end{bmatrix} z_3 - \sum_{i=1}^{r_f} m_{3i}(z_3, \theta)h_i(\theta) \text{ is SOS}
\]

for any vectors \( z_1, z_2 \) and \( z_3 \) and some positive numbers \( \epsilon_1, \epsilon_2 \).
• The computational complexity of LPV analysis and synthesis conditions based on SOS decomposition is polynomial in time,

• Less conservative than other relaxation methods for polynomial LPV systems,

• If the LPV systems have affine parameter dependency, then the corresponding LPV problems can be solved exactly using affine parameter-dependent Lyapunov functions.
LPV Applications

Aerospace industry related:
- aircraft control,
- aircraft fault tolerance,
- missile autopilot,
- spacecraft attitude control.

Other application areas:
- nuclear reactor,
- automotive engine control.
- robotic manipulator, etc.
My Current Research

- Missile autopilot design,
- High angle-of-attack aircraft control (sponsored by NASA LaRC)
  - Actuator saturation control
  - LPV control switching
  - Robust LPV control synthesis algorithms,
- Development of advanced LPV analysis and control techniques (sponsored by NSF)
  - Exploring powerful tool of SOS, not only for numerical calculation, but also for performance improvement
  - Unmanned aerial vehicle (UAV) control design automation.
Summary

LPV control methodology provides

- simplified, systematic controller design approach,
- guaranteed stability and performance properties,
- convex synthesis conditions, in fact, resulting LMI (or semidefinite programming) problem can be solved efficiently using interior-point algorithms,
- easy to treat some practical issues in control design, such as, variable time delay, input saturation.

Our future work will focus on SOS-based nonlinear $\mathcal{H}_\infty$ control techniques for polynomial nonlinear systems.

It could bridge the gap between linear and nonlinear systems, and provide solid foundation to the development of next generation advanced control techniques.
Consider robust stability of a parameter-dependent system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 \\
(-2 - p(t))x_1 - x_2
\end{bmatrix} = f(x, p) \quad 0 < p(t) < k
\]

Using two types of Lyapunov functions:

- homogeneous non-quadratic Lyapunov functions \( V_n(x) = \sum_{i=0}^{2n} a_i x_1^i x_2^{2n-i}, \quad n = 2, \cdots, 5 \) with the stability condition
  
  \[- \frac{\partial V}{\partial x} f(x, p) \geq 0 \quad \text{for} \quad p = 0, k\]

- parameter-dependent quadratic Lyapunov function \( V(x, p) = (a_1 + a_2 p)x_1^2 + (b_1 + b_2 p)x_1 x_2 + (c_1 + c_2 p)x_2^2 \) with its stability criterion
  
  \[- \frac{\partial V}{\partial x} f(x, p) \pm \nu \frac{\partial V}{\partial p} - q_i(x) p(k - p) \geq 0 \quad \text{with} \quad q_i(x) = x^T Q_i x \geq 0\]
**Conclusion:**

- Non-quadratic Lyapunov functions has the potential to expand stability margins as \( n \) increases,

- Using parameter-dependent Lyapunov functions, significant improvement is achievable when parameter variation rate \( \nu \) decreases.